Research Article

A D-N Alternating Algorithm for Solving 3D Exterior Helmholtz Problems

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The nonoverlapping domain decomposition method, which is based on the natural boundary reduction, is applied to solve the exterior Helmholtz problem over a three-dimensional domain. The basic idea is to introduce a spherical artificial boundary; the original unbounded domain is changed into a bounded subdomain and a typical unbounded region; then, a Dirichlet-Nuemann (D-N) alternating method is presented; the finite element method and natural boundary element methods are alternately applied to solve the problems in the bounded subdomain and the typical unbounded subdomain. The convergence of the D-N alternating algorithm and its discretization are studied. Some numerical experiments are presented to show the performance of this method.

1. Introduction

Many scientific and engineering problems can be reduced to exterior boundary value problems of partial differential equations. Although the numerical methods for solving boundary value problems, such as the finite element method and the finite difference method, are very effective on a bounded domain, yet we often find it difficult to be applied to unbounded problem directly. To solve such problems in infinite region numerically, there are a variety of numerical methods, cf. [1–3] and references therein for more details.

There have been studied for Dirichlet-Nuemann alternating algorithm, or D-N alternating algorithm on 2D unbounded domain problems extensively, such as Poisson equation [4] and Helmholtz equation [5]. Wu and Yu [6] studied the natural integral equations of 3D Helmholtz problems. Jia et al. [7] investigated the coupled natural boundary element-finite element method for solving 3D exterior Helmholtz problem.

In this paper, based on the natural boundary reduction, a D-N alternating algorithm is devised for the numerical solution of three dimensional Helmholtz problem in an infinite region with an inner spheroid boundary. Firstly, we turn

the D-N alternating algorithm into the Richardson iterative algorithm which is equivalent to the original method. Secondly, we prove the convergence of algorithm in the general exterior domain. Then, we give the weak form and discretization of the original equations and prove the convergence of the discretization form. Finally, some numerical examples are presented to illustrate the feasibility and efficiency of this method.

2. D-N Alternating Algorithm

Let (r, θ, φ) denote the spherical coordinates; we consider the following 3D exterior Helmholtz problem:

$$\begin{aligned} -\Delta u - k^2 u &= f, \quad \text{in } \Omega^c, \\ \frac{\partial u}{\partial n} &= g, \quad \text{on } \Gamma_R, \end{aligned} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^3$, $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$, and $\Gamma_R = \partial\Omega$. k denotes the wave number related to the wavelength λ of the incident wave through $k = 2\pi/\lambda$. And assume that g is a known function. In order to assure the existence and uniqueness of the solution of (1), problem (1) satisfies the

Sommerfeld radiation condition at infinity, which imposes that the scattered wave is outgoing:

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0, \quad (2)$$

where $i = \sqrt{-1}$, $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. Take a sphere $\Gamma_1 = \{(r, \theta, \varphi) \mid r = R_1, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$ which surrounds Γ_R , such that $\text{dist}(\Gamma_1, \Gamma_R) > 0$. Then the artificial boundary Γ_1 divides the original unbounded region into two subregions Ω_{ext} and Ω_{int} , where Ω_{ext} is the outside subregion and Ω_{int} is the bounded subregion surrounded by Γ_R and Γ_1 , Ω_{ext} and Ω_{int} are nonoverlapping domains. Then the original problem is turned into two subproblems over subdomains Ω_{ext} and Ω_{int} . For exterior boundary value problem (1), we construct the following D-N alternating algorithm:

(i) Step 1. Select the initial $\lambda^0 \in H^{1/2}(\Gamma_1)$, $n = 0$.

(ii) Step 2. Solve the Dirichlet problem in Ω_{ext} :

$$\begin{aligned} -\Delta u_2^n - k^2 u_2^n &= 0, \quad \text{in } \Omega_{\text{ext}}, \\ u_2^n &= \lambda^n, \quad \text{on } \Gamma_1, \\ \lim_{r \rightarrow \infty} r \left(\frac{\partial u_2^n}{\partial r} - iku_2^n \right) &= 0. \end{aligned} \quad (3)$$

(iii) Step 3. Solve the Neumann problem in Ω_{int} :

$$\begin{aligned} -\Delta u_1^n - k^2 u_1^n &= f, \quad \text{in } \Omega_{\text{int}}, \\ \frac{\partial u_1^n}{\partial n_1} &= g, \quad \text{on } \Gamma_R, \\ \frac{\partial u_1^n}{\partial n_1} &= -\frac{\partial u_2^n}{\partial n_2}, \quad \text{on } \Gamma_1. \end{aligned} \quad (4)$$

(iv) Step 4. Input θ_n , and let

$$\lambda^{n+1} = \theta_n u_1^n|_{\Gamma_1} + (1 - \theta_n) \lambda^n. \quad (5)$$

(v) Step 5. Let $n = n + 1$, and go to Step 2,

where u_1^n and u_2^n are the n th approximate solutions in Ω_{int} and Ω_{ext} , respectively. n_1 and n_2 denote the unit outward normals of Γ_1 with respect to two neighboring subdomains; θ_n denotes the n th relaxation factor and λ^0 is an arbitrary function in $H^{1/2}(\Gamma_1)$. Note that, on interface Γ_1 , only the value of the normal derivative of the solution of (3) is needed in solving (4). So it is unnecessary to solve (3). Actually, we can obtain $\partial u_2^n / \partial n_2$ directly from λ^n by making use of the following natural integral equation [6]:

$$\frac{\partial u_2^n}{\partial n_2} = \mathcal{K} \lambda^n, \quad (6)$$

where $\mathcal{K}$ is the natural integral operator of Helmholtz equation in Ω_{ext} :

$$\frac{\partial u_2^n}{\partial n_2} = -\frac{\partial u_2^n}{\partial r}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{kh_l'(kR_1)}{h_l(kR_1)} \Lambda_l^m Y_l^m(\theta, \varphi), \quad (7)$$

