Research Article

The Improved Moving Least-Square Ritz Method for the One-Dimensional Sine-Gordon Equation

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Analysis of the one-dimensional sine-Gordon equation is performed using the improved moving least-square Ritz method (IMLS-Ritz method). The improved moving least-square approximation is employed to approximate the 1D displacement field. A system of discrete equations is obtained by application of the Ritz minimization procedure. The effectiveness and accuracy of the IMLS-Ritz method for the sine-Gordon equation are investigated by numerical examples in this paper.

1. Introduction

It is well known that many physical phenomena in one or higher-dimensional space can be described by a soliton model. Many of these models are based on simple integrable models such as Korteweg-de Vries equation and the nonlinear Schrödinger equation. Solitons have found to model among others shallow-water waves, optical fibres, Josephson-junction oscillators, and so forth. Equations which also lead to solitary waves are the sine-Gordon. The sine-Gordon equation arises in extended rectangular Josephson junctions, which consist of two layers of super conducting materials separated by an isolating barrier. A typical arrangement is a layer of lead and a layer of niobium separated by a layer of niobium oxide. A quantum particle has a nonzero significant probability of being able to penetrate to the other side of a potential barrier that would be impenetrable to the corresponding classical particle. This phenomenon is usually referred to as quantum tunneling [1–3].

A numerical study for sine-Gordon equation has been proposed including the finite difference schemes [4–6], the finite element methods [7, 8], the modified Adomian decomposition method [9], the boundary integral equation approach [10], radius basis function (RBF) [11], the meshless local Petrov-Galerkin (MLPG) [12], the discrete singular convolution [13], meshless local boundary integral equation method (LBIE) [14], the dual reciprocity boundary element

method (DRBEM) [15], and the mesh-free kp-Ritz method [16].

The meshless method is a new and interesting numerical technique. Important meshless methods have been developed and proposed, such as smooth particle hydrodynamics methods (SPH) [17], radial basis function (RBF) [18], element free Galerkin method (EFG) [19], meshless local Petrov-Galerkin method (MLPG) [20], reproducing kernel particle method (RKPM) [21–23], the boundary element free method (BEFM) [24–27], the complex variable meshless method [28–36], improved element free Galerkin method (IEFG) [37–41], and the improved meshless local Petrov-Galerkin method [42–46].

The moving least-square (MLS) technique was originally used for data fitting. Nowadays, the MLS technique has been employed as the shape functions of the meshless method or element-free Galerkin (EFG) method [47]. Though EFG method is now a very popular numerical computational method, a disadvantage of the method is that the final algebraic equations system is sometimes ill-conditioned. Sometimes a poor solution will be obtained due to the ill-conditioned system. The improved moving least-square (IMLS) approximation has been proposed [48–52] to overcome this disadvantage. In the IMLS, the orthogonal function system with a weight function is chosen to be the basis functions. The algebraic equations system in the IMLS approximation will be no more ill-conditioned.

The Ritz [53] approximation technique is a generalization of the Rayleigh [54] method and it has been widely used in computational mechanics. The element-free kp-Ritz method is firstly developed and implemented for the free vibration analysis of rotating cylindrical panels by Liew et al. [55]. The kp-Ritz method was widely applied and used in many kinds of problems, such as free vibration of two-side simply-supported laminated cylindrical panels [56], nonlinear analysis of laminated composite plates [57], Sine-Gordon equation [16], 3D wave equation [58], and biological population problem [59].

A new numerical method which is named the IMLS-Ritz method for the sine-Gordon equation is presented in this paper. In this paper, the unknown function is approximated by these IMLS approximation; a system of nonlinear discrete equations is obtained by the Ritz minimization procedure, and the boundary conditions are enforced by the penalty method. Numerical examples are presented to validate the accuracy and efficiency of the proposed method.

