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Research Article

Portfolio Selection Based on Distance between Fuzzy Variables

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This paper researches portfolio selection problem in fuzzy environment. We introduce a new simple method in which the distance between fuzzy variables is used to measure the divergence of fuzzy investment return from a prior one. Firstly, two new mathematical models are proposed by expressing divergence as distance, investment return as expected value, and risk as variance and semivariance, respectively. Secondly, the crisp forms of the new models are also provided for different types of fuzzy variables. Finally, several numerical examples are given to illustrate the effectiveness of the proposed approach.

1. Introduction

Portfolio selection is to select a combination of securities among a large number of candidate securities which is the best to meet the investors' goal. Markowitz [1] applied probability theory to portfolio selection problem and proposed the famous mean-variance model, in which expected return and variance were used to describe investment return and risk, respectively. Since then, variance has been widely accepted as a risk measure, and a great number of extensions have been proposed [2–5]. However, the mean-variance model has limited generality since variance considers high returns as equally undesirable as low returns. Variance becomes a deficient measure of risk when security returns are asymmetrical. Thus, Markowitz proposed semivariance as an improvement measure of risk, and numerous models have been developed based on semivariance such as models proposed in [6–10].

Generally speaking, the extension of Markowitz model is defined by minimizing the risk and maximizing the investment return. However, Kapur and Kesavan [11] introduced an entropy maximization model and a cross entropy minimization model. The objective of the entropy model is to maximize the uncertainty of random return and that of the cross entropy model is to minimize the divergence of the random return from a prior one. After that, many scholars accepted and explored these new models [12–15].

In the above literatures, security returns are considered as random variables. Since the security market is complex, in many cases, security returns are hard to be well reflected by historical data. Therefore, many researches argued that we should find another theory to solve the portfolio selection problem in this situation. With the introduction of fuzzy set theory and credibility theory, many scholars began to employ them to describe and study fuzzy portfolio selection problems. Numerous models containing fuzzy variables are proposed. For example, Bilbao-Terol et al. [16], Gupta et al. [17], and Zhang et al. [18] extended the mean-variance model from different angles. Huang [19] developed fuzzy mean-variance models and further proposed fuzzy mean-semivariance portfolio selection models [20]. Li et al. [21] employed skewness to describe asymmetry of fuzzy returns and further established fuzzy mean-variance-skewness models. Bhattacharyya et al. [22] proposed fuzzy mean-variance-skewness portfolio selection models by interval analysis. Huang [23] has used the entropy method to fuzzy environment to provide the fuzzy diversification models. In 2008, Li and Liu [24] proposed a concept of fuzzy entropy for measuring the uncertainty of fuzzy variables. Based on this concept, Huang [25] researched Kapur entropy maximization model in fuzzy environment. Moreover, the fuzzy cross entropy was given in [26] for measuring the divergence of fuzzy variables from a prior one. According to [26], Qin et al. [27] extended the Kapur cross entropy minimization model to fuzzy environment. These models in [27] were solved by using a hybrid intelligent algorithm which is designed by integrating numerical integration, fuzzy simulation, and genetic algorithm.

Distance between fuzzy variables is an important concept in fuzzy theory. Many scholars gave different definitions of distance between fuzzy variables, such as Hamming distance, Euclidean distance, and Minkowski distance. Recently, Tang et al. [28] gave a kind of definition of distance based on expected value operator of fuzzy variable. We define a new distance between fuzzy variables based on distance measure for interval numbers in this paper. Comparing to the distance measure of [28], the proposed distance measure can be calculated more easily.

In this paper, our motivation is that the divergence of fuzzy investment return from a prior one is measured by using the proposed distance between fuzzy variables. Based on this idea, we establish two distance minimization models by defining investment return as expected value and risk as variance and semivariance, respectively. In addition, several crisp and simple equivalents of the optimization models are also proposed for different types of fuzzy variables. Finally, we compare our method with the methods presented by Chen et al. [29] and Wu and Liu [30] to demonstrate the effectiveness of the proposed approach.

The remainder of the paper is organized as follows. Some preliminary concepts of credibility theory are briefly recalled in Section 2. The concept of distance between fuzzy variables is introduced in Section 3. In Section 4, we will propose two new models by minimizing distance between fuzzy variables. In Section 5, the crisp forms of the new models will be presented. Section 6 gives several numerical examples to illustrate availability of the proposed approach. Finally, a brief summary is given in Section 7.

2. Necessary Knowledge about Credibility Theory

After Zadeh [31] initiated the concept of fuzzy set by membership function in 1965, he further indicated possibility theory [32]. Many research scholars, such as Dubois and Prade [33, 34], made their great contribution to its development. In 2002, B. Liu and Y.-K. Liu [35] defined a credibility measure to describe a fuzzy event. In order to develop an axiom system similar to the theory of probability, Liu founded the credibility theory in [36], which is a branch of mathematics for studying fuzzy phenomena. Further developments can be found in [37, 38].

Let ξ be a fuzzy variable with membership function μ. The credibility measure is defined as [35]

$$\text{Cr}\{\xi \in B\} = \frac{1}{2}\left(\operatorname*{sub}_{x\in B}\mu(x) + 1 - \sup_{x\in B^C}\mu(x)\right) \quad (1)$$

for any set B of real numbers. It is easy to see that credibility is self-dual.

In order to make a more general definition of expected value of a fuzzy variable, according to the credibility measure, B. Liu and Y.-K. Liu [35] defined the expected value of ξ as

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \geq r\}\,dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\}\,dr \quad (2)$$

provided that at least one of the two integrals is finite. If the fuzzy variable ξ has a finite expected value, then its variance is defined as [35]

$$V[\xi] = E\left[(\xi - E[\xi])^2\right]. \quad (3)$$

Let ξ be a fuzzy variable with finite expected value. Then the semivariance of ξ is defined as [20]

$$\text{SV}[\xi] = E\left[\left((\xi - E[\xi])^-\right)^2\right], \quad (4)$$

where

$$(\xi - E[\xi])^- = \begin{cases} \xi - E[\xi], & \text{if } \xi \leq E[\xi] \\ 0, & \text{if } \xi > E[\xi]. \end{cases} \quad (5)$$

Generally speaking, expected value is used to measure the return and variance or semivariance is used to reflect the risk in portfolio selection problem.