$h_l(\cdot)$ is the first-kind Hankel function of order l and

$$\begin{aligned} \Lambda_l^m &= \int_0^{2\pi} \int_0^\pi \lambda^n(\theta', \varphi') \overline{Y_l^m(\theta', \varphi')} \sin \theta' d\theta' d\varphi', \\ Y_l^m(\theta, \varphi) &= \sqrt{\frac{(l-m)!}{(l+m)!} \frac{2l+1}{4\pi}} \cdot P_l^m(\cos \theta) e^{im\varphi}, \\ \sum_{l=0}^{\infty} \sum_{m=-l}^l (1+l^2)^{1/2} |\Lambda_l^m|^2 &< +\infty, \end{aligned} \quad (8)$$

and $P_l^m(\cdot)$ denotes the associated Legendre function of the first kind, $\overline{Y_l^m(\cdot)}$ denotes the complex conjugate of $Y_l^m(\cdot)$.

3. Equivalent Iterative Method

To discuss the convergence of the D-N alternating algorithm in Section 2, we first establish a law of deciding. Let u be the exact solution of (1). Denote $\lambda = \gamma_0 u = u|_{\Gamma_1}$, $u_1 = u|_{\Omega_{\text{int}}}$, $u_2 = u|_{\Omega_{\text{ext}}}$; it is easy to know $\lambda \in H^{1/2}(\Gamma_1)$, and u_1 and u_2 are solutions of the following equations, respectively:

$$\begin{aligned} -\Delta u_1 - k^2 u_1 &= f, \quad \text{in } \Omega_{\text{int}}, \\ \frac{\partial u_1}{\partial n_1} &= g, \quad \text{on } \Gamma_R, \end{aligned} \quad (9)$$

$$\begin{aligned} u_1 &= \lambda, \quad \text{on } \Gamma_1, \\ -\Delta u_2 - k^2 u_2 &= 0, \quad \text{in } \Omega_{\text{ext}}, \\ u_2 &= \lambda, \quad \text{on } \Gamma_1, \end{aligned} \quad (10)$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u_2}{\partial r} - iku_2 \right) = 0.$$

By Green formula, we have

$$\frac{\partial u_1}{\partial n_1} \Big|_{\Gamma_1} = -\frac{\partial u_2}{\partial n_2} \Big|_{\Gamma_1}. \quad (11)$$

According to superimposition principle, the solutions of (9) and (10) can be denoted

$$u_s = \mathcal{R}_s \lambda + \mathcal{T}_s f, \quad s = 1, 2, \quad (12)$$

where $\mathcal{T}_s f$ ($s = 1, 2$) satisfy, respectively,

$$\begin{aligned} -\Delta \mathcal{T}_1 f - k^2 \mathcal{T}_1 f &= f, \quad \text{in } \Omega_{\text{int}}, \\ \frac{\partial \mathcal{T}_1 f}{\partial n_1} &= g, \quad \text{on } \Gamma_R, \\ \mathcal{T}_1 f &= 0, \quad \text{on } \Gamma_1, \\ -\Delta \mathcal{T}_2 f - k^2 \mathcal{T}_2 f &= 0, \quad \text{in } \Omega_{\text{ext}}, \\ \mathcal{T}_2 f &= 0, \quad \text{on } \Gamma_1, \end{aligned} \quad (13)$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial \mathcal{T}_2 f}{\partial r} - ik \mathcal{T}_2 f \right) = 0.$$

It is easy to verify that

$$\frac{\partial \mathcal{T}_2 f}{\partial n_1} = 0. \quad (14)$$

Extend elements of Ω_1 and Ω_2 by zeros to $\mathcal{R}_1 \lambda$ and $\mathcal{R}_2 \lambda$, respectively; therefore $\mathcal{R}_1 \lambda$ satisfies

$$\begin{aligned} -\Delta \mathcal{R}_1 \lambda - k^2 \mathcal{R}_1 \lambda &= 0, \quad \text{in } \Omega_{\text{int}}, \\ \frac{\partial \mathcal{R}_1 \lambda}{\partial n_1} &= 0, \quad \text{on } \Gamma_R, \\ \mathcal{R}_1 \lambda &= \lambda, \quad \text{on } \Gamma_1, \end{aligned} \quad (15)$$

while $\mathcal{R}_2 \lambda$ satisfies

$$\begin{aligned} -\Delta \mathcal{R}_2 \lambda - k^2 \mathcal{R}_2 \lambda &= 0, \quad \text{in } \Omega_{\text{ext}}, \\ \mathcal{R}_2 \lambda &= \lambda, \quad \text{on } \Gamma_1, \\ \lim_{r \rightarrow \infty} r \left(\frac{\partial \mathcal{R}_2 \lambda}{\partial r} - ik \mathcal{R}_2 \lambda \right) &= 0, \end{aligned} \quad (16)$$

where $\mathcal{R}_2$ is the Possion integral operator of Helmholtz equation in the exterior spherical domain Ω_{ext} . Define

$$\mathcal{S}_i = \frac{\partial \mathcal{R}_i}{\partial n_i}, \quad i = 1, 2, \quad (17)$$