2. IMLS-Ritz Formulation for the Sine-Gordon Equation

Consider the following one-dimensional sine-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \sin(u), \quad a \leq x \leq b, \quad 0 \leq t < \infty \quad (1a)$$

with initial conditions

$$u(x, 0) = \varphi_1(x), \quad (1b)$$

$$\frac{\partial u(x, 0)}{\partial t} = \varphi_2(x) \quad (1c)$$

and boundary conditions

$$u(a, t) = h_1(t), \quad u(b, t) = h_2(t), \quad (1d)$$

where Γ denotes the domain of x , Γ_u denotes the boundaries, and $\varphi_1(x)$ and $\varphi_2(x)$ are wave modes or kinks and velocity, respectively. Parameter β is the so-called dissipative term, assumed to be a real number with $\beta \geq 0$.

The weighted integral form of (1a) is obtained as follows:

$$\int_{\Gamma} w \cdot \left[\frac{\partial^2 u}{\partial x^2} - \sin(u) - \frac{\partial^2 u}{\partial t^2} - \beta \frac{\partial u}{\partial t} \right] d\Gamma = 0. \quad (2)$$

The weak form of (2) is

$$\int_{\Gamma} \left[\nabla^T w \nabla u + w \sin(u) + w \frac{\partial^2 u}{\partial t^2} + \beta w \frac{\partial u}{\partial t} \right] d\Gamma = 0. \quad (3)$$

The energy functional $\Pi(u)$ can be written as

$$\begin{aligned} \Pi(u) = & \frac{1}{2} \int_{\Gamma} \nabla^T u \cdot \nabla u d\Omega \\ & + \int_{\Gamma} \left[u \frac{\partial^2 u}{\partial t^2} + \beta u \frac{\partial u}{\partial t} + u \sin(u) \right] d\Gamma. \end{aligned} \quad (4)$$

In the improved moving least-square approximation [19], define a local approximation by

$$\mathbf{u}^h(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{j=1}^m \mathbf{p}_j(\bar{\mathbf{x}}) \mathbf{a}_j(\mathbf{x}) \equiv \mathbf{p}^T(\bar{\mathbf{x}})(\mathbf{x}). \quad (5)$$

This defines the quadratic form

$$J = \sum_{i=1}^n w(\mathbf{x} - \mathbf{x}_i) \left[u^h(\mathbf{x}, \mathbf{x}_i) - u(\mathbf{x}_i) \right]^2. \quad (6)$$

Equation (6) can be rewritten in the vector form

$$J = (\mathbf{p}\mathbf{a} - \mathbf{u})^T \mathbf{W}(\mathbf{x})(\mathbf{p}\mathbf{a} - \mathbf{u}). \quad (7)$$

To find the coefficients $\mathbf{a}(\mathbf{x})$, we obtain the extremum of J by

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{A}(\mathbf{x}) \mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x}) \mathbf{u} = 0 \quad (8)$$

which results in the equation system

$$\mathbf{A}(\mathbf{x}) \mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x}) \mathbf{u}. \quad (9)$$

If the functions $p_1(\mathbf{x}), p_2(\mathbf{x}), \dots, p_m(\mathbf{x})$ satisfy the conditions

$$(p_k, p_j) = \sum_{i=1}^n w_i p_k(\mathbf{x}_i) p_j(\mathbf{x}_i) = \begin{cases} 0 & k \neq j \\ A_k & k = j \end{cases} \quad (10)$$

$(k, j = 1, 2, \dots, m),$

then $p_1(\mathbf{x}), p_2(\mathbf{x}), \dots, p_m(\mathbf{x})$ is called a weighted orthogonal function set with a weight function $\{w_i\}$ about points $\{\mathbf{x}_i\}$. The weighted orthogonal basis function set $\mathbf{p} = (p_i)$ can be formed with the Schmidt method [39–41],

$$\begin{aligned} p_1 &= 1, \\ p_i &= r^{i-1} - \sum_{k=1}^{i-1} \frac{(r^{i-1}, p_k)}{(p_k, p_k)} p_k, \quad i = 2, 3, \dots \end{aligned} \quad (11)$$

