Research Article

Multiple Soliton Solutions for a New Generalization of the Associated Camassa-Holm Equation by Exp-Function Method

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The Exp-function method is generalized to construct N-soliton solutions of a new generalization of the associated Camassa-Holm equation. As a result, one-soliton, two-soliton, and three-soliton solutions are obtained, from which the uniform formulae of N-soliton solutions are derived. It is shown that the Exp-function method may provide us with a straightforward, effective, and alternative mathematical tool for generating N-soliton solutions of nonlinear evolution equations in mathematical physics.

1. Introduction

The investigation of the traveling wave solutions to nonlinear evolution equations (NLEEs) plays an important role in mathematical physics. A lot of physical models have supported a wide variety of solitary wave solutions. In the recent years, much efforts have been spent on this task and many significant methods have been established such as inverse scattering transform [1], Bäcklund and Darboux transform [2], Hirota bilinear method [3], homogeneous balance method [4], Jacobi elliptic function method [5], tanh-function method [6], Exp-function method [7], simple equation method [8], F-expansion method [9, 10], improved F-expansion method [11], and extended F-expansion method [12]. Here, we study a new generalization of the associated Camassa-Holm equation.

The Camassa-Holm (CH) equation

$$U_T + 2k^2 U_X - U_{XXT} + 3UU_X = 2U_X U_{XX} + UU_{XXX}, \quad (1)$$

where k is a nonzero real constant, was derived as a model for shallow water waves by Camassa and Holm in 1993 [13]. This equation is integrable with the following Lax pair:

$$\begin{aligned} \psi_{xx} &= \lambda(u - u_{xx} + k^2)\psi + \frac{1}{4}\psi, \\ \psi_t &= \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi. \end{aligned} \quad (2)$$

Considerable interest was paid to the CH equation in recent decades about its integrability and exact solutions [14–20]. Schiff and Fisher showed that the Camassa-Holm equation possessed the Bäcklund transformations and an infinite number of local conserved quantities by using the loop group approach [15, 16]. Parker gave explicit multisoliton solutions for the CH equation by taking the Hirota bilinear method and a coordinate transformation [18]. Its structure and dynamics were investigated in the different parameter regimes. According to [18], there is a reciprocal transformation, $(T, X) \rightarrow (t, x)$, such that

$$\begin{aligned} dx &= RdX - URdT, & dt &= dT, \\ R &= \sqrt{U - U_{XX} + k^2}. \end{aligned} \quad (3)$$

Let us apply the reciprocal transformation to the Lax pair (2) and define the following potential function $u(x, t)$:

$$u = \frac{1}{2} \frac{R_{xx}}{R} - \frac{1}{4} \left(\frac{R_x}{R} \right)^2 + \frac{1}{4R^2} - \frac{1}{4k^2}; \quad (4)$$

then (1) is transformed into the following associated Camassa-Holm (ACH) equation:

$$u_t + 2k^3 u_x + 4k^2 uu_t + 2k^2 u_x \partial_x^{-1} u_t - k^2 u_{xxt} = 0. \quad (5)$$

Hone showed in [21] how the ACH equation (5) is related to Schrodinger operators and the KdV equation and described how to construct solutions of the ACH equation from tau-functions of the KdV hierarchy, including rational, N-soliton, and elliptic solutions.

Recently, integrable negative order flows, mixed equations, and the relationship of different hierarchies attracted much attention, including continuous and discrete cases, such as the negative KdV, mixed KdV, and Volterra lattice equations. In [22], Luo et al. introduced a new generalization of the associated Camassa-Holm equation; namely,

$$u_t + \alpha (u_{xxx} - 6uu_x) + \beta (2k^3 u_x + 4k^2 uu_t + 2k^2 u_x \partial_x^{-1} u_t - k^2 u_{xxt}) = 0, \quad (6)$$

where α and β are two arbitrary constants.

Apparently, (6) is reduced to the ACH equation (5) when we take $\alpha = 0$, $\beta = 1$. For $\alpha = 1$, $\beta = 0$, (6) gives the KdV equation. So, (6) may be called the ACH-KdV equation. In [22], Luo et al. show that (6) is integrable in the sense of Lax pair and construct some exact solutions of (6) by Darboux transformation through Lax pair.

The Exp-function method [7] proposed by He and Wu in 2006 provides us with a straightforward and effective method for obtaining generalized solitary wave solutions and periodic solutions of NLEEs. The method was used by many researchers to study various NLEEs. More recently, Marinakis [23] did very interesting work to generalize the Exp-function method for constructing N-soliton solutions of NLEEs. Marinakis chose the famous Korteweg-de Vries (KdV) equation to illustrate the generalized work and successfully obtained the known 2-soliton and 3-soliton solutions in a simple and straightforward way.