$\mathcal{S}_2$ is $\mathcal{K}$, which is the natural integral operator of Helmholtz equation in the exterior spherical domain Ω_{ext} . Substituting (12) in (11), we have

$$\frac{\partial (\mathcal{R}_1 - \mathcal{R}_2) \lambda}{\partial n_1} = \frac{\partial (\mathcal{T}_2 - \mathcal{T}_1) f}{\partial n_1} = -\frac{\partial \mathcal{T}_1 f}{\partial n_1}. \quad (18)$$

Namely,

$$\frac{\partial (\mathcal{R}_1 - \mathcal{R}_2) \lambda}{\partial n_1} = -\frac{\partial \mathcal{T}_1 f}{\partial n_1}. \quad (19)$$

Define the right-hand term in (19) $\chi = -\partial \mathcal{T}_1 f / \partial n_1$; obviously, χ , a function independent of λ , can be solved independently by the problem in subdomains in advance. Then, (19) is equivalent to equation

$$\mathcal{S} \lambda = \chi, \quad (20)$$

where $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}_1 + \mathcal{K}$ is the Poincaré-Steklov operator [8] on Γ_1 . Note that $\lambda \in H^{1/2}(\Gamma_1)$, $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}_1 + \mathcal{K}$; it comes that

$$\mathcal{S}_1: H^{1/2}(\Gamma_1) \longrightarrow H^{-1/2}(\Gamma_1). \quad (21)$$

Then,

$$\mathcal{S}: H^{1/2}(\Gamma_1) \longrightarrow H^{-1/2}(\Gamma_1). \quad (22)$$

Theorem 1. $\mathcal{S}_i : H^{1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_1)$, $i = 1, 2$, is symmetric positive operator.

Proof. We note that $H^{1/2}(\Gamma_1) \subset L^2(\Gamma_1) \subset H^{-1/2}(\Gamma_1)$; therefore, for any $\lambda, \mu \in H^{1/2}(\Gamma_1) \subset L^2(\Gamma_1)$. By Green formula, we know

$$\langle \mathcal{S}_i \lambda, \mu \rangle = \int_{\Gamma_1} \mathcal{S}_i \lambda \mu \, ds = \int_{\Gamma_1} \frac{\partial \mathcal{R}_i \lambda}{\partial n_i} \mu \, ds = a_i(\mathcal{R}_i \lambda, \mathcal{R}_i \mu), \quad (23)$$

where

$$a_i(u, v) = \sum_{s=1}^3 \int_{\Omega_{\text{ext}}} \frac{\partial u}{\partial x_s} \cdot \frac{\partial v}{\partial x_s} \, d\mathbf{x} + \int_{\Omega_{\text{ext}}} k^2 uv \, d\mathbf{x}. \quad (24)$$

This proves $\mathcal{S}_i$ is symmetric positive operator. $\square$

Corollary 2. There exists $\mathcal{S}_i^{-1}$, which is the inverse operator of $\mathcal{S}_i$.

Corollary 3. The bilinear form $a_i(\cdot, \cdot)$ on a normed linear space V_k is said to be coercive on $H^{1/2}(\Gamma_1)$; that is, there exists a positive constant C such that

$$\langle \mathcal{S}_k \lambda, \lambda \rangle \geq C \|\lambda\|_{H^{1/2}(\Gamma_1)}^2, \quad (25)$$

where $V_k = \{v \mid v \in H_0^1(\Omega_k), v|_{\partial\Omega \cap \partial\Omega_k} = 0, k = 1, 2, \}$.

By Poincaré-Steklov operator, solving the original problem (1) is rewritten as solving the operator equation (20). Generally speaking, it is easier to solve $\mathcal{S}_1^{-1}$ than to solve the inverse of $\mathcal{S}$. In fact,

$$\mathcal{S}_1 \lambda = h, \quad h \in H^{-1/2}(\Gamma_1) \quad (26)$$

is equivalent to finding $u_1 \in V_1$, such that

$$a_1(u_1, v) = \int_{\Gamma_1} h v \, ds, \quad \forall v \in V_1. \quad (27)$$

Let

$$\lambda = u_1|_{\Gamma_1}. \quad (28)$$

It is easy to verify that

$$\lambda = \mathcal{S}_1^{-1} h \in H^{1/2}(\Gamma_1). \quad (29)$$

Now we introduce $\mathcal{S}_1$ as preprocessor and preconditioned Richardson iterative method. Given $\lambda^0 \in H^{1/2}(\Gamma_1)$, the procedure of iterative

$$\mathcal{S}_1(\lambda^{n+1} - \lambda^n) = \theta_n(\chi - \mathcal{S} \lambda^n), \quad (30)$$

where

$$\chi = -\frac{\partial}{\partial n_1}(\mathcal{T}_1 f). \quad (31)$$

Following [9], to analyze the convergence of algorithm (30), what we need to do is just to estimate the eigenvalues of operator $\mathcal{S}_1^{-1} \mathcal{S}$, namely, to estimate the upper bound and the lower bound of $\langle \mathcal{S} \lambda, \lambda \rangle / \langle \mathcal{S}_1 \lambda, \lambda \rangle$.

