Construction of New Exact Solutions for the \((3 + 1)\)-Dimensional Burgers System

Zitian Li

College of Mathematics and Information Science, Qujing Normal University, Qujing, Yunnan 655011, China

Correspondence should be addressed to Zitian Li; lizitian88@163.com

Received 9 July 2014; Accepted 6 October 2014; Published 12 October 2014

Academic Editor: Miguel A. F. Sanjuan

Copyright © 2014 Zitian Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By means of a variable separation method and a generalized direct ansatz function approach, new exact solutions including cross kink-wave solutions, doubly periodic kinky-wave solutions, and breather type of two-solitary wave solutions for the \((3 + 1)\)-dimensional Burgers system are obtained. Moreover, the mechanical features are also investigated.

1. Introduction

Searching for explicit solutions of nonlinear evolution equations by using various different methods is useful and meaningful in physical science and nonlinear science. Many powerful methods have been presented, such as inverse scattering transform [1], Hirota’s bilinear form method [2], two-soliton method [3], homoclinic test technique [4], Bäcklund transformation method [5], and three-wave type of ansatz approach [6]. And much work has been focused on the various extensions and application of the known algebraic methods to construct the solutions of nonlinear evolution equations [7, 8].

In this letter, we will present exact solutions of the \((3 + 1)\)-dimensional Burgers system [9]:

\[
\begin{align*}
    u_t & = 2u u_x + 2v u_x + 2w u_z + u_{xx} + u_{yy} + u_{zz}, \\
    v_x & = v_y, \\
    w_z & = w_y,
\end{align*}
\]  

\( (1) \)

which was derived from the inverse transformation of heat-conduction equation, where \( u(x, y, z, t) \), \( v(x, y, z, t) \), \( w(x, y, z, t) \) are three physical field function and \( u : \mathbb{R}^x \times \mathbb{R}^y \times \mathbb{R}^z \rightarrow \mathbb{R} \). Recently, (1) was widely researched [10–12]. When \( z = x, w = u \), (1) degenerated into the \((2 + 1)\)-dimensional Burgers equation and the soliton-like solutions were obtained by using the extended Riccati equation mapping method [13], and many other results were found in the literature [14, 15]. In our work, we will give some new exact solutions for (1) by using the variable separation hypothesis and the direct ansatz function method all together.

2. Exact Solutions to the \((3 + 1)\)-Dimensional Burgers System

We firstly suppose the solution of system (1) has the following ansatz:

\[
\begin{align*}
    u & = u_0 + (\ln \phi)_y, \\
    v & = v_0 + (\ln \phi)_x, \\
    w & = w_0 + (\ln \phi)_z,
\end{align*}
\]  

\( (2) \)

where \( u_0, v_0 \), and \( w_0 \) are real constants and \( \phi = \phi(x, y, z, t) \) has the following variable separation form:

\[
\phi(x, y, z, t) = a_0 + a_1 f(y, t) + a_2 g(x, z),
\]  

\( (3) \)

with \( a_i (i = 0, 1, 2) \) constants and \( f(y, t), g(x, z) \) are unknown function about \( \{y, t\} \) and \( \{x, z\} \), respectively. Substituting (2) and (3) into (1), we have

\[
a_0 \left[ f_{yy} - 2u_0 f_{yy} - f_{yyy}\right] + a_1 \left[ f_{yy} - 2u_0 f_{yy} - f_{yyy}\right] - f_y \left( f_{tt} - 2u_0 f_{ty} - f_{yyyy}\right) = 0,
\]  

\( (4) \)
\[ f_y (g_{xx} + g_{zz}) + 2 f_y (g_0 g_x + w_0 g_z) + g (f_{yy} - 2 u_0 f_y y - f_{yy} y) = 0. \] (5)

Now let us consider (5), since \( g(x, z) \) is only a function about variables \( x \) and \( z \), letting

\[ \frac{f_y - 2 u_0 f_{yy} - f_{yy}}{f_y} = C \text{ (constant)}, \] (6)

and by using the wave transformation \( \xi = ax + bz + \xi_0 \), (5) can be changed into

\[ g''(\xi) + \frac{2 (a v_0 + b w_0)}{a^2 + b^2} g'(\xi) + \frac{C}{a^2 + b^2} g(\xi) = 0, \] (7)

where \( a, b, \) and \( \xi_0 \) are arbitrary constants. And when \( a_0 \) and \( C \) are taken as \( a_0 = 0, C \neq 0 \) or \( a_0 \neq 0, C = 0 \), we find that (4) is automatically satisfied.

When \( a v_0 + b w_0 = 0 \), solutions of (7) are well known as follows:

(i) \( g(x, z) = c_1 \cos \left[ \sqrt{\frac{C}{a^2 + b^2}} (ax + bz + \xi_0) \right] \\
+ c_2 \sin \left[ \sqrt{\frac{C}{a^2 + b^2}} (ax + bz + \xi_0) \right], \quad C > 0, \)

(ii) \( g(x, z) = c_1 (ax + bz + \xi_0) + c_2, \quad C = 0, \)

(iii) \( g(x, z) = c_1 \cosh \left[ \sqrt{-\frac{C}{a^2 + b^2}} (ax + bz + \xi_0) \right] \\
+ c_2 \sinh \left[ \sqrt{-\frac{C}{a^2 + b^2}} (ax + bz + \xi_0) \right], \quad C < 0, \)

where \( c_1, c_2 \) are arbitrary constants.

Therefore, we only need to solve (6) from which we can obtain the solutions of system (1).

**Example 1.** Let the test function be

\[ f(y, t) = b_1 e^{ky + bt} + b_0 + b_2 e^{-(ky + bt)}, \] (9)

where \( k, l, b_1, b_0, \) and \( b_2 \) are constants to be determined.