Example 1. Suppose that $\xi = (a, b, c)$ is a triangular fuzzy variable, and its membership function is given by

$$\mu(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & \text{if } a \leq x \leq b \\ \dfrac{(x-c)}{(b-c)}, & \text{if } b \leq x \leq c \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

According to (1), we have

$$\text{Cr}\{\xi \leq x\} = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{2(b-a)}, & a \leq x \leq b \\ \dfrac{x+c-2b}{2(c-b)}, & b \leq x \leq c \\ 1, & x \geq c, \end{cases}$$
$$\text{Cr}\{\xi \geq x\} = \begin{cases} 1, & x \leq a \\ \dfrac{2b-a-x}{2(b-a)}, & a \leq x \leq b \\ \dfrac{c-x}{2(c-b)}, & b \leq x \leq c \\ 0, & x \geq c. \end{cases} \quad (7)$$

By (2) and (4), it is easy to prove that

$$E[\xi] = e = \frac{a+2b+c}{4},$$
$$\mathrm{SV}[\xi] = \begin{cases} \dfrac{(e-a)^3}{6(b-a)}, & \text{if } b-a \ge c-b \\ \big((a-b)(c-b)(a+2b-3e) \\ \quad +(b-e)^2(3c-4b+e)\big) \\ \times(6(c-b))^{-1}, & \text{if } b-a < c-b. \end{cases} \quad (8)$$

If ξ is a symmetric triangular fuzzy variable with $b-a = c-b$, then $E[\xi] = b$ and

$$V[\xi] = \frac{(c-a)^2}{24}. \quad (9)$$

Example 2. Suppose that $\xi = (a, b, c, d)$ is a trapezoidal fuzzy variable, and its membership function is given by

$$\mu(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & \text{if } a \le x \le b \\ 1, & \text{if } b \le x \le c \\ \dfrac{(x-d)}{(c-d)}, & \text{if } c \le x \le d. \end{cases} \quad (10)$$

According to (1), we obtain

$$\mathrm{Cr}\{\xi \le x\} = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{2(b-a)}, & a \le x \le b \\ \dfrac{1}{2}, & b \le x \le c \\ \dfrac{x+d-2c}{2(d-c)}, & c \le x \le d \\ 1, & x \ge d, \end{cases} \quad (11)$$

$$\mathrm{Cr}\{\xi \ge x\} = \begin{cases} 1, & x \le a \\ \dfrac{2b-a-x}{2(b-a)}, & a \le x \le b \\ \dfrac{1}{2}, & b \le x \le c \\ \dfrac{d-x}{2(d-c)}, & c \le x \le d \\ 0, & x \ge c. \end{cases}$$

From (2) and (4), it is easy to obtain that

$$E[\xi] = e = \frac{a+b+c+d}{4},$$
$$\mathrm{SV}[\xi] = \begin{cases} \dfrac{(e-a)^3}{6(b-a)}, & \text{if } e < b \\ \dfrac{(b-a)(3e-2b-a)+3(e-b)^2}{6}, & \text{if } b \le e \le c \\ \dfrac{(b-a)(3e-2b-a)+3(c-b)(2e-b-c)}{6} \\ \quad +\dfrac{(c-e)^2(3d-4c+e)}{6(d-c)}, & \text{if } e > c. \end{cases} \quad (12)$$

If ξ is a symmetric trapezoidal fuzzy variable with $b-a = d-c$, then $E[\xi] = (a+d)/2$ and $V[\xi] = (1/24)(3(d-b)^2 + (d-c)^2)$.

3. Distance Measure for Fuzzy Variables

3.1. Distance Measure for Interval Numbers

Definition 3 (see [39]). For any real numbers a_1 and a_2, let $A = [a_1, a_2] = \{x \mid a_1 \le x \le a_2\}$; then A is called an interval number.

Definition 4 (see [40]). Suppose that $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are two interval numbers; the distance between A and B can be defined as follows:

$$D(A, B) = \sqrt{\frac{(a_1-b_1)^2 + (a_2-b_2)^2}{2}}. \quad (13)$$

Equation (13) satisfies the properties of the distance metric. In other words, let M be a set of interval numbers; then, for every $A, B, C \in M$, the following conditions hold:

(i) $D(A, B) \ge 0, \quad D(A, B) = 0$ and iff $A = B$; (14)

(ii) $D(A, B) = D(B, A)$; (15)

(iii) $D(A, B) \le D(A, C) + D(C, B)$. (16)

3.2. Distance Measure for Fuzzy Variables. Suppose that ξ is fuzzy variable with membership function $\mu(x)$. Let $u_\lambda = \{x \mid \mu(x) \ge \lambda\}$ be the λ-cut of ξ for any real number $\lambda \in [0, 1]$; then $u_\lambda = [\mu_L^{-1}(\lambda), \mu_R^{-1}(\lambda)]$ is an interval number, where $\mu_L^{-1}(\lambda) = \min\{x \mid \mu(x) \ge \lambda\}$ and $\mu_R^{-1}(\lambda) = \max\{x \mid \mu(x) \ge \lambda\}$.

Example 5. Let $\xi = (a, b, c)$ be a triangular fuzzy variable with membership function $\mu(x)$ given by (6); then the λ-cut of ξ is $u_\lambda = [\lambda(b-a) + a, \lambda(b-c) + c]$ for every $\lambda \in [0, 1]$.

Example 6. Suppose that $\xi = (a, b, c, d)$ is a trapezoidal fuzzy variable with membership function $\mu(x)$ given by (10); then the λ-cut of ξ is $u_\lambda = [\lambda(b-a) + a, \lambda(c-d) + d]$ for any real number $\lambda \in [0, 1]$.

According to the definition of the distance between interval numbers, we can obtain the definition of the distance between two fuzzy variables.