Lemma 4. *There exist two positive constants σ and τ , such that*

$$\sigma = \inf_{\lambda \in H^{1/2}(\Gamma_1)} \frac{\langle \mathcal{S}\lambda, \lambda \rangle}{\langle \mathcal{S}_1\lambda, \lambda \rangle}, \quad \tau = \sup_{\lambda \in H^{1/2}(\Gamma_1)} \frac{\langle \mathcal{S}\lambda, \lambda \rangle}{\langle \mathcal{S}_1\lambda, \lambda \rangle}. \quad (32)$$

Lemma 5. *For any $\lambda \in H^{1/2}(\Gamma_1)$, one has $\sigma \leq \langle \mathcal{S}\lambda, \lambda \rangle / \langle \mathcal{S}_1\lambda, \lambda \rangle \leq \tau$.*

Theorem 6. *If one puts $\varepsilon = 2/\tau$ and $0 < \theta_n < \varepsilon$, preconditioned Richardson iterative algorithm (30) is convergent. Particularly, optimal relaxation factor is $\theta_{\text{opt}} = 2/(\sigma + \tau)$. Correspondingly, optimal compression ratio of the iteration is $\delta_{\text{opt}} = (\tau - \sigma)/(\sigma + \tau)$.*

Theorem 7. *The D-N alternating algorithm (3)–(5) is equivalent to the preconditioned Richardson iterative method (30).*

Proof. Let $e_1^n = u_1 - u_1^n$, let $e_2^n = u_2 - u_2^n$, and let $v^n = \lambda - \lambda^n$. Following (3)–(10), we have

$$\begin{aligned} -\Delta e_2^n - k^2 e_2^n &= 0, \quad \text{in } \Omega_{\text{ext}}, \\ e_2^n &= v^n, \quad \text{on } \Gamma_1, \end{aligned} \quad (33)$$

$$\begin{aligned} \lim_{r \rightarrow \infty} r \left(\frac{\partial e_2^n}{\partial r} - i k e_2^n \right) &= 0, \\ -\Delta e_1^n - k^2 e_1^n &= 0, \quad \text{on } \Omega_{\text{int}}, \\ e_1^n &= 0, \quad \text{on } \Gamma_R, \end{aligned} \quad (34)$$

$$\begin{aligned} \frac{\partial e_1^n}{\partial n_1} &= -\frac{\partial e_2^n}{\partial n_2}, \quad \text{on } \Gamma_1, \\ v^{n+1} &= \theta_n e_1^n|_{\Gamma_1} + (1 - \theta_n) v^n. \end{aligned} \quad (35)$$

Meanwhile,

$$e_2^n = \mathcal{R}_2 v^n, \quad e_1^n = \mathcal{R}_1 (e_1^n). \quad (36)$$

Substituting the above equation to the third equation of (34), we can obtain

$$\mathcal{S}_1 (e_1^n) = -\mathcal{S}_2 (v^n) = -\mathcal{K} (v^n). \quad (37)$$

Therefore,

$$\begin{aligned} \mathcal{S}_1 (v^{n+1} - v^n) &= \mathcal{S}_1 (\theta_n e_1^n|_{\Gamma_1} - \theta_n v^n) \\ &= -\theta_n (\mathcal{K} v^n) - \theta_n (\mathcal{S}_1 v^n) = -\theta_n \mathcal{S} v^n; \end{aligned} \quad (38)$$

thus,

$$\mathcal{S}_1 (\lambda^{n+1} - \lambda^n) = \theta_n \mathcal{S} (\lambda - \lambda^n). \quad (39)$$

It comes that

$$\begin{aligned} \mathcal{S}\lambda &= \frac{\partial}{\partial n_1} (\mathcal{R}_1 \lambda) + \frac{\partial}{\partial n_2} (\mathcal{R}_2 \lambda) \\ &= \frac{\partial}{\partial n_1} (u_1 - \mathcal{T}_1 f) + \frac{\partial u_2}{\partial n_2} = -\frac{\partial \mathcal{T}_1 f}{\partial n_1} = \chi. \end{aligned} \quad (40)$$

Substituting the above equation in (39), we have

$$\mathcal{S}_1 (\lambda^{n+1} - \lambda^n) = \theta_n (\chi - \mathcal{S}\lambda^n). \quad (41)$$

□

4. Convergence of the Algorithm

Considering a special case in which $\Gamma_0 = \{(r, \theta, \varphi) \mid r = R_0, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$ and the interface $\Gamma_1 = \{(r, \theta, \varphi) \mid r = R_1, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$, following [10], we have the following.

Lemma 8. *Suppose*

$$f_n(x) = \frac{n}{2n+1} + \frac{n+1}{2n+1} x^{n+1}, \quad (42)$$

$$x \in [0, 1], \quad n \in \mathbb{Z}^+ = \{0, 1, 2, \dots\},$$

and set

$$U(x) = \sup_{n \in \mathbb{Z}^+} \{f_n(x)\}, \quad L(x) = \inf_{n \in \mathbb{Z}^+} \{f_n(x)\}; \quad (43)$$

then, we have

$$U(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\ x, & \frac{1}{2} < x \leq 1, \end{cases} \quad (44)$$

$$L(x) = \begin{cases} f_0(x), & 0 \leq x \leq x_0, \\ f_{k+1}(x), & x_k < x \leq x_{k+1}, \end{cases}$$

where x_k is the only root of equation $f_{k+1}(x) - f_k(x) = 0$ in interval $(0, 1)$ and $x_k < x_{k+1}$, k is a nonnegative integer.

