Research Article

A Nonlinear k - ε Turbulence Model Applicable to High Pressure Gradient and Large Curvature Flow

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Most of the RANS turbulence models solve the Reynolds stress by linear hypothesis with isotropic model. They can not capture all kinds of vortexes in the turbomachineries. In this paper, an improved nonlinear k - ε turbulence model is proposed, which is modified from the RNG k - ε turbulence model and Wilcox's k - ω turbulence model. The Reynolds stresses are solved by nonlinear methods. The nonlinear k - ε turbulence model can calculate the near wall region without the use of wall functions. The improved nonlinear k - ε turbulence model is used to simulate the flow field in a curved rectangular duct. The results based on the improved nonlinear k - ε turbulence model agree well with the experimental results. The calculation results prove that the nonlinear k - ε turbulence model is available for high pressure gradient flows and large curvature flows, and it can be used to capture complex vortexes in a turbomachinery.

1. Introduction

Boussinesq hypothesis is a first-order-closure model of turbulence which assumes an explicit algebraic relationship between Reynolds stresses and mean-velocity gradients. The Boussinesq hypothesis is a popular linear eddy-viscosity model, which is used in most Reynolds averaged Navies-Stokes equations to predict the shear stress. The k - ε model and k - ω model are the most common turbulence models [1], which are a two-equation model including two extra transport equations to represent the turbulent properties of the flow. The k - ε model and k - ω model do not perform well in cases of large adverse pressure gradients [2]. The use of a k - ω formulation in the inner parts of the boundary layer makes the model directly usable all the way down to the wall through the viscous sublayer [3]. The SST k - ω model can be used as a Low-Re turbulence model without any extra damping functions. The SST formulation also switches to k - ε model in the free stream and thereby avoids the common k - ω problem that the model is too sensitive to the inlet free-stream turbulence properties [4]. However, practical engineering flows exhibit complex mean strain associated, for example, with

high pressure gradients, separation, impingement, streamline curvature, and swirl. The linear hypothesis is not suitable to capture the flow of anisotropy.

One way to model the anisotropy is to use the Reynolds stress transport equations. Craft and Launder [5] discussed different Reynolds stress turbulence models and find out that higher order pressure strain models which consider the stress redistribution near walls can predict the lateral spreading rate well. Lübcke et al. [6] used another way to simulate the anisotropy. They presented an explicit Reynolds-stress closure which offers a physically sound extension of the most prominent linear Boussinesq viscosity models with modest computational effort. Pettersson Reif [7] also presented a promising explicit Reynolds stress closure. Ji et al. [8, 9] studied unsteady turbulence flows around a twisted hydrofoil using partially averaged Navier-Stokes (PANS) methods. The subfilter scale stress in PANS methods was a positive semidefinite tensor exhibiting much of the same properties as Reynolds stress, and it was a nonlinear stress model.

Durbin [10] introduced the elliptic relaxation approach within the framework of the linear eddy-viscosity formulation, the so-called v^2 - f model. The nonlinear v^2 - f turbulence

FIGURE 1: A curved rectangular duct.

FIGURE 2: Mesh of the rectangular duct.

model differs from the family of conventional nonlinear eddy-viscosity models. In this approach, elliptic wall effects are accounted for indirectly through the solution of modified Helmholtz equation [11–13]. The v^2 - f turbulence model is a seven-equation model, which consumes more computational time and resources. Nonlinear turbulence models often arise from scientific research, modelling of nonlinear phenomena, and optimal control of complex systems [14, 15]. Nonlinear turbulence model can be also used in the study of heat transfer [16] and phase-change process [17]. The nonlinear solutions are more accurate in the solution of complex phenomena.

In this paper, an improved nonlinear k - ε turbulence model was proposed, which was modified from RNG k - ε turbulence model and Wilcox's k - ω turbulence model. The Reynolds stresses were solved by nonlinear methods which were proposed by Ehrhard [18]. The nonlinear k - ε turbulence model was used to simulate the flow in a curved rectangular duct, and the results of the internal flow were compared with experimental data.