Definition 7. Suppose that ξ and η are fuzzy variables with membership functions $\mu(x)$ and $\nu(x)$, and the λ-cut of ξ and η is $u_\lambda = [\mu_L^{-1}(\lambda), \mu_R^{-1}(\lambda)]$ and $\nu_\lambda = [\nu_L^{-1}(\lambda), \nu_R^{-1}(\lambda)]$ for all $\lambda \in [0,1]$, respectively. Then the distance measure between ξ and η can be defined by

$$D(\xi, \eta) = \sqrt{\int_0^1 D^2(u_\lambda, \nu_\lambda)\, d\lambda}. \tag{17}$$

Theorem 8. *Suppose that ξ, η, and ζ are fuzzy variables. Let $D(\cdot,\cdot)$ be distance; then the $D(\cdot,\cdot)$ satisfies the following properties of a distance metric:*

(a) *$D(\xi,\eta) \geq 0$, and $D(\xi,\eta) = 0$ if and only if $\xi = \eta$;*

(b) *$D(\xi,\eta) = D(\eta,\xi)$;*

(c) *$D(\xi,\eta) \leq D(\xi,\zeta) + D(\zeta,\eta)$.*

Proof. Based on the properties (i) and (ii) of the interval numbers distance, the parts (a) and (b) follow immediately from Definition 7. Now we prove the part (c). Suppose that membership function of ζ is $\omega(x)$; then the λ-cut of ζ is $w_\lambda = [\omega_L^{-1}(\lambda), \omega_R^{-1}(\lambda)]$ for any real number $\lambda \in [0,1]$. Based on the inequality (16), we have

$$\begin{gathered} 0 \leq D(u_\lambda, \nu_\lambda) \leq D(u_\lambda, w_\lambda) + D(w_\lambda, \nu_\lambda), \\ D^2(u_\lambda, \nu_\lambda) \leq (D(u_\lambda, w_\lambda) + D(w_\lambda, \nu_\lambda))^2, \\ \int_0^1 D^2(u_\lambda, \nu_\lambda)\, d\lambda \leq \int_0^1 (D(u_\lambda, w_\lambda) + D(w_\lambda, \nu_\lambda))^2 d\lambda, \\ \sqrt{\int_0^1 D^2(u_\lambda, \nu_\lambda)\, d\lambda} \leq \sqrt{\int_0^1 (D(u_\lambda, w_\lambda) + D(w_\lambda, \nu_\lambda))^2 d\lambda}. \end{gathered} \tag{18}$$

According to Minkowski inequality, we can obtain

$$\begin{aligned} &\sqrt{\int_0^1 (D(u_\lambda, w_\lambda) + D(w_\lambda, \nu_\lambda))^2 d\lambda} \\ &\quad \leq \sqrt{\int_0^1 D^2(u_\lambda, w_\lambda)\, d\lambda} + \sqrt{\int_0^1 D^2(w_\lambda, \nu_\lambda)\, d\lambda}. \end{aligned} \tag{19}$$

Form the above inequalities, we can get

$$\begin{aligned} \sqrt{\int_0^1 D^2(u_\lambda, \nu_\lambda)\, d\lambda} \leq{}& \sqrt{\int_0^1 D^2(u_\lambda, w_\lambda)\, d\lambda} \\ &+ \sqrt{\int_0^1 D^2(w_\lambda, \nu_\lambda)\, d\lambda}. \end{aligned} \tag{20}$$

Thus

$$D(\xi, \eta) \leq D(\xi, \zeta) + D(\zeta, \eta). \tag{21}$$

The theorem is proved. □

4. Distance Minimization Models

Let x_i be the investment proportions in securities i and ξ_i the fuzzy returns of the ith securities, $i = 1, 2, \ldots, n$, respectively. Suppose that η is a prior fuzzy investment return for an investor, and his/her objective is to minimize the divergence of the fuzzy investment return from η. In addition, the return remains above the minimum return level and the risk remains below the maximum risk level. In this paper, we use the distance to measure the degree of divergence and use the expected value to reflect the return. The main problem is how to measure the risk. If the fuzzy security returns ξ_i are symmetrical, we use variance to measure risk; then we have the following model:

$$\begin{aligned} \text{minimize} \quad & D(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n, \eta) \\ \text{subject to} \quad & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \geq \alpha \\ & V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \leq \beta \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{22}$$

The first constraint ensures the expected return is no less than some given value α, and the second one assures that risk does not exceed some given level β the investor can bear. The last two constraints imply that all capital will be invested in n securities.

Remark 9. Suppose that the security returns ξ_i $(i = 1, 2, \ldots, n)$ are fuzzy variables. It follows from Extension Principle of Zadeh that $\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$, which is also a fuzzy variable. Let the membership functions of ξ and η be $u(x)$ and $\nu(x)$, respectively. Then the objective of model (22) can be calculated as

$$D(\xi, \eta) = \sqrt{\int_0^1 D^2(u_\lambda, \nu_\lambda)\, d\lambda}. \tag{23}$$

Though it is usually adopted that the security returns are symmetrical, there do exist empirical evidences [41] indicating that many security returns are not symmetrically distributed. In the case where the fuzzy security returns are asymmetrical, we can use semivariance to replace variance. The semivariance is more suitable to measure risk because it only punishes the investment return below the expected value; thus we have the model as follows:

$$\begin{aligned} \text{minimize} \quad & D(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n, \eta) \\ \text{subject to} \quad & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \geq \alpha \\ & \mathrm{SV}[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \leq \beta \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{24}$$

where α and β are the predetermined confidence levels accepted by the investor.

5. Crisp Forms

In this section, we propose the crisp equivalents of the optimization models. In order to simplify models, the objective function $D(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n, \eta)$ of models (22) and (24) is replaced by $D^2(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n, \eta)$ in crisp forms.