Equation (9) can be rewritten as

$$\begin{bmatrix} (p_1, p_1) & 0 & \cdots & 0 \\ 0 & (p_2, p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (p_m, p_m) \end{bmatrix} \begin{bmatrix} a_1(\mathbf{x}) \\ a_2(\mathbf{x}) \\ \vdots \\ a_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} (p_1, u_I) \\ (p_2, u_I) \\ \vdots \\ (p_m, u_I) \end{bmatrix}. \quad (12)$$

The coefficients $a_i(\mathbf{x})$ can be directly obtained as follows:

$$a_i(\mathbf{x}) = \frac{(p_i, u_I)}{(p_i, p_i)}; \quad (i = 1, 2, \dots, m); \quad (13)$$

that is,

$$\mathbf{a}(\mathbf{x}) = \bar{\mathbf{A}}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{u}, \quad (14)$$

where

$$\bar{\mathbf{A}}(\mathbf{x}) = \begin{bmatrix} \frac{1}{(p_1, p_1)} & 0 & \cdots & 0 \\ 0 & \frac{1}{(p_2, p_2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{(p_m, p_m)} \end{bmatrix}. \quad (15)$$

From (12), the approximation function $u^h(\mathbf{x})$ can be rewritten as

$$u^h(\mathbf{x}) = \bar{\Phi}(\mathbf{x}) \mathbf{u} = \sum_{I=1}^n \bar{\Phi}_I(\mathbf{x}) u_I, \quad (16)$$

where $\bar{\Phi}(\mathbf{x})$ is the shape function and

$$\bar{\Phi}(\mathbf{x}) = (\bar{\Phi}_1(\mathbf{x}), \bar{\Phi}_2(\mathbf{x}), \dots, \bar{\Phi}_n(\mathbf{x})) = \mathbf{p}^T(\mathbf{x}) \bar{\mathbf{A}}(\mathbf{x}) \mathbf{B}(\mathbf{x}). \quad (17)$$

Taking derivatives of (17), we can obtain the first derivatives of shape function

$$\bar{\Phi}_{I,i}(\mathbf{x}) = \sum_{j=1}^m \left[p_{j,i}(\bar{\mathbf{A}}\mathbf{B})_{jI} + p_j(\bar{\mathbf{A}}_i\mathbf{B} + \bar{\mathbf{A}}\mathbf{B}_i)_{jI} \right]. \quad (18)$$

Imposing boundary conditions by penalty method, the total energy functional for this problem will be obtained:

$$\begin{aligned} \Pi^*(u) = & \frac{1}{2} \int_{\Gamma} \nabla^T u \cdot \nabla u \, d\Omega \\ & + \int_{\Gamma} \left[u \frac{\partial^2 u}{\partial t^2} + \beta u \frac{\partial u}{\partial t} + u \sin(u) \right] d\Gamma \\ & + \frac{\alpha}{2} \int_S (u - \bar{u})^2 dS. \end{aligned} \quad (19)$$

By (16), we can derive the approximation function

$$\begin{aligned} u^h(x, t) &= \sum_{I=1}^n \Phi_I(x) \cdot T_I(t) = \Phi(x) \cdot \mathbf{T}, \\ \frac{\partial u(x, t)}{\partial t} &= \frac{\partial}{\partial t} \sum_{I=1}^n \Phi_I(x) \cdot T_I(t) \\ &= \sum_{I=1}^n \Phi_I(x) \cdot \frac{\partial T_I(t)}{\partial t} = \Phi(x) \dot{\mathbf{T}}, \\ \frac{\partial^2 u(x, t)}{\partial t^2} &= \frac{\partial^2}{\partial t^2} \sum_{I=1}^n \Phi_I(x) \cdot T_I(t) \\ &= \sum_{I=1}^n \Phi_I(x) \cdot \frac{\partial^2 T_I(t)}{\partial t^2} = \Phi(x) \ddot{\mathbf{T}}. \end{aligned} \quad (20)$$

Substituting (20) into (19) and applying the Ritz minimization procedure to the energy function $\Pi^*(u)$, we obtain

$$\begin{aligned} \frac{\partial \Pi^*(u)}{\partial \Delta} = 0, \quad \Delta = T_I(t), \quad \frac{\partial T_I(t)}{\partial t}, \quad \frac{\partial^2 T_I(t)}{\partial t^2}, \\ I = 1, 2, \dots, n. \end{aligned} \quad (21)$$