For $\lambda(\theta, \varphi) \in H^{1/2}(\Gamma_1)$, in the L^2 -sense, the following expansion holds

$$\lambda(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \Lambda_l^m Y_l^m(\theta, \varphi), \quad (45)$$

where $P_l^m(x)$ denotes the associated Legendre function of the first kind; $\overline{Y_l^m(\theta, \varphi)}$ denotes the complex conjugate of Y_l^m . On the one hand,

$$\langle \mathcal{K} \lambda, \lambda \rangle = R_1 \sum_{l=0}^{\infty} \sum_{m=-l}^l (1+l) |\Lambda_l^m|^2 \quad (46)$$

satisfies

$$R_1 \left\{ \sum_{l=0}^{\infty} \sum_{m=-l}^l (1+l^2)^{1/2} |\Lambda_l^m|^2 \right\}^{1/2} < +\infty; \quad (47)$$

$$\overline{\Lambda_l^m} = (-1)^{|m|} \Lambda_l^{-m},$$

on the other hand, set

$$H_1^N \lambda = \sum_{l=0}^N \sum_{m=-l}^l \frac{R_1^{l+1} \Lambda_l^m}{R_1^{2l+1} - R_0^{2l+1}} \left(r^l - \frac{R_0^{2l+1}}{r^{l+1}} \right) Y_l^m(\theta, \varphi), \quad (48)$$

$$R_0 < r < R_1.$$

Following [11] and (48), we have

$$\lim_{N \rightarrow \infty} \|H_1 \lambda - H_1^N \lambda\|_{H^1(\Omega_1)} = 0. \quad (49)$$

According to Green formula and orthogonality of $Y_l^m(\theta, \varphi)$, it comes that

$$\begin{aligned}
 \langle \mathcal{S}_1 \lambda, \lambda \rangle &= \int_{\Omega_1} \nabla H_1 \lambda \cdot \nabla H_1 \lambda \, d\mathbf{x} \\
 &= \lim_{M \rightarrow \infty, N \rightarrow \infty} \int_{\Omega_1} \nabla H_1^M \lambda \cdot \nabla H_1^M \lambda \, d\mathbf{x} \\
 &= \lim_{M \rightarrow \infty, N \rightarrow \infty} \int_{\Gamma_1} \frac{\partial H_1^M \lambda}{\partial r} H_1^N \lambda \, ds \\
 &= \lim_{M \rightarrow \infty, N \rightarrow \infty} \sum_{l=0}^{\min\{M, N\}} \sum_{m=-l}^l \frac{l R_1^{2l+1} + (l+1) R_0^{2l+1}}{R_1^{2l+1} - R_0^{2l+1}} \\
 &\quad \times R_1 \Lambda_l^m \cdot \overline{\Lambda_l^m}. \tag{50}
 \end{aligned}$$

Lemma 9. Suppose

$$\begin{aligned}
 \langle \mathcal{S}_1 \lambda, \lambda \rangle &= R_1 \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{l R_1^{2l+1} + (l+1) R_0^{2l+1}}{R_1^{2l+1} - R_0^{2l+1}} |\Lambda_l^m|^2; \\
 \langle \mathcal{K} \lambda, \lambda \rangle &= R_1 \sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1) |\Lambda_l^m|^2,
 \end{aligned} \tag{51}$$

then,

$$\frac{1}{U(R_0/R_1)} \leq \frac{\langle \mathcal{S} \lambda, \lambda \rangle}{\langle \mathcal{S}_1 \lambda, \lambda \rangle} \leq \frac{1}{L(R_0/R_1)}. \tag{52}$$

Theorem 10. If relaxation factor θ_n satisfies

$$0 < \theta_n < 2L \left(\frac{R_0}{R_1} \right), \tag{53}$$

then the D-N alternating algorithm (3)–(5) converges. Particularly, the optimal relaxation factor is

$$\theta_{opt} = \frac{2U(R_0/R_1)L(R_0/R_1)}{U(R_0/R_1) + L(R_0/R_1)}, \tag{54}$$

while the optimal compression ratio of the iteration is

$$\delta_{opt} = \frac{U(R_0/R_1) - L(R_0/R_1)}{U(R_0/R_1) + L(R_0/R_1)}. \tag{55}$$

Remark 11. We have analyzed the convergence for exterior spherical domain. Following [10], the convergence analysis for the general exterior domain can be extended from above content. Obviously, the optimal relaxation factor and optimal compression ratio of the iteration depend on R_1 and the geometry of Ω^c in some way, which is confirmed by our numerical results.

5. The Weak Form and Discretization

The weak form of problem (4) is Find $u_1^n \in H_g^1(\Omega_{\text{int}})$, such that

$$D_{\Omega_{\text{int}}}(u_1^n, v) = \int_{\Omega_{\text{int}}} f v \, d\mathbf{x} - \int_{\Gamma_1} v \frac{\partial u_2^n}{\partial n_2} \, ds, \quad \forall v \in H_0^1(\Omega_{\text{int}}), \tag{56}$$

where

$$\begin{aligned}
 D_{\Omega_{\text{int}}}(u_1^n, v) &= \int_{\Omega_{\text{int}}} (\nabla u_1^n \cdot \nabla v + k^2 u_1^n v) \, d\mathbf{x}, \\
 H_g^1(\Omega_{\text{int}}) &= \{v \mid v \in H^1(\Omega_{\text{int}}), v|_{\Gamma_R} = g\}, \\
 H_0^1(\Omega_{\text{int}}) &= \{v \mid v \in H^1(\Omega_{\text{int}}), v|_{\Gamma_0} = 0\}.
 \end{aligned} \tag{57}$$