In the present paper, we would like to generalize the Exp-function method for constructing N-soliton solutions of ACH-KdV equation (6).

The rest of this paper is organized as follows. In Section 2, we give the description of the Exp-function method for constructing N-soliton solutions of NLEEs. In Section 3, we apply the method to (6). In Section 4, some conclusions and discussions are given.

2. Basic Idea of the Exp-Function Method for N-Soliton Solutions of NLEEs

In this section, we recall the Exp-function method [23] for N-soliton solutions of NLEEs. For a given NLEE, say, in two variables x and t ,

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad (7)$$

the Exp-function method for one-soliton solution is based on the assumption

$$u(x, t) = \frac{\sum_{i_1=0}^{p_1} a_{i_1} e^{i_1 \xi_1}}{\sum_{j_1=0}^{q_1} a_{j_1} e^{j_1 \xi_1}}, \quad \xi_1 = w_1 x - c_1 t + \lambda_1, \quad (8)$$

where a_{i_1} , b_{j_1} , k_1 , c_1 , λ_1 , and w_1 are unknown constants and the values of p_1 and q_1 can be determined by balancing the linear term of highest order in (7) with the highest order nonlinear term.

In order to seek N-soliton solutions for integer $N > 1$, we generalize (8) to the following form:

$$u(x, t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} e^{\sum_{g=1}^N i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} a_{j_1 j_2 \cdots j_N} e^{\sum_{g=1}^N j_g \xi_g}}, \quad (9)$$

$$\xi_g = w_g x - c_g t + \lambda_g;$$

given the value of $N = 2$, it becomes

$$u(x, t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} e^{\sum_{g=1}^2 i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} a_{j_1 j_2} e^{\sum_{g=1}^2 j_g \xi_g}}, \quad (10)$$

which can be used to construct two-soliton solution.

When $N = 3$, (9) changes into

$$u(x, t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \sum_{i_3=0}^{p_3} a_{i_1 i_2 i_3} e^{\sum_{g=1}^3 i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{q_3} a_{j_1 j_2 j_3} e^{\sum_{g=1}^3 j_g \xi_g}}, \quad (11)$$

which can be used to obtain three-soliton solution.

Substituting (10) into (7) and using Mathematica, then equating to zero each coefficient of the same order power of the exponential functions yields a set of equations. Solving the set of equations, we can determine the 2-soliton solution and the following 3-soliton solution by means of (11), provided they exist. If possible, we may conclude with the uniform formula of N-soliton solutions for any $N \geq 1$.

3. Multisoliton Solutions of the ACH-KdV Equation

In this section we apply Exp-function method to the ACH-KdV equation (6). We first remove the integral term in (6) by introducing the potential

$$u(x, t) = q_x(x, t); \quad (12)$$

then substituting (12) into (6) yields

$$q_{xt} + \alpha (q_{xxx} - 6q_x q_{xx}) + \beta (2k^3 q_{xx} + 4k^2 q_x q_{xt} + 2k^2 q_{xx} q_t - k^2 q_{xxt}) = 0. \quad (13)$$

Suppose that (13) admits the one-soliton solution of the form

$$q(x, t) = \frac{a_1 e^{\xi_1}}{1 + b_1 e^{\xi_1}}, \quad (14)$$

where $\xi_1 = w_1 x - c_1 t + \lambda_1$, w_1 , c_1 are undetermined constants, and λ_1 is an arbitrary constant. Obviously, (14) is included in the same form as (7). Substituting (14) into (13) and then

FIGURE 1: Figures of solution (18) and with $\alpha = 1$, $\beta = -2$, $k = 1$, $w_1 = 1$, $b_1 = 1$, $\lambda_1 = 0$. (a) Spatial plots in the intervals $x \in [-5, 5]$ and $t \in [-5, 5]$; (b) plan plots with $t = 0$, in the interval $x \in [-10, 10]$.

equating to zero each coefficient of the same order power of $e^{i\xi_i}$ ($i = 1, 2, 3, 4$) yield a set of equations for a_1 , b_1 , w_1 , and c_1 as follows:

$$\begin{aligned}
 &\beta k^2 a_1 c_1 w_1^3 + 2\beta k^3 a_1 w_1^2 + \alpha a_1 w_1^4 - a_1 c_1 w_1 = 0, \\
 &-11\beta k^2 a_1 b_1 c_1 w_1^3 + 2\beta k^3 a_1 b_1 w_1^2 - 6\beta k^2 a_1^2 c_1 w_1^2 \\
 &-11\alpha a_1 b_1 w_1^4 - 6\alpha a_1^2 w_1^3 - a_1 b_1 c_1 w_1 = 0, \\
 &11\beta k^2 a_1 b_1^2 c_1 w_1^3 - 2\beta k^3 a_1 b_1^2 w_1^2 + 6\beta k^2 a_1^2 b_1 c_1 w_1^2 \\
 &+ 11\alpha a_1 b_1^2 w_1^4 + 6\alpha a_1^2 b_1 w_1^3 + a_1 b_1^2 c_1 w_1 = 0, \\
 &-\beta k^2 a_1 b_1^3 c_1 w_1^3 - 2\beta k^3 a_1 b_1^3 w_1^2 \\
 &-\alpha a_1 b_1^3 w_1^4 + a_1 b_1^3 c_1 w_1 = 0.
 \end{aligned} \quad (15)$$