2. Model Development

In Wilcox's k - ω turbulence model, the turbulence kinetic energy transport equation is given by the following equation:

$$\rho \frac{Dk}{Dt} = \frac{\partial}{\partial x_j} \left((\mu + \sigma^* \mu_t) \frac{\partial k}{\partial x_j} \right) + \rho \tau_{ij} \frac{\partial u_i}{\partial x_j} - \rho \beta^* k \omega \quad (1)$$

and the turbulence specific dissipation rate equation is

$$\rho \frac{D\omega}{Dt} = \frac{\partial}{\partial x_j} \left((\mu + \sigma \mu_t) \frac{\partial \omega}{\partial x_j} \right) + \rho \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \rho \beta \omega^2. \quad (2)$$

The relationship between turbulence dissipation and turbulence specific dissipation rate is

$$\varepsilon = \beta^* k \omega. \quad (3)$$

It can be calculated that

$$\begin{aligned} \frac{\partial \omega}{\partial x_j} &= \frac{\partial}{\partial x_j} \left(\frac{\varepsilon}{\beta^* k} \right) = \frac{1}{\beta^* k} \frac{\partial \varepsilon}{\partial x_j} - \frac{\varepsilon}{\beta^* k^2} \frac{\partial k}{\partial x_j} \\ \frac{D\varepsilon}{Dt} &= \frac{D}{Dt} (\beta^* k \omega) = \beta^* \omega \frac{Dk}{Dt} + \beta^* k \frac{D\omega}{Dt}. \end{aligned} \quad (4)$$

Then, the improved equation for turbulence dissipation can be derived:

$$\begin{aligned} \rho \frac{D\varepsilon}{Dt} &= \rho \beta^* \omega \frac{Dk}{Dt} + \rho \beta^* k \frac{D\omega}{Dt} \\ &= (\mu + \sigma \mu_t) \frac{\partial}{\partial x_j} \left(\frac{\partial \varepsilon}{\partial x_j} \right) - 2\mu \frac{1}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varepsilon}{\partial x_j} \\ &\quad + \left(2\sigma^* \beta^* \rho + \beta^* \rho \frac{\mu}{\mu_t} \right) \left(\frac{\partial k}{\partial x_j} \right)^2 - \sigma \mu_t \frac{1}{\varepsilon} \left(\frac{\partial \varepsilon}{\partial x_j} \right)^2 \\ &\quad + (\alpha + 1) \rho \frac{\varepsilon}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \left(1 + \frac{\beta}{\beta^*} \right) \rho \frac{\varepsilon^2}{k} \end{aligned} \quad (5)$$

$$\begin{aligned} D_\varepsilon &= (\mu + \sigma \mu_t) \frac{\partial}{\partial x_j} \left(\frac{\partial \varepsilon}{\partial x_j} \right) - 2\mu \frac{1}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varepsilon}{\partial x_j} \\ &\quad + \left(2\sigma^* \beta^* \rho + \beta^* \rho \frac{\mu}{\mu_t} \right) \left(\frac{\partial k}{\partial x_j} \right)^2 - \sigma \mu_t \frac{1}{\varepsilon} \left(\frac{\partial \varepsilon}{\partial x_j} \right)^2 \end{aligned} \quad (6)$$

$$P_\varepsilon = (\alpha + 1) \rho \frac{\varepsilon}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j} \quad (7)$$

$$H_\varepsilon = - \left(1 + \frac{\beta}{\beta^*} \right) \rho \frac{\varepsilon^2}{k}, \quad (8)$$

where D_ε is the diffusion term, P_ε represents the generation of turbulent dissipation due to the mean velocity gradients, and H_ε denotes the damping term for turbulent dissipation rate.

For k - ε turbulence model, the equation for turbulence kinetic energy is given by the following equation:

$$\rho \frac{Dk}{Dt} = \frac{\partial}{\partial x_j} \left((\mu + \sigma^* \mu_t) \frac{\partial k}{\partial x_j} \right) + \rho \tau_{ij} \frac{\partial u_i}{\partial x_j} - \rho \varepsilon, \quad (9)$$

FIGURE 3: Grid dependency studies (Pa).

FIGURE 4: Transverse velocity.

FIGURE 6: Axial velocity.

FIGURE 5: Longitudinal velocity.

where the kinematic eddy viscosity

$$\mu_t = \rho C_{\mu k \epsilon} \frac{k^2}{\epsilon}. \quad (10)$$

The closure coefficients are as follows:

$$\begin{aligned} \alpha &= \frac{5}{9}, & \beta &= \frac{3}{40}, & \beta^* &= \frac{9}{100}, \\ \sigma &= \frac{1}{2}, & \sigma^* &= \frac{1}{2}, & C_{\mu k \epsilon} &= \frac{9}{100}. \end{aligned} \quad (11)$$

The turbulence kinetic energy equation (5) and turbulence dissipation equation (9) form the improved $k-\epsilon$ turbulence model.

The combination of RNG $k-\epsilon$ turbulence model and Wilcox's $k-\omega$ turbulence model does not need to make use of wall functions, because it is valid up to solid walls.