Obviously, $v|_{\Gamma_1} \in H^{1/2}(\Gamma_1)$. Following (6), the weak form (56) can be rewritten as Find $u_1^n \in H_g^1(\Omega_{\text{int}})$, such that

$$\begin{aligned}
 D_{\Omega_{\text{int}}}(u_1^n, v) &= \int_{\Omega_{\text{int}}} f v \, d\mathbf{x} - \widehat{D}^c(\lambda^n, v), \\
 \forall v &\in H_0^1(\Omega_{\text{int}}), \quad i = 1, 2,
 \end{aligned} \tag{58}$$

where

$$\begin{aligned}
 \widehat{D}^c(\lambda^n, v) &= \langle \mathcal{K} \lambda^n, v \rangle = - \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{k R_1^2 h_l'(kR)}{h_l(kR)} (\lambda^n)_l^m \overline{V_l^m}, \\
 V_l^m &= \int_0^{2\pi} \int_0^\pi v(\theta', \varphi') \overline{Y_l^m(\theta', \varphi')} \sin \theta' \, d\theta' \, d\varphi', \\
 (\lambda^n)_l^m &= \int_0^{2\pi} \int_0^\pi \lambda^n(\theta', \varphi') Y_l^m(\theta', \varphi') \sin \theta' \, d\theta' \, d\varphi'.
 \end{aligned} \tag{59}$$

(58) and (5) are the weak form of D-N method (3)–(5). In the following, we consider the discretization of this weak form. Discretize Ω_{int} into a finite number of element domains. Let $S_h(\Omega_{\text{int}})$ denote the linear subspace of $H_0^1(\Omega_{\text{int}})$ corresponding to this partition. Define M_0 , M_1 , and M_i to be the sets of all nodes belonging to Γ_R , Γ_1 , and Ω_{int} , respectively. Denote U_A^n the approximate value and $L_A(\mathbf{x})$ a basis function at node A . Obviously,

$$S_h(\Omega_{\text{int}}) = \text{span} \{L_{A^i}(\mathbf{x}), L_{A^1}(\mathbf{x}) \mid A^i \in M_i, A^1 \in M_1\}. \tag{60}$$

The finite element approximation of u_1^n can be expressed as

$$\begin{aligned}
 u_{1h}^n &= \sum_{A^i \in M_i} U_{A^i}^n L_{A^i}(\mathbf{x}) + \sum_{A^1 \in M_1} U_{A^1}^n L_{A^1}(\mathbf{x}) \\
 &\quad + \sum_{A^0 \in M_0} g(\mathbf{x}_{A^0}) L_{A^0}(\mathbf{x}).
 \end{aligned} \tag{61}$$

By (61) and (5), we can obtain the following discrete D-N alternating algorithm:

$$\begin{pmatrix} Q^{11} & Q^{1i} & Q^{10} \\ Q^{i1} & Q^{ii} & Q^{i0} \\ Q^{01} & Q^{0i} & Q^{00} \end{pmatrix} \begin{pmatrix} U_1^n \\ U_i^n \\ U_0^n \end{pmatrix} = \begin{pmatrix} -B\Lambda^n \\ 0 \\ F^0 \end{pmatrix}, \tag{62}$$

$$\Lambda^{n+1} = \theta_n U_{A^1}^n + (1 - \theta_n) \Lambda^n, \tag{63}$$

where

$$\begin{aligned}\Lambda^n &= [\lambda^n (A^1)]^T, & U_s^n &= [U_{A^s}^n]^T, \\ Q^{st} &= [D_{\Omega_{\text{int}}} (L_{A^s}, L_{B^t})], \\ & s, t = 0, 1, i, \\ Q^{01} &= Q^{10} = 0, & B &= [b(\gamma_0 N_{A^1}, \gamma_0 N_{B^1})], \\ F^0 &= \left[\int_{\Gamma_R} g N_{B^0} ds \right]^T, \\ A^s, B^s &\in M_s, \quad s = 0, 1, i.\end{aligned}\quad (64)$$

6. Convergence Analysis for the Discretization Form

In the following, we consider the convergence of the discretization form.

Theorem 12. *The discrete D-N alternating algorithms (62) and (63) are equivalent to the following preconditioned Richardson iteration:*

$$S_h^{(1)} (\Lambda^{n+1} - \Lambda^n) = \theta_n (F^1 - \mathcal{S}_h \Lambda^n), \quad (65)$$

where

$$\begin{aligned}S_h^{(1)} &= Q^{11} - Q^{1i} (Q^{ii} - Q^{i0} (Q^{00})^{-1} Q^{0i})^{-1} Q^{i1}, \\ S_h &= S_h^{(1)} + B, \\ F^1 &= Q^{1i} (Q^{ii} - Q^{i0} (Q^{00})^{-1} Q^{0i})^{-1} Q^{i0} (Q^{00})^{-1} F^0.\end{aligned}\quad (66)$$

Proof. Let $[U_1, U_i, U_0]^T$ be the solution of the equation

$$\begin{pmatrix} Q^{11} + B & Q^{1i} & Q^{10} \\ Q^{i1} & Q^{ii} & Q^{i0} \\ Q^{01} & Q^{0i} & Q^{00} \end{pmatrix} \begin{pmatrix} U_1 \\ U_i \\ U_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F^0 \end{pmatrix}. \quad (67)$$

Simplified to

$$\begin{aligned}& [Q^{11} - Q^{1i} (Q^{ii} - Q^{i0} (Q^{00})^{-1} Q^{0i})^{-1} Q^{i1} + B] U_1 \\ &= Q^{1i} (Q^{ii} - Q^{i0} (Q^{00})^{-1} Q^{0i})^{-1} Q^{i0} (Q^{00})^{-1} F^0,\end{aligned}\quad (68)$$

namely,

$$S_h U_1 = F^1. \quad (69)$$

Then, (67) can be rewritten as

$$\begin{pmatrix} Q^{11} & Q^{1i} & Q^{10} \\ Q^{i1} & Q^{ii} & Q^{i0} \\ Q^{01} & Q^{0i} & Q^{00} \end{pmatrix} \begin{pmatrix} U_1 \\ U_i \\ U_0 \end{pmatrix} = \begin{pmatrix} -B U_1 \\ 0 \\ F^0 \end{pmatrix}. \quad (70)$$