By introducing means of computer algebra (i.e., Maple), we obtain

\[ u_1 = u_0 + \left( \frac{2ka_1 \sqrt{b_1 b_2} \sinh (ky + 2ku_0 t + r)}{a_1 b_0 + 2a_1 \sqrt{b_1 b_2} \cosh (ky + 2ku_0 t + r)} \right. \\
+ \left. a_2 \left[ c_1 \cosh B (ax + bz + \xi_0) \\
+ c_2 \sinh B (ax + bz + \xi_0) \right] \right) ^{-1}. \]

\[ v_1 = v_0 + \left( \frac{aa_2 B [c_1 \sinh B (ax + bz + \xi_0) \\
+ c_2 \cosh B (ax + bz + \xi_0)]}{a_1 b_0 + 2a_1 \sqrt{b_1 b_2} \cosh (ky + 2ku_0 t + r)} \right. \\
+ \left. a_2 \left[ c_1 \cosh B (ax + bz + \xi_0) \\
+ c_2 \sinh B (ax + bz + \xi_0) \right] \right) ^{-1}. \]

which is a cross kink-wave solution, where \( r = \ln |b_1/b_2| \), \( B = \sqrt{k^2/(a^2 + b^2)} \) (see Figure 1: F_1).

**Example 2.** Similarly, if \( f(y, t) \) was taken as

\[ f(y, t) = \left( h_1 e^{ky + bt} + h_0 + h_2 e^{-(ky + bt)} \right)^2, \] (11)

where \( k, l, h_0, h_1, \) and \( h_2 \) are constants to be determined later, then, we derived a periodic soliton solution and a multiple-soliton solution as follows:

\[ u_2 = u_0 + \left( \frac{2a_i h_1 k [h_1 e^{ky + (2ku_0 + 3k^2 t) t} + h_0] e^{ky + (2ku_0 + 3k^2 t) t}}{a_1 [h_1 e^{ky + (2ku_0 + 3k^2 t) t} + h_0]^2} \right. \\
+ \left. a_2 \left[ c_1 \cos A (ax + bz + \xi_0) \\
+ c_2 \sin A (ax + bz + \xi_0) \right] \right) ^{-1}, \]

where \( k, l, b_1, b_0, \) and \( b_2 \) are constants to be determined.
where $A = \sqrt{2k^2/(a^2 + b^2)}$ and $B = \sqrt{4k^2/(a^2 + b^2)}$ (see Figure 1: $F_6$, Figure 2: $F_7$).

Example 3. Furthermore, when choosing $f(y, t)$ to be
\begin{equation}
  f(y, t) = L \cos[k_1 y + l_1 t] + e^{k_2 z + l_2 t} + H_0,
\end{equation}

where $A = \sqrt{2k^2/(a^2 + b^2)}$ and $B = \sqrt{4k^2/(a^2 + b^2)}$ (see Figure 1: $F_6$, Figure 2: $F_7$).
with $L, H_0, k_1, k_2, l_1,$ and $l_2$ constants, we obtained a breather type of two-solitary wave solution as follows:

$$u_4 = u_0 + \left( a_1 k_2 \left[ \sinh (\xi_1) + \cosh (\xi_1) \right] - a_1 k_2 L \sin (\xi_2) \right) \times \left( a_1 H_0 + a_1 \left[ \cosh (\xi_1) + \sinh (\xi_1) \right] + a_1 L \cos (\xi_2) \right) \nonumber + a_2 \left[ c_1 \cos (\xi_3) + c_2 \sin (\xi_3) \right] \right)^{-1},$$

$$v_4 = v_0 - \left( a_2 a B \left[ c_1 \sin \left[ B (ax + bz + \xi_0) \right] \right] - c_2 \cos \left[ B (ax + bz + \xi_0) \right] \right) \times \left( a_1 H_0 + a_1 \left[ \cosh (\xi_1) + \sinh (\xi_1) \right] + a_1 L \cos (\xi_2) \right) \nonumber + a_2 \left[ c_1 \cos (\xi_3) + c_2 \sin (\xi_3) \right] \right)^{-1},$$

$$w_4 = w_0 - \left( a_2 b B \left[ c_1 \sin \left[ B (ax + bz + \xi_0) \right] \right] \nonumber - c_2 \cos \left[ B (ax + bz + \xi_0) \right] \right) \times \left( a_1 H_0 + a_1 \left[ \cosh (\xi_1) + \sinh (\xi_1) \right] + a_1 L \cos (\xi_2) \right) \nonumber + a_2 \left[ c_1 \cos (\xi_3) + c_2 \sin (\xi_3) \right] \right)^{-1},$$

where $\xi_1 = k_2 y + (k_1^2 + k_2^2 + 2k_2 u_0)t, \xi_2 = k_1 y + 2k_1 u_0 t, \xi_3 = B(ax + bz + \xi_0)$.

The solution represented by (14) is breather type of two-solitary wave solution which contains a periodic wave and two solitary waves, whose amplitude periodically oscillates with the evolution of time (see Figure 2: $F_8$).

3. Conclusion

In this paper, by using the variable separation method and the generalized direct ansatz method the $(3 + 1)$-dimensional Burgers system is investigated. New exact solutions including cross kink-wave solution, doubly kinky-wave solution, and breather type of two-solitary wave solutions are obtained; these solutions enrich the structures of solutions of the $(3 + 1)$-dimensional Burgers system. Moreover, the mechanical features are also investigated.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by Natural Science Foundation of Yunnan Province under Grant no. 2013FZ113.

References


Submit your manuscripts at
http://www.hindawi.com