Theorem 10. *Assume that each security return is the symmetrical triangular fuzzy variable denoted by $\xi_i = (a_i, b_i, c_i)$ $(i = 1, 2, \ldots, n)$. Let the prior fuzzy investment return $\eta = (a', b', c')$ be a triangular fuzzy variable; then the model (22) can be transformed into the following crisp form:*

$$\begin{aligned}
\text{minimize} \quad & \left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + 2\left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 \\
& + \left(\sum_{i=1}^{n} c_i x_i - c'\right)^2 \\
& + \left(\sum_{i=1}^{n} b_i x_i - b'\right)\left(\sum_{i=1}^{n} (a_i + c_i) x_i - a' - c'\right) \\
\text{subject to} \quad & \sum_{i=1}^{n} b_i x_i \geq \alpha \\
& \left(\sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} a_i x_i\right)^2 \leq 24\beta \\
& x_1 + x_2 + \cdots + x_n = 1 \\
& x_i \geq 0, \quad i = 1, 2 \ldots, n.
\end{aligned} \tag{25}$$

Proof. Since $\xi_i = (a_i, b_i, c_i)$ $(i = 1, 2, \ldots, n)$ are all symmetrical triangular fuzzy variables, it follows from Extension Principle of Zadeh that $\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n = (\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n} c_i x_i)$, which is also a symmetrical triangular fuzzy variable. According to (2) and (3), we have

$$\begin{aligned}
E[\xi] &= \sum_{i=1}^{n} b_i x_i, \\
V[\xi] &= \frac{1}{24}\left(\sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} a_i x_i\right)^2.
\end{aligned} \tag{26}$$

In addition, according to Example 5, for any real number $\lambda \in [0, 1]$, we have the λ-cut of ξ

$$\begin{aligned}
u_\lambda = \Bigg[& \lambda\left(\sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} a_i x_i\right) + \sum_{i=1}^{n} a_i x_i, \\
& \lambda\left(\sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} c_i x_i\right) + \sum_{i=1}^{n} c_i x_i \Bigg]
\end{aligned} \tag{27}$$

and λ-cut of η

$$v_\lambda = \left[\lambda\left(b' - a'\right) + a', \lambda\left(b' - c'\right) + c'\right]. \tag{28}$$

Thus, it is known from Definition 7 that

$$\begin{aligned}
& D^2\left(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n, \eta\right) \\
& = \int_0^1 D^2\left(u_\lambda, v_\lambda\right) d\lambda \\
& = \frac{1}{6}\left[\left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + 2\left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 + \left(\sum_{i=1}^{n} c_i x_i - c'\right)^2\right] \\
& \quad + \frac{1}{6}\left[\left(\sum_{i=1}^{n} b_i x_i - b'\right)\left(\sum_{i=1}^{n} (a_i + c_i) x_i - a' - c'\right)\right].
\end{aligned} \tag{29}$$

The proof is completed. □

When the investment return $\xi = (a, b, c)$ is an asymmetrical triangular fuzzy variable, the investors focus on the case $c - b < b - a$. Therefore, we only consider this situation in this paper.

Theorem 11. *Suppose that security returns $\xi_i = (a_i, b_i, c_i)$ $(i = 1, 2, \ldots, n)$ are asymmetrical triangular fuzzy variables. Let the prior fuzzy investment return $\eta = (a', b', c')$ be a triangular fuzzy variable. Then the model (24) can be converted into the following crisp form:*

$$\begin{aligned}
\text{minimize} \quad & \left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + 2\left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 \\
& + \left(\sum_{i=1}^{n} c_i x_i - c'\right)^2 + \left(\sum_{i=1}^{n} b_i x_i - b'\right) \\
& \times \left(\sum_{i=1}^{n} (a_i + c_i) x_i - a' - c'\right) \\
\text{subject to} \quad & \left(\sum_{i=1}^{n} a_i x_i + 2\sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i\right) \geq 4\alpha \\
& \left(\sum_{i=1}^{n} c_i x_i + 2\sum_{i=1}^{n} b_i x_i - 3\sum_{i=1}^{n} a_i x_i\right)^3 \\
& \quad \leq 384\beta\left(\sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} a_i x_i\right) \\
& \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} b_i x_i \\
& \quad < \sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} a_i x_i \\
& x_1 + x_2 + \cdots + x_n = 1 \\
& x_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{30}$$

Proof. According to (2) and (4), it can be proved that, for an asymmetrical triangular fuzzy investment return $\xi = (\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n} c_i x_i)$, its expected value and semivariance are

$$E[\xi] = e = \frac{1}{4}\left(\sum_{i=1}^{n} a_i x_i + 2\sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i\right),$$
$$\mathrm{SV}[\xi] = \frac{(e - \sum_{i=1}^{n} a_i x_i)^3}{6(\sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} a_i x_i)}, \quad (31)$$

respectively, when $\sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} b_i x_i < \sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} a_i x_i$. Furthermore, the objectives of model (30) and model (25) are the same. Thus the proof is completed. □

According to above proof, we can also obtain Theorems 12–15 for different types of fuzzy variables.

Theorem 12. *Assume that each security return is the symmetrical trapezoidal fuzzy variable denoted by $\xi_i = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, \ldots, n)$. Let the prior fuzzy investment return $\eta = (a', b', c', d')$ be trapezoidal fuzzy variable. Then the model (22) can be converted into the following crisp form:*

$$\begin{aligned}
\text{minimize} \quad & \left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + \left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 \\
& + \left(\sum_{i=1}^{n} c_i x_i - c'\right)^2 + \left(\sum_{i=1}^{n} d_i x_i - d'\right)^2 \\
& + \left(\sum_{i=1}^{n} a_i x_i - a'\right)\left(\sum_{i=1}^{n} b_i x_i - b'\right) \\
& + \left(\sum_{i=1}^{n} c_i x_i - c'\right)\left(\sum_{i=1}^{n} d_i x_i - d'\right) \\
\text{subject to} \quad & \left(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} d_i x_i\right) \geq 2\alpha \\
& 3\left(\sum_{i=1}^{n} (d_i - b_i) x_i\right)^2 + \left(\sum_{i=1}^{n} (d_i - c_i) x_i\right)^2 \leq 24\beta \\
& x_1 + x_2 + \cdots + x_n = 1 \\
& x_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \quad (32)$$