In the matrix form, the results can be expressed as

$$\mathbf{C}\ddot{\mathbf{T}} + \beta\mathbf{C}\dot{\mathbf{T}} + \mathbf{K}\mathbf{T} = \mathbf{F}, \quad (22)$$

where

$$\begin{aligned} C_{IJ} &= \int_{\Gamma} \Phi_I(x) \Phi_J(x) \, d\Gamma, \\ K_{IJ} &= \int_{\Gamma} \nabla \Phi_I^T(\mathbf{x}) \nabla \Phi_J(\mathbf{x}) \, d\Gamma + \alpha (\Phi_I \Phi_J|_{x=a} + \Phi_I \Phi_J|_{x=b}), \\ F_I &= - \int_{\Gamma} \Phi_I(x) \sin(T) \, d\Gamma + \alpha (\Phi_I \bar{u}|_{x=a} + \Phi_I \bar{u}|_{x=b}). \end{aligned} \quad (23)$$

Making time discretization of (22) by the center difference method, we get

$$\begin{aligned} \mathbf{C} \frac{\mathbf{T}^{(k+2)} - 2\mathbf{T}^{(k+1)} + \mathbf{T}^{(k)}}{\Delta t^2} \\ + \beta \mathbf{C} \frac{\mathbf{T}^{(k+1)} - \mathbf{T}^{(k)}}{\Delta t} + \mathbf{K} \frac{\mathbf{T}^{(k+1)} + \mathbf{T}^{(k)}}{2} = \mathbf{F}, \end{aligned} \quad (24)$$

where

$$\begin{aligned} \mathbf{T}^{(k)} = \mathbf{T}(t_k) &= (T_1(t_k), T_2(t_k), \dots, T_n(t_k)), \\ F_I &= - \int_{\Gamma} \Phi_I(x) \, d\Gamma \cdot \sin(\bar{T}) + \alpha (\Phi_I \bar{u}|_{x=a} + \Phi_I \bar{u}|_{x=b}). \end{aligned} \quad (25)$$

The numerical solution of the one-dimensional sine-Gordon equation will be obtained by solving the above iteration equation.

3. Numerical Examples and Analysis

To verify the efficiency and accuracy of the proposed IMLS-Ritz method for the sine-Gordon equation, two examples are studied and the numerical results are presented. The weight function is chosen to be cubic spline and the bases are chosen to be linear in all examples.

Example 1. Consider the sine-Gordon Equation (1a)–(1d) without nonlinear term $\sin(u)$ over the region $-1 \leq x \leq 1$ with initial conditions

$$\begin{aligned} u(x, 0) &= \varphi_1(x) = \sin(\pi x), \\ u_t(x, 0) &= \varphi_2(x) = 0 \end{aligned} \quad (26)$$

with boundary conditions

$$u(-1, t) = u(1, t) = 0. \quad (27)$$

The exact solution is

$$u(x, t) = \frac{1}{2} (\sin \pi(x+t) + \sin \pi(x-t)). \quad (28)$$

TABLE 1: The comparisons of exact solution with numerical solutions by IELG and EFG methods with 41 nodes at $t = 0.1$ with $dt = 0.001$ and $d_{\max} = 3.0$ (Example 1).

Node number	Exact solution	IMLS-Ritz method	EFG method
21	0.1488	0.1479	0.1473
22	0.2939	0.2918	0.2909
23	0.4318	0.4297	0.4275
24	0.5590	0.5583	0.5536
25	0.6725	0.6732	0.6661
26	0.7694	0.7687	0.7653
27	0.8474	0.8471	0.8399
28	0.9045	0.9032	0.8989
29	0.9393	0.9389	0.9380
30	0.8474	0.8478	0.8487
31	0.7694	0.7685	0.7625
32	0.6725	0.6732	0.6753

FIGURE 1: Numerical solution and exact solution of $u(x, t)$ when $t = 0.1, 0.2, 0.3, 0.4$ (Example 1).