By (62) and (70),

$$\begin{pmatrix} Q^{11} & Q^{1i} & Q^{10} \\ Q^{i1} & Q^{ii} & Q^{i0} \\ Q^{01} & Q^{0i} & Q^{00} \end{pmatrix} \begin{pmatrix} U_1^n - U_1 \\ U_i^n - U_i \\ U_0^n - U_0 \end{pmatrix} = \begin{pmatrix} -B (\Lambda^n - U_1) \\ 0 \\ 0 \end{pmatrix}. \quad (71)$$

We have

$$S_h^{(1)} (U_1^n - U_1) = B (U_1 - \Lambda^n). \quad (72)$$

Therefore,

$$\begin{aligned}S_h^{(1)} (\Lambda^{n+1} - \Lambda^n) &= \theta_n S_h^{(1)} (U_1^n - \Lambda^n) \\ &= \theta_n [S_h^{(1)} (U_1^n - U_1) + S_h^{(1)} (U_1 - \Lambda^n)] \\ &= \theta_n (B + S_h^{(1)}) (U_1 - \Lambda^n) \\ &= \theta_n S_h (U_1 - \Lambda^n) \\ &= \theta_n (F^1 - S_h \Lambda^n).\end{aligned}\quad (73)$$

□

Theorem 13. *Let α be spectral radius of $(S_h^{(1)})^{-1} S_h$, which is iterative matrix of preconditioned Richardson iteration. Then, there is a positive constant β , which is independent of finite element mesh parameter h of subdomain Ω_{int} , such that $\alpha \leq \beta$.*

Theorem 14. *Eigenvalues of iterative matrix $(S_h^{(1)})^{-1} S_h$ are real and greater than 1.*

Theorem 15. *Put $\theta_n = \eta$ ($n = 0, 1, \dots$); then, there exists a constant δ ($0 < \delta < 1$), which is independent of finite element mesh parameter h of subdomain Ω_{int} . For $0 < \eta < \delta$, the preconditioned Richardson iteration converges, the discrete D-N alternating algorithm (62)-(63) converges and the convergence rate is independent of mesh parameter h of subdomain Ω_{int} .*

Proof. By (73), we have

$$\begin{aligned}U_1 - \Lambda^{n+1} &= [I - \eta (S_h^{(1)})^{-1} S_h] (U_1 - \Lambda^n) \\ &= [I - \eta (S_h^{(1)})^{-1} S_h]^{n+1} (U_1 - \Lambda^0);\end{aligned}\quad (74)$$

it comes that

$$\|U_1 - \Lambda^{n+1}\|_2 \leq \delta^{n+1} \|U_1 - \Lambda^0\|_2, \quad (75)$$

where

$$\delta = \|I - \eta (S_h^{(1)})^{-1} S_h\|_2. \quad (76)$$

Following Theorem 13 and Theorem 14, there exists a constant δ ($0 < \delta < 1$), which is independent of h . For $0 < \eta < \delta$, spectral radius of $I - \eta (S_h^{(1)})^{-1} S_h$ is less than 1, and spectral norm $\delta < 1$; therefore,

$$\lim_{n \rightarrow +\infty} \|U_1 - \Lambda^{n+1}\|_2 = 0. \quad (77)$$

It follows that the preconditioned Richardson iteration converges; then, the discrete D-N alternating algorithm (62)-(63)

TABLE 1: Maximum node-error on $\overline{\Omega}_i$ for $R_1 = 2.0$.

θ	Iterative times					
	0	1	2	3	4	12
0.3	0.33062	0.18917	0.11558	0.07835	0.05963	0.04858
0.4	0.33062	0.14107	0.07438	0.05198	0.04806	0.04713
0.5	0.33062	0.08362	0.05926	0.04816	0.04823	0.04818
0.6	0.33062	0.06345	0.09187	0.06537	0.06823	0.09684
0.7	0.33062	0.11998	0.15164	0.06356	0.08451	0.06529

TABLE 2: Maximum node-error on $\overline{\Omega}_i$ for $R_1 = 6.0$.

θ_n	Iterative times					
	0	1	2	3	4	12
0.3	0.13381	0.05442	0.04968	0.04930	0.04927	0.04927
0.4	0.13381	0.03435	0.05269	0.04849	0.49448	0.04927
0.5	0.13381	0.04332	0.06801	0.03922	0.05465	0.04939
0.6	0.13381	0.11119	0.14317	0.07156	0.11630	0.08039
0.7	0.13381	0.16126	0.24338	0.22273	0.31370	0.63435

converges and the convergence rate is independent of mesh parameter h of subdomain Ω_{int} . $\square$

7. Numerical Examples

To test the effectiveness of the method in this paper, we give two numerical examples, using the discrete D-N alternating algorithm in Section 2. In the two examples, the exact solutions are known. The purpose of these examples is to check the convergence in terms of iteration and mesh size h .

Suppose that $\Omega^c = \{(x, y, z) \mid |x| > 1, |y| > 1, |z| > 1\}$ is the exterior domain of cube $[-1, 1] \times [-1, 1] \times [-1, 1]$. The artificial boundary is $\Gamma_1 = \{(r, \theta, \varphi) \mid r = R_1, R_1 > \sqrt{3}\}$, $r = \sqrt{x^2 + y^2 + z^2}$. Split the bounded domain Ω_{int} into finite element mesh as follows. Firstly, make every inner boundary edge into N_1 equivalent parts and connect nodes by using segments which are parallel to the corresponding coordinate axis; then, the grid located in the inner boundary Γ_1 is generated. Secondly, along the radial direction, rays starting from each node of located in the inner boundary intersect artificial boundaries at some nodes, and all the other nodes are produced by dividing segments along the radial direction into N_2 equivalent parts. At last, we get an eight-node trilinear isoparametric finite elements of Ω_{int} .