Theorem 13. *Assume that each security return is the symmetrical trapezoidal fuzzy variable denoted by $\xi_i = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, \ldots, n)$. Let the prior fuzzy investment return $\eta = (a', b', c')$ be triangular fuzzy variable. Then the model (22) can be transformed into the following crisp form:*

$$\begin{aligned}
\text{minimize} \quad & \left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + \left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 \\
& + \left(\sum_{i=1}^{n} c_i x_i - b'\right)^2 + \left(\sum_{i=1}^{n} d_i x_i - c'\right)^2 \\
& + \left(\sum_{i=1}^{n} a_i x_i - a'\right)\left(\sum_{i=1}^{n} b_i x_i - b'\right) \\
& + \left(\sum_{i=1}^{n} c_i x_i - b'\right)\left(\sum_{i=1}^{n} d_i x_i - c'\right) \\
\text{subject to} \quad & \left(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} d_i x_i\right) \geq 2\alpha \\
& 3\left(\sum_{i=1}^{n} (d_i - b_i) x_i\right)^2 + \left(\sum_{i=1}^{n} (d_i - c_i) x_i\right)^2 \leq 24\beta \\
& x_1 + x_2 + \cdots + x_n = 1 \\
& x_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \quad (33)$$

Theorem 14. *Suppose that security returns $\xi_i = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, \ldots, n)$ are asymmetrical trapezoidal fuzzy variables. Let the prior fuzzy investment return $\eta = (a', b', c', d')$ be trapezoidal fuzzy variables. Then the model (24) can be transformed into the following crisp form:*

$$\begin{aligned}
\text{minimize} \quad & \left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + \left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 \\
& + \left(\sum_{i=1}^{n} c_i x_i - c'\right)^2 + \left(\sum_{i=1}^{n} d_i x_i - d'\right)^2 \\
& + \left(\sum_{i=1}^{n} a_i x_i - a'\right)\left(\sum_{i=1}^{n} b_i x_i - b'\right) \\
& + \left(\sum_{i=1}^{n} c_i x_i - c'\right)\left(\sum_{i=1}^{n} d_i x_i - d'\right) \\
\text{subject to} \quad & \left(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} d_i x_i\right) \geq 4\alpha \\
& V(x) \leq \beta \\
& x_1 + x_2 + \cdots + x_n = 1 \\
& x_i \geq 0, \quad i = 1, 2 \ldots, n,
\end{aligned} \quad (34)$$

where

$$V(x) = \begin{cases} \dfrac{\left(e-\sum_{i=1}^{n} a_i x_i\right)^3}{6\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right)} & \text{if } e < \sum_{i=1}^{n} b_i x_i \\ \dfrac{1}{6}\left[\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right) \times\left(3e-\left(\sum_{i=1}^{n}(a_i+2b_i)x_i\right)\right) + 3\left(e-\sum_{i=1}^{n} b_i x_i\right)^2\right] & \text{if} \sum_{i=1}^{n} b_i x_i \le e \le \sum_{i=1}^{n} c_i x_i \\ \dfrac{1}{6}\left[\left(\sum_{i=1}^{n}(b_i-a_i)x_i\right) \times\left(3e-\left(\sum_{i=1}^{n}(a_i+2b_i)x_i\right)\right) + 3\left(\sum_{i=1}^{n}(c_i-b_i)x_i\right) \times\left(2e-\sum_{i=1}^{n} b_i x_i-\sum_{i=1}^{n} c_i x_i\right)\right] + \left(\left(\sum_{i=1}^{n} c_i x_i - e\right)^2 \times\left(3\sum_{i=1}^{n} d_i x_i - 4\sum_{i=1}^{n} c_i x_i + e\right)\right) \times\left(6\left(\sum_{i=1}^{n}(d_i-c_i)x_i\right)\right)^{-1} & \text{if } e > \sum_{i=1}^{n} c_i x_i, \end{cases}$$

$$e = \frac{1}{4}\left(\sum_{i=1}^{n}(a_i+b_i+c_i+d_i)x_i\right). \tag{35}$$

Theorem 15. *Suppose that security returns $\xi_i = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, \ldots, n)$ are asymmetrical trapezoidal fuzzy variables. Let the prior fuzzy investment return $\eta = (a', b', c')$ be triangular fuzzy variable. Then the model (24) can be transformed into the following crisp form:*

$$\begin{aligned} \text{minimize} \quad & \left(\sum_{i=1}^{n} a_i x_i - a'\right)^2 + \left(\sum_{i=1}^{n} b_i x_i - b'\right)^2 + \left(\sum_{i=1}^{n} c_i x_i - b'\right)^2 + \left(\sum_{i=1}^{n} d_i x_i - c'\right)^2 \\ & + \left(\sum_{i=1}^{n} a_i x_i - a'\right)\left(\sum_{i=1}^{n} b_i x_i - b'\right) + \left(\sum_{i=1}^{n} c_i x_i - b'\right)\left(\sum_{i=1}^{n} d_i x_i - c'\right) \\ \text{subject to} \quad & \left(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} d_i x_i\right) \ge 4\alpha \\ & V(x) \le \beta \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{36}$$

Table 1: The symmetrical fuzzy returns of 10 securities.

Security i	Fuzzy return ξ_i	Security i	Fuzzy return ξ_i
1	(−0.4, 1.5, 3.4)	2	(−0.1, 1.2, 2.5)
3	(−0.2, 2.0, 4.2)	4	(−0.5, 1.2, 2.9)
5	(−0.6, 1.4, 3.4)	6	(−0.1, 1.8, 3, 7)
7	(−0.3, 1.6, 3.5)	8	(−0.1, 2.2, 4.5)
9	(−0.7, 1.0, 2.7)	10	(−0.2, 1.8, 3.8)

6. Numerical Examples

In this section, some numerical examples are given to illustrate the availability of the two new models. Examples 1–3 consider the case in which there are 10 or 30 securities from different industries. Let ξ_i be the return of the ith security determined as $\xi_i = (p_i' + f_i - p_i)/p_i$, where p_i' is the estimated closing price of the ith security in the next period, p_i the closing price of the ith security at present, and f_i the estimated dividends of the ith security during the next period. It is clear that p_i' and f_i are unknown at present. In other words, the predictions of security returns have to be given mainly based on expert's judgments and estimations.