The IMLS-Ritz method is applied to solve the above equation with penalty factor $\alpha = 10^4$ and time step length $\Delta t = 0.001$, $d_{\max} = 3.0$. In Figure 1, the numerical solution and exact solution are plotted at times $t = 0.1, 0.2, 0.3$, and 0.4 , respectively. In Figures 2, 3, and 4, the graphs of error function $u^h(x, t) - u(x, t)$ are plotted at times $t = 0.1, 0.2$, and 0.3 , respectively, where $u(x, t)$ is the exact solution and numerical solution $u^h(x, t)$ is obtained by using the IMLS-Ritz method. Table 1 shows the comparison of exact solutions and numerical solutions by IMLS-Ritz method and EFG method. From the results of Table 1, we can draw the conclusion that IMLS-Ritz method has higher accuracy than the EFG method. The surfaces of the numerical solution with the IMLS-Ritz method and exact solutions are plotted in Figures 5 and 6.

FIGURE 2: Error function $u^h(x, t) - u(x, t)$, where $u(x, t)$ is an exact solution and numerical solution $u^h(x, t)$ is obtained by using the IMLS-Ritz method with $dt = 0.001$ at $t = 0.1$ (Example 1).

Example 2. Consider the case $\beta = 0$ in (1a)–(1d) over the rectangular region $-20 \leq x \leq 20$ and initial condition

$$\begin{aligned} u(x, 0) &= \varphi_1(x) = 0, \\ u_t(x, 0) &= \varphi_2(x) = 4\operatorname{sech}(x) \end{aligned} \quad (29)$$

which derives the analytic solution

$$u(x, t) = 4\tan^{-1}(\operatorname{sech}(xt)) \quad (30)$$

and the boundary conditions can be obtained from (30).

The IMLS-Ritz method is applied to solve the above equation with penalty factor $\alpha = 10^5$ and time step length $\Delta t = 0.01$, $d_{\max} = 2.2$. Figure 7 depicts the numerical and exact solution when $t = 1, 5, 10, 20$, and 30 , respectively. In Figures 8, 9, and 10, the graphs of error function $u^h(x, t) - u(x, t)$ are plotted at times $t = 1, 5$, and 10 , respectively, where

FIGURE 3: Error function $u^h(x, t) - u(x, t)$, where $u(x, t)$ is an exact solution and numerical solution $u^h(x, t)$ is obtained by using the IMLS-Ritz method with $dt = 0.001$ at $t = 0.2$ (Example 1).

FIGURE 4: Error function $u^h(x, t) - u(x, t)$, where $u(x, t)$ is an exact solution and numerical solution $u^h(x, t)$ is obtained by using the IMLS-Ritz method with $dt = 0.001$ at $t = 0.3$ (Example 1).

FIGURE 5: The surface of exact solution (Example 1).

FIGURE 6: The surface of numerical solution with IMLS-Ritz method (Example 1).

FIGURE 7: Numerical solution and exact solution of $u(x, t)$ when $t = 1, 5, 10, 20, 30$ (Example 2).

$u(x, t)$ is the exact solution and numerical solution $u^h(x, t)$ is obtained by using the IMLS-Ritz method. Table 2 shows the comparison of exact solutions and numerical solutions by IMLS-Ritz method and EFG method. From the results of Table 2, it is shown that IMLS-Ritz method has higher accuracy than the EFG method. The surfaces of the numerical solution with the IMLS-Ritz method and exact solution are plotted in Figures 11 and 12.

Example 3. Consider the case in (1a)–(1d) over the rectangular region $-20 \leq x \leq 20$ and initial condition

$$u(x, 0) = \varphi_1(x) = 4 \arctan(e^\eta),$$

$$u_t(x, 0) = \varphi_2(x) = -\frac{4ce^\eta}{\sqrt{1-c^2}(1+e^{2\eta})} \quad (31)$$

FIGURE 8: Error function $u^h(x, t) - u(x, t)$ with $dt = 0.01$ at $t = 1$ (Example 2).