The Reynolds stress anisotropy tensor which was proposed by Ehrhard [18] was combined with the improved

FIGURE 7: Pressure distribution on the convex surface (Pa).

k - ϵ turbulence model. Ehrhard developed a nonlinear k - ϵ turbulence model which was calibrated by the use of simple two-dimensional shear and complex flows with recirculation zones. The final form of the nonlinear k - ϵ turbulence model is given by the nonlinear solution shown as follows.

The Reynolds stress can be calculated by

$$\tau_{ij} = -\rho \overline{u_i' u_j'} \quad (12)$$

The nonlinear solution of Reynolds stress is

$$\begin{aligned} \overline{u_i' u_j'} = & \frac{2}{3} k \delta_{ij} - 2 C_\mu \nu^2 T S_{ij} + C_1 C_\mu \nu^2 T^2 \left(S_{ik} S_{kj} - \frac{1}{3} S_{kl} S_{kl} \delta_{ij} \right) \\ & + C_2 C_{\mu ke} \nu^2 T^2 \left(\Omega_{ik} S_{kj} - \Omega_{jk} S_{ki} \right) \\ & + C_3 C_\mu \nu^2 T^2 \left(\Omega_{ik} \Omega_{jk} - \frac{1}{3} \Omega_{lk} \Omega_{lk} \delta_{ij} \right) \\ & + C_4 C_{\mu ke} \nu^2 T^3 \left(S_{ki} \Omega_{lj} - S_{kj} \Omega_{li} \right) S_{kl} \\ & + C_5 C_\mu \nu^2 T^3 S_{ij} S_{kl} S_{kl} + C_6 C_\mu \nu^2 T^3 S_{ij} \Omega_{kl} \Omega_{kl}, \end{aligned} \quad (13)$$

where,

$$\begin{aligned} C_\mu &= \min \left(\frac{1}{0.9 S^{1.4} + 0.4 \Omega^{1.4} + 3.5}, 0.15 \right) \\ S &= \frac{k}{\epsilon} \sqrt{2 S_{ij} S_{ij}} \\ \Omega &= \frac{k}{\epsilon} \sqrt{2 \Omega_{ij} \Omega_{ij}} \\ S_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ \Omega_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \end{aligned} \quad (14)$$

The closure coefficients are as follows:

$$\begin{aligned} C_1 &= -0.2, & C_2 &= 0.4, & C_3 &= 2.0 - \exp(-(S - \Omega)^2), \\ C_4 &= -32.0 C_\mu^2, & C_5 &= -16.0 C_\mu^2, & C_6 &= 16.0 C_\mu^2, \end{aligned} \quad (15)$$

FIGURE 8: Velocity distribution near the convex surface (m/s).

where ν is the turbulence velocity scale and T is the turbulence time scale.

3. Test Case

To prove that the nonlinear $k-\epsilon$ turbulence model is available for capturing complex flow, a three-dimensional (3-D) curved rectangular duct was studied and the results of the internal flow were compared with experimental data. The nonlinear $k-\epsilon$ turbulence model was performed by Fluent software, using user define function (UDF). The computational results of the nonlinear $k-\epsilon$ turbulence model were also compared with the results performed by RNG $k-\epsilon$ turbulence model and SST $k-\omega$ turbulence model. Experimental data of the internal flow in the curved rectangular duct were performed by Yakinthos et al. [19].

The structure of the curved rectangular duct was shown in Figure 1. $H = 0.203$ m, and the radius of the inner circle was

0.608 m. Velocity at the inlet of the tunnel was $u_o = 16$ m/s, and the Reynolds number was $Re = u_o H / \nu = 224,000$. Mesh was generated by ICFM software, which was shown in Figure 2. Grid dependency is studied by two kinds of number of meshes, and the results of pressure distribution on the local convex surface are shown in Figure 3. It can be seen that the result based on 4.3 million cells is the same as 6.8 million cells. All the following calculations are performed by the mesh with 6.8 million cells. y^+ on the surface in the computational domain is less than 2.

The transverse velocity (perpendicular to the measuring line and along the flow direction), axial velocity (perpendicular to the measuring line and flow direction), and longitudinal velocity (along the measuring line) on the measuring curve were calculated.

Velocity inlet and pressure outlet were used. No slip boundary conditions were used to solve the near wall region. Couple method was used to correct the pressure during the calculation. Second-order upwind scheme was used

FIGURE 9: Normal velocity distribution near the convex surface (m/s).

to resolve the Navies-Stocks equations. Convergence was determined by the residual error less than 0.0001.

4. Results and Discussions

4.1. Velocity Distribution on the Measuring Line. Figure 4 shows results of the transverse velocity on the measuring line. The result of the nonlinear $k-\epsilon$ turbulence model is in good agreement with experimental result. The transverse velocity of experimental result on the measuring line has a hump characteristic, and it can be captured by the nonlinear $k-\epsilon$ turbulence model. The results of RNG $k-\epsilon$ model and SST $k-\omega$ model could not obtain the nonlinear phenomena.