Example 1. Assume that each security return is the symmetrical triangular fuzzy variable denoted by $\xi_i = (a_i, b_i, c_i)$ $(i = 1, 2, \ldots, 10)$, where the parameters a_i, b_i, and c_i are determined based on the estimated values of financial experts. The data set is given in Table 1. Suppose that the minimum expected return the investor can accept is 1.95 and the bearable maximum risk is 1.0. In addition, the prior fuzzy investment return is $\eta = (-0.2, 1.9, 4.0)$. From the model (25), we can obtain a simple and crisp optimization model and employ fmincon in MATLAB 7.1 to solve this model. The numerical results are given in Table 2.

In order to obtain the minimized distance of the investment return from η when the portfolio satisfies the return and risk constraints, the investor should allocate his or her money according to Table 2. The corresponding objective value is 0.0150; the expected return and variance of the portfolio are 1.951 and 0.739, respectively. Furthermore, the investment return is $\xi = (-0.15, 1.95, 4.05)$. The graphic comparison of

Table 2: Investment proportion of 10 securities (%).

Security i	1	2	3	4	5	6	7	8	9	10
Allocation of money	0.3	2.9	47.9	0.6	0.2	26.7	0.5	20.1	0.3	0.6

Table 3: The asymmetrical fuzzy returns of 10 securities.

Security i	Fuzzy return ξ_i	Security i	Fuzzy return ξ_i
1	$(-0.4, 2.7, 3.4)$	2	$(-0.1, 1.9, 2.6)$
3	$(-0.2, 3.0, 4.0)$	4	$(-0.5, 2.0, 2.9)$
5	$(-0.6, 2.2, 3.3)$	6	$(-0.1, 2.5, 3.6)$
7	$(-0.3, 2.4, 3.5)$	8	$(-0.1, 3.3, 4.5)$
9	$(-0.7, 1.1, 2.7)$	10	$(-0.2, 2.1, 3.8)$

Table 4: Investment proportion of 10 securities (%).

Security i	1	2	3	4	5	6	7	8	9	10
Allocation of money	0	0.4	6.8	0	1.6	0	0	35.2	1.2	54.9

Table 5: The trapezoidal fuzzy returns of 30 securities.

Security i	Fuzzy return ξ_i	Security i	Fuzzy return ξ_i
1	(1.000, 1.003, 1.007, 1.008)	2	(1.001, 1.004, 1.009, 1.012)
3	(1.001, 1.004, 1.009, 1.012)	4	(0.996, 1.008, 1.009, 1.022)
5	(0.995, 1.007, 1.012, 1.023)	6	(0.994, 1.015, 1.024, 1.026)
7	(0.999, 1.019, 1.023, 1.038)	8	(1.005, 1.026, 1.032, 1.046)
9	(1.008, 1.021, 1.035, 1.046)	10	(1.017, 1.020, 1.026, 1.060)
11	(1.010, 1.027, 1.038, 1.055)	12	(1.013, 1.033, 1.045, 1.058)
13	(1.011, 1.039, 1.044, 1.062)	14	(1.015, 1.030, 1.044, 1.071)
15	(1.037, 1.053, 1.084, 1.096)	16	(1.042, 1.047, 1.070, 1.113)
17	(1.034, 1.055, 1.088, 1.100)	18	(1.039, 1.051, 1.088, 1.105)
19	(1.031, 1.067, 1.091, 1.108)	20	(1.048, 1.061, 1.067, 1.135)
21	(1.020, 1.089, 1.095, 1.107)	22	(1.043, 1.063, 1.092, 1.121)
23	(1.038, 1.073, 1.078, 1.132)	24	(1.051, 1.063, 1.104, 1.124)
25	(1.044, 1.076, 1.095, 1.134)	26	(1.039, 1.088, 1.097, 1.139)
27	(1.036, 1.089, 1.114, 1.127)	28	(1.054, 1.076, 1.089, 1.155)
29	(1.058, 1.070, 1.101, 1.150)	30	(1.056, 1.071, 1.126, 1.132)

the obtained investment return ξ and the prior one η is shown in Figure 1.

From Figure 1, we see that the obtained investment return ξ is close to the prior one η. It shows that the new approach is feasible.

Example 2. In this example, all data is from [27]. The asymmetrical fuzzy returns of 10 securities are shown in Table 3. The maximum risk level and the minimum return level are 0.7 and 2.25, respectively. The prior fuzzy return is $\eta = (-0.2, 2.3, 4)$. From the model (30), we can obtain a simple and crisp optimization model and employ fmincon in MATLAB 7.1 to solve this model. The numerical results are given in Table 4.

In order to obtain the minimized distance of the investment return from the prior return η when the portfolio satisfies the return and risk constraints, the investor should assign his or her capital according to Table 4. The corresponding objective value is 0.165; the expected return and semivariance of the portfolio are 2.253 and 0.612, respectively. In addition, the investment return is $\xi = (-0.18, 2.58, 4.03)$. $\zeta = (-0.20, 2.63, 3.95)$ denotes the investment return calculated by [27]. The graphic comparison of the investment returns ξ, ζ and the prior one η is shown in Figure 2.

From Figure 2, we can see that the investment returns obtained by our model and model of [27] are similar. However, solving our model is easier than solving the model of [27].

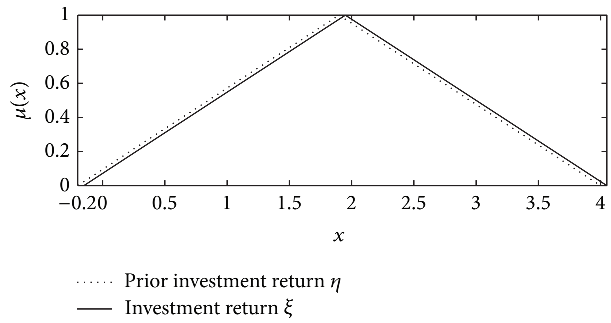

FIGURE 1: Comparison of investment return ξ and the prior one η.