FIGURE 9: Error function $u^h(x, t) - u(x, t)$ with $dt = 0.01$ at $t = 5$ (Example 2).

and the boundary conditions can be derived from the following exact solitary wave solution:

$$u(x, t) = 4 \arctan\left(e^{(x-ct)/\sqrt{1-c^2}}\right), \quad (32)$$

where $\eta = x/\sqrt{1-c^2}$ and c is the velocity of solitary wave.

The IMLS-Ritz method is applied to solve the above equation with penalty factor $\alpha = 10^7$ and time step length $\Delta t = 0.01$, $d_{\max} = 2.7$, $c = 0.5$. Table 3 shows the comparison of exact solutions and numerical solutions by IMLS-Ritz method and EFG method. From the results of Table 3, it is shown that IMLS-Ritz method has higher accuracy than the EFG method. The surfaces of the numerical solution with the IMLS-Ritz method and exact solution are plotted in Figures

FIGURE 10: Error function $u^h(x, t) - u(x, t)$ with $dt = 0.01$ at $t = 10$ (Example 2).

FIGURE 11: The surface of exact solution (Example 2).

FIGURE 12: The surface of numerical solution with IMLS-Ritz method (Example 2).

TABLE 2: The comparisons of exact solution with numerical solutions by IIEFG and EFG methods with 40 nodes at $t = 1$ with $dt = 0.01$ and $d_{\max} = 2.2$ (Example 2).

Node number	Exact solution	IMLS-Ritz method	EFG method
10	0.0001	0.0001	0.0001
11	0.0004	0.0004	0.0004
12	0.0010	0.0010	0.0010
13	0.0027	0.0027	0.0026
14	0.0073	0.0072	0.0074
15	0.0198	0.0197	0.0196
16	0.0539	0.0536	0.0535
17	0.1464	0.1463	0.1462
18	0.3960	0.3958	0.3956
19	1.0392	1.0396	1.0442
20	2.3000	2.2993	2.2989
21	3.1416	3.1408	3.1406

TABLE 3: The comparisons of exact solution with numerical solutions by IIEFG and EFG methods with 81 nodes at $t = 1$ with $dt = 0.001$ and $d_{\max} = 2.7$ (Example 3).

Node number	Exact solution	IMLS-Ritz method	EFG method
30	0.0039	0.0037	0.0036
31	0.0070	0.0068	0.0059
32	0.0124	0.0125	0.0131
33	0.0222	0.0231	0.0229
34	0.0395	0.0402	0.0423
35	0.0703	0.0710	0.0722
36	0.1252	0.1251	0.1253
37	0.2228	0.2225	0.2231
38	0.3960	0.3948	0.3952
39	0.7004	0.6978	0.6988
40	1.2212	1.2209	1.2301
41	2.0462	2.0471	2.0483

FIGURE 13: The surface of exact solution (Example 3).

FIGURE 14: The surface of numerical solution with IMLS-Ritz method (Example 3).

13 and 14. In Figure 15, the graph of error function $u^h(x, t) - u(x, t)$ is plotted at time $t = 1$, where $u(x, t)$ is the exact

solution and the numerical solution $u^h(x, t)$ is obtained by using the IMLS-Ritz method.

FIGURE 15: Error function $u^h(x, t) - u(x, t)$ with $dt = 0.01$ at $t = 2$ (Example 3).

From these figures, it is shown that numerical results obtained by the IMLS-Ritz method are in good agreement with the exact solutions.

4. Conclusion

This paper presents a numerical method, named the IMLS-Ritz method, for the one-dimensional sine-Gordon equation. The IMLS approximation is employed to approximate the 1D displacement field. A system of discrete equations is obtained through application of the Ritz minimization. In the IMLS approximation, the basis function is chosen as the orthogonal function system with a weight function. The IMLS approximation has greater computational efficiency and precision than the MLS approximation, and it does not lead to an ill-conditioned system of equations. The numerical results show that the technique is accurate and efficient.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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