Denote e the maximum node-error on $\overline{\Omega}_i$:

$$e(n) = \sup_{P_i \in \overline{\Omega}_i} |u(P_i) - u_{1h}^n(P_i)|; \quad (78)$$

e_h denotes the maximum node-error of the adjacent two-steps on nodes

$$e_h(n) = \sup_{P_i \in \overline{\Omega}_i} |u_{1h}^{n+1}(P_i) - u_{1h}^n(P_i)|; \quad (79)$$

TABLE 3: The relation between convergence rate and relaxation factor.

θ_n		Iterative times					
		0	1	2	3	4	12
0.2	e	0.13381	0.07449	0.05900	0.05302	0.05072	0.04928
	e_h		0.02358	0.05223	0.01157	0.00256	0.00057
	q_h			4.51432	4.51329	4.51206	4.51101
0.3	e	0.13381	0.54416	0.04968	0.04330	0.04927	0.04927
	e_h	0	0.50323	0.0806	0.01294	0.00209	0.00031
	q_h	0	0	6.2432	6.2301	6.1952	6.1809
0.4	e	0.13381	0.03434	0.05268	0.04849	0.04945	0.04927
	e_h	0	0.01512	0.00176	0.00021	0.00002	0
	q_h	0	0	8.5961	8.5532	8.3201	7.9326

q_h is the convergence rate

$$q_h(n) = \frac{e_h(n-1)}{e_h(n)}. \quad (80)$$

We substitute $\sum_{l=0}^M$ for $\sum_{l=0}^{\infty}$ in the computing of the entries of K_h . Denote by m the total number of nodes on $\overline{\Omega}_{\text{int}}$. By computing, the results are as follows.

Example 16. In the problem (1), we take

$$g = \sqrt{\frac{3}{4\pi}} \left(\frac{k^2 i - 2i - 2k}{k^2} \right) e^{ik} \cos \theta; \quad (81)$$

the exact solution is

$$u(r, \theta) = -\sqrt{\frac{3}{4\pi}} \frac{kr + i}{k^2 r^2} e^{ikr} \cos \theta. \quad (82)$$

Let $k = 2.0$, $N_1 = N_2 = 2$, $m = 78$, $M = 4$, the numerical results are shown in Tables 1 and 2.

Tables 1 and 2 show that the D-N alternating algorithm (62)-(63) is convergent. Table 1 shows that the speed of convergence is faster while $\theta_n = 0.5$. Conversely, Table 2 shows the different phenomenon that the speed of convergence is faster while $\theta_n = 0.3$. It proves Theorem 10 and shows relaxation factor θ_n is dependent on R_1 . But it is valuable to know that $\theta_n = 0.5$ unsatisfied the requirement of Theorem 10 ($2L(R_0/R_1) = 0.5$), but the D-N alternating algorithm (62)-(63) is convergent, which shows (53) is the sufficient condition of Theorem 10.

Example 17. In the problem (1), we take

$$g = \sqrt{\frac{15}{32\pi}} \left(\frac{k^3 + 4k^2 i - 9k - 9i}{k^3} \right) e^{i(k+\varphi)} \sin 2\theta; \quad (83)$$

the exact solution is

$$u(r, \theta) = -\sqrt{\frac{15}{32\pi}} \left(\frac{3i}{k^3 r^3} + \frac{3}{k^2 r^2} - \frac{i}{kr} \right) e^{i(kr+\varphi)} \sin 2\theta. \quad (84)$$

Let $k = 2.0$, $R_1 = 6.0$, $N_1 = N_2 = 2$, $m = 78$, $M = 4$, the numerical results are shown in Table 3.

TABLE 4: The relation between convergence rate and mesh.

Mesh	Iterative times						
	0	1	2	3	4	12	
I	e	0.16651	0.12025	0.11342	0.11354	0.11339	0.11327
	e_h		0.11547	0.10300	0.08651	0.07381	0.06525
	q_h			1.12105	1.19058	1.17205	1.13114
II	e	0.13381	0.03434	0.05268	0.04849	0.04945	0.04927
	e_h		0.02834	0.01814	0.01162	0.00745	0.00478
	q_h			1.56223	1.56134	1.56001	1.56018
III	e	0.08874	0.01239	0.01753	0.01623	0.01648	0.01644
	e_h		0.01056	0.00449	0.00191	0.00083	0.00036
	q_h			2.35073	2.35327	2.30124	2.29801

Table 3 shows that in the scope of relaxation factor, the bigger the relaxation factor θ_n is, the faster the convergence rate is.

While we put $\theta_n = 0.4$ and $R_1 = 6.0$ and denote Mesh I: $N_1 = N_2 = 1$, $m = 16$, $M = 4$; Mesh II: $N_1 = N_2 = 2$, $m = 78$, $M = 4$; Mesh III: $N_1 = N_2 = 4$, $m = 490$, $M = 4$. The numerical results are shown in Table 4.

Table 4 shows that the convergence rate of the discrete D-N alternating algorithm is independent of mesh parameter h . At the same time, the closer we refine the mesh, the smaller the maximum node-error will be. The maximum node-error is roughly of $O(h)$ order.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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