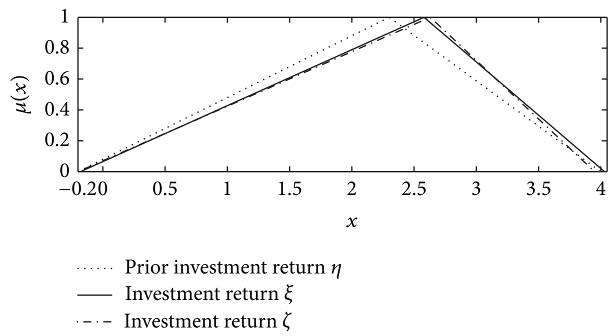

FIGURE 2: Comparison of the investment returns ξ, ζ and the prior one η.

TABLE 6: Investment proportion of 30 securities (%).

Security i	14	18	23	26	27	28	29
Allocation of money	10.17	0.12	4.9	33.01	17.85	0.23	33.69

TABLE 7: Investment proportion of 30 securities under different risk level constraints with $\alpha = 1.0732$ (%).

β	0.000245	0.000255	0.000265	0.000275	0.000315	0.000345	0.000475
Obj.	0.0012	$9.7681e-004$	$4.7702e-004$	$1.6635e-004$	$1.5111e-005$	$1.3800e-005$	$9.7857e-006$
x_1	19.01	5.45	3.25	3.16	0	0	0
x_2	1.08	2.62	0.64	1.10	0	0	0
x_9	0	0.61	0	0	0	0	0
x_{10}	4.02	17.96	12.63	11.91	0.61	0.50	0
x_{13}	0	0.8	0.11	0	0	0	0
x_{14}	0	0	0	0	2.44	1.87	10.17
x_{18}	0	0	0	0	1.12	0.35	0.12
x_{19}	0	0	0	0	1.30	10.83	0
x_{20}	0	0	0	0	3.67	0	0
x_{21}	0	0	0	0	0	22.29	0
x_{22}	0	0	0	0	0.40	0.50	0
x_{23}	0	0	0	0	30.63	1.79	4.9
x_{25}	0	0	0	0	1.11	0.32	0
x_{26}	0	0	32.94	26.41	29.73	22.44	33.01
x_{27}	0	0	0	0	1.45	1.53	17.85
x_{28}	0	0	0	32.16	0.63	0.18	0.23
x_{29}	0	0	0	1.49	9.98	36.48	33.69
x_{30}	75.88	72.55	50.43	23.76	16.91	0.91	0

Table 8: Investment proportion of 30 securities under different return level constraints with $\beta = 0.000475$ (%).

α	1.0732	1.0865	1.0885	1.0898	1.919	1.0953	1.0962
Obj.	$9.7857e-006$	$1.0206e-005$	$3.7685e-005$	$1.5338e-004$	$2.0983e-004$	$4.6784e-004$	0.0011
x_{14}	10.17	4.9	0.48	0.38	0.23	1.40	0
x_{16}	0	0	0.14	0.40	0	0	0
x_{18}	0.12	0	0.17	0	0	0	0
x_{19}	0	8.53	0.32	0	0	0	0
x_{20}	0	0	14.92	17.33	1.35	2.15	0
x_{21}	0	0	8.98	0.46	0	0.49	0
x_{22}	0	0	0	1.0	0	0	0
x_{23}	4.9	16.17	3.63	1.51	0.41	0	0
x_{25}	0	0	0.23	2.43	1.38	0	0
x_{26}	33.01	20.37	20.5	20.14	55.28	4.25	0
x_{27}	17.85	17.74	17.15	26.63	0	0	0
x_{28}	0.23	1.51	0	0	10.2	19.29	0.11
x_{29}	33.69	30.65	18.06	0	0	0	0
x_{30}	0	0.14	15.4	29.73	31.16	72.42	99.89

Table 9: Optimal portfolios produced by model (23) of [29] under different risk level constraints (%).

s_0	0.002	0.0025	0.0045	0.0050	0.0075	0.0095	0.0125
x_1	93.34	92.49	79.11	74.8	58.37	44.54	21.99
x_2	3.46	0.14	0.06	0.52	0.74	1.44	3.87
x_3	0	0	0	0.45	0.29	0	0.34
x_4	0	0	0	0.88	0	0.32	0
x_5	0	0	0	0	0.35	0	0
x_{30}	3.18	7.36	20.83	23.34	40.24	53.68	73.75

Example 3. Assume that each security return is trapezoidal fuzzy variable denoted by $\xi_i = (a_i, b_i, c_i, d_i)$ $(i = 1, 2, \ldots, 30)$. The data set from [30] is shown in Table 5. The maximum risk level and the minimum return level are 0.000475 and 1.0732, respectively. In addition, the prior fuzzy return is $\eta = (1.040, 1.075, 1.095, 1.135)$ for an investor. From the model (34), we can obtain a simple and crisp optimization model and use gravitation search algorithm (GSA) [42] to solve this model. The numerical results are given in Table 6.

The results show that among 30 securities, satisfying the constraints, in order to minimize distance of the investment return from the prior return η, the investor should allocate his or her money according to Table 6. The corresponding objective value is $9.7857e-006$. In addition, the investment return is $\xi = (1.0424, 1.0754, 1.0950, 1.1333)$.

In order to examine the sensitivity of the predetermined confidence level, we adjust the β value and do the experiment. The results are shown in Table 7. It is seen that as maximum risk level increases, the optimal objective will decrease.

In addition, we also examine the sensitivity of the return level α to optimal objective in the same way. The results are given in Table 8. The results indicate that as expected return level increases, the minimal distance will increase.

Furthermore, in order to examine the availability of the new approach, we compare the proposed method with the methods of [29, 30]. Based on the data in Table 5, we use GSA to solve the models (18) and (23) of [29] for different return level r_0 and risk level s_0. The numerical results are given in Tables 9 and 10. In addition, Tables 11 and 12 show the results of models (21) and (22) of [30], which are from [30].

From Tables 7 to 12, it is seen that the computational results about optimal allocation proportion to 30 securities are different, and the optimal portfolios produced by our model are more diversified than the optimal portfolio produced by models of [29, 30].