The longitudinal velocity on the measuring line is shown in Figure 5. The result of RNG $k-\epsilon$ model and SST $k-\omega$ model is smaller than the result of nonlinear $k-\epsilon$ turbulence model at the region of dominant flow. Longitudinal velocity on the measuring line agrees well with the experiment result. The result of longitudinal velocity by nonlinear $k-\epsilon$ turbulence model can also capture the nonlinear phenomena when $0 < h/H < 0.3$.

The axial velocity on the measuring line is shown in Figure 6. The maximum point of the axial velocity by the nonlinear $k-\epsilon$ model is very close to the result of experimental data, while the results based on RNG $k-\epsilon$ model and SST

$k-\omega$ model have large error at this point. The nonlinear characteristic of the axial velocity when $0 < h/H < 0.3$ can also be captured by the nonlinear $k-\epsilon$ model.

4.2. Pressure and Velocity Distribution Near the Convex Surface. Pressure distributions near the convex surface of the curved rectangular duct are shown in Figure 7. The results based on the nonlinear $k-\epsilon$ model have a large region with negative pressure, while the results by RNG $k-\epsilon$ model can only capture a small region with negative pressure. The pressure on the convex surface by SST $k-\omega$ model does not have negative value. The velocity distribution on the convex surface is shown in Figure 8. Results by RNG $k-\epsilon$ model are almost the same with SST $k-\omega$ model. Results of the nonlinear $k-\epsilon$ model are much different from the other two turbulence models. In consideration of the velocity results on the measuring line, the pressure and velocity distribution on the convex surface by the nonlinear $k-\epsilon$ model are more close the real flow.

Normal velocity distributions near the convex surface are shown in Figure 9. The normal velocity calculated by the nonlinear $k-\epsilon$ model has negative value on the surface near the convex surface. The results based on SST $k-\omega$ model and RNG $k-\epsilon$ model could not have this phenomenon. The

FIGURE 10: Velocity distribution on the convex surface (m/s).

nonlinear solution of Reynolds stress can predict the complex flow near the wall.

4.3. Limit Streamline Near the Convex Surface. Figure 10 shows the results of limit streamline near the convex surface. Results of RNG k - ϵ model and SST k - ω model have flow separation phenomena which can be seen from Figure 10(b). The flow also shows symmetric distribution and no vortex near the convex surface by RNG k - ϵ model and SST k - ω model. The results of the limit streamline near the convex surface based on the nonlinear k - ϵ model show great difference from the other two models. Both flow separation and reverse flow can be captured by the nonlinear k - ϵ model. The flow separation at the front of the curved rectangular duct is the same as the other turbulence model, but the results of RNG

k - ϵ model and SST k - ω model could not predict the reverse flow at the curved region.

4.4. Reynolds Stress Close to the Convex Surface. One of the Reynolds stresses, which is shear stress, is shown in Figure 11. The distribution of shear stress close to the convex surface calculated by the nonlinear k - ϵ model is greatly different from the results of the other two turbulence models. The results of shear stress based on RNG k - ϵ model and SST k - ω model close to the convex surface are all above 0, while the result performed by the nonlinear k - ϵ model has negative value. The range of the value of shear stress by RNG k - ϵ model and SST k - ω model is very small, and it has uniform distribution close to the convex surface. So the nonlinear k - ϵ model can calculate the nonlinear distribution of Reynolds stresses close

FIGURE 11: Shear stress distributions on the convex surface (Pa-s).

to the wall. It can be used to predict the complex flow with nonlinear phenomena.

5. Conclusions

This paper presents an improved nonlinear k - ε turbulence model to include sensitivity to high pressure gradient flow and large curvature flow. The nonlinear k - ε turbulence model incorporates the advantage of Wilcox's k - ω model for the prediction of near wall region and the k - ε model for the main-stream flow. The improved nonlinear k - ε turbulence model improves accuracy by incorporating a nonlinear solution of Reynolds stress, which is realized through the turbulence velocity scale and turbulence time scale. The resulting formulation for the nonlinear solution is relatively simple and satisfies principles of frame invariance and realizability.

Results for the curved rectangular duct indicate that the improved model shows a nontrivial improvement compared with the original k - ω model and RNG k - ε model. Results of velocity on the measuring line are in great agreement with experimental result. The model is able to capture the flow separation and reverse flow, and it can predict the nonlinear flow caused by flow separation. Due to its simplicity and realizability, this model can provide a method for flow prediction of complex flows with significant effects of streamline curvature.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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