7. Conclusions

In this paper, a concept of distance between fuzzy variables was introduced for measuring the divergence of fuzzy investment returns from a prior one. By defining the risk as variance and semivariance, two distance minimization models were proposed. In addition, crisp equivalents of the optimization models have also been provided. Finally, the results of the numerical examples illustrated the availability of the new method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Table 10: Optimal portfolios produced by model (18) of [29] under different return level constraints (%).

r_0	1.0732	1.0885	1.0898	1.0919	1.0953	1.0962
x_1	6.69	0	0	0	0	0
x_2	1.64	0	0	0	0	0
x_{20}	61.01	13.96	3.13	0	2.83	0
x_{21}	30.6	1.73	3.75	0	0.61	0
x_{25}	0	2.83	0.14	1.52	0	0
x_{26}	0	80.78	92.65	75.03	0	0
x_{28}	0	0	0	0	8.4	1.34
x_{30}	0	0	0	23.44	88.06	98.66

Table 11: Optimal portfolios produced by model (21) of [30] under different risk level constraints (%).

φ	0.000025	0.000475	0.000654	0.000779	0.000858	0.000932	0.000989
x_1	91.326	11.338	0	0	0	0	0
x_{15}	0	44.196	27.026	0	0	0	0
x_{20}	6.045	24.411	33.617	34.866	20.38	3.097	0
x_{21}	2.629	20.055	28.889	28.094	10.128	0	0
x_{26}	0	0	0	8.539	40.933	62.456	9.515
x_{28}	0	0	0	0	0	4.976	36.367
x_{30}	0	0	10.469	28.501	28.56	29.471	54.118

Table 12: Optimal portfolios produced by model (22) of [30] under different return level constraints (%).

ρ	1.0732	1.0898	1.0919	1.0953	1.0962	1.09625
x_1	0	0	0	0	0	0
x_{15}	44.39	0	0	0	0	0
x_{20}	29.887	15.421	4.392	0	0	0
x_{21}	25.732	3.979	0	0	0	0
x_{26}	0	52.021	62.61	0	0	0
x_{28}	0	0	3.419	34.545	1.818	0
x_{30}	0	28.58	29.58	65.454	98.182	100

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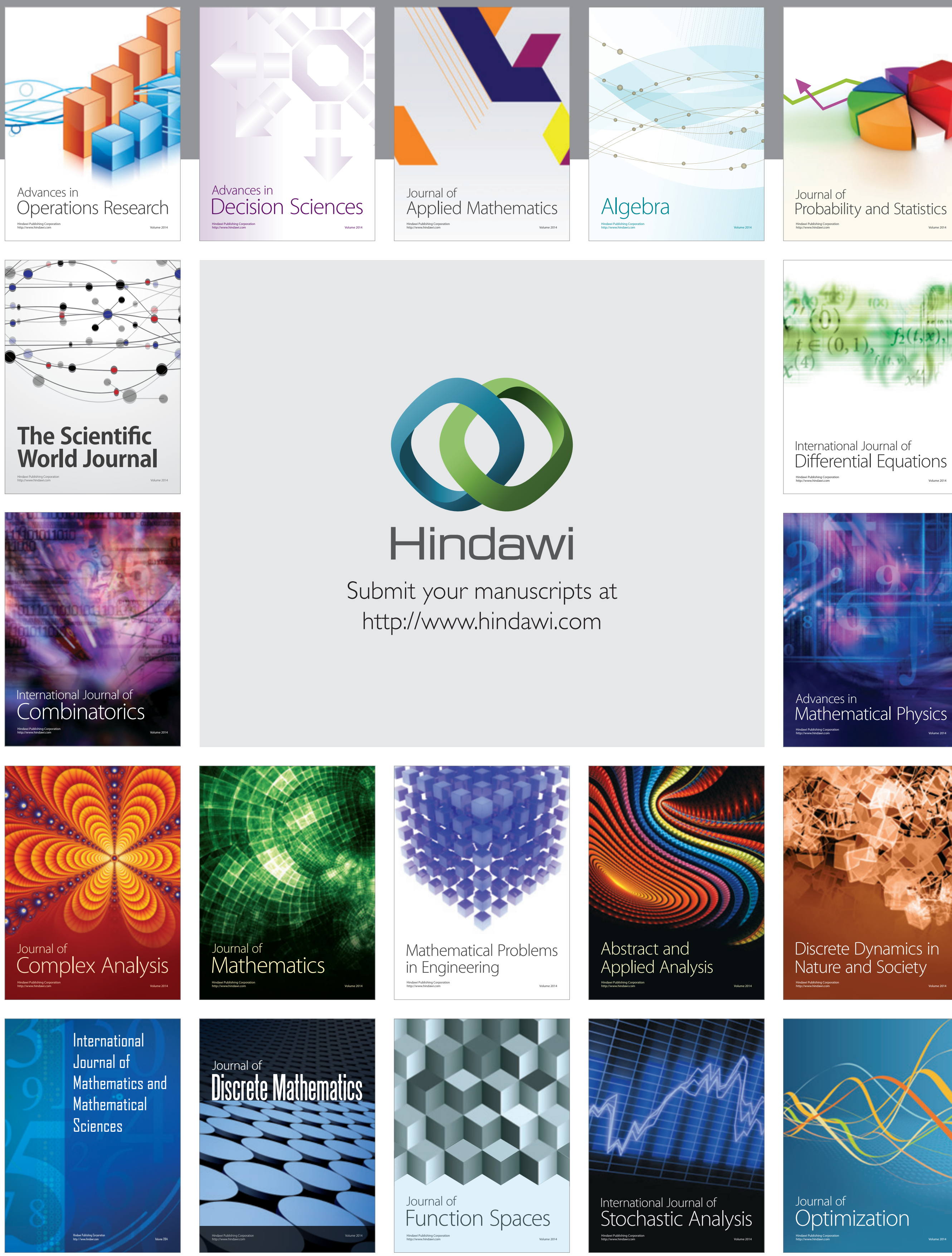

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