Solving these equations by Maple, one has

$$\begin{aligned}
 a_1 &= -2b_1 w_1, & b_1 &= b_1, & w_1 &= w_1, \\
 c_1 &= \frac{w_1 (2\beta k^3 + \alpha w_1^2)}{1 - \beta k^2 w_1^2}.
 \end{aligned} \quad (16)$$

Substituting (16) into (14), we have

$$q(x, t) = \frac{-2b_1 w_1 e^{\xi_1}}{1 + b_1 e^{\xi_1}}. \quad (17)$$

Using potential (12), we can get the one-soliton solutions of (6) as follows:

$$u(x, t) = q_x(x, t) = -\frac{2b_1 w_1^2 e^{\xi_1}}{(1 + b_1 e^{\xi_1})^2}, \quad (18)$$

where $\xi_1 = w_1 x - (w_1(2\beta k^3 + \alpha w_1^2)/(1 - \beta k^2 w_1^2))t + \lambda_1$ and w_1 , λ_1 , and b_1 are arbitrary constants. The one-soliton solution (18) is shown in Figure 1.

Next, we suppose that (13) has the 2-soliton solution in the form

$$q(x, t) = \frac{a_1 e^{\xi_1} + a_2 e^{\xi_2} + a_3 e^{\xi_1 + \xi_2}}{1 + b_1 e^{\xi_1} + b_2 e^{\xi_2} + b_3 e^{\xi_1 + \xi_2}}, \quad (19)$$

where $\xi_1 = w_1 x - c_1 t + \lambda_1$; $\xi_2 = w_2 x - c_2 t + \lambda_2$; $a_1, a_2, a_3, b_1, b_2, b_3, w_1$, and w_2 are constants to be determined; and λ_1 and λ_2 are arbitrary constants. Obviously, (19) has the same form as (10). Substituting (19) into (13) and using manipulations similar to those illustrated above, we obtain

$$\begin{aligned}
 a_1 &= -2b_1 w_1, & a_2 &= -2b_2 w_2, \\
 a_3 &= -2b_1 b_2 (w_1 + w_2) A_{12}, & b_3 &= b_1 b_2 A_{12}, \\
 c_1 &= \frac{w_1 (2\beta k^2 + \alpha w_1^2)}{1 - \beta k^2 w_1^2}, & c_2 &= \frac{w_2 (2\beta k^2 + \alpha w_2^2)}{1 - \beta k^2 w_2^2},
 \end{aligned} \quad (20)$$

where

$$A_{12} = \frac{(w_1 - w_2)^2}{(w_1 + w_2)^2}. \quad (21)$$

Substituting (20) into (19), we have

$$q(x, t) = -2 \frac{b_1 w_1 e^{\xi_1} + b_2 w_2 e^{\xi_2} + b_1 b_2 (w_1 + w_2) A_{12} e^{\xi_1 + \xi_2}}{1 + b_1 e^{\xi_1} + b_2 e^{\xi_2} + b_1 b_2 A_{12} e^{\xi_1 + \xi_2}}. \quad (22)$$

FIGURE 2: Figures of solution (23) and with $\alpha = 1$, $\beta = -1$, $w_1 = 1$, $w_2 = 1.1$, $b_1 = 1$, $b_2 = 1$, $k = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$. (a) Spatial plots in the intervals $x \in [-10, 15]$ and $t \in [-10, 10]$; (b)–(d) plan plots and with $t = -10, 0, 10$, $x \in [-5, 20]$, $x \in [-15, 15]$, $x \in [-15, 15]$, respectively.

Using potential (12), we can get the two-soliton solutions of (6) as follows:

$$u(x, t) = -2 \left((b_1^2 b_2 w_2^2 A_{12} e^{2\xi_1 + \xi_2} + b_1 b_2^2 w_1^2 A_{12} e^{\xi_1 + 2\xi_2} + 2b_1 b_2 (w_1 - w_2)^2 e^{\xi_1 + \xi_2} + b_1 w_1^2 e^{\xi_1} + b_2 w_2^2 e^{\xi_2}) \times (1 + b_1 e^{\xi_1} + b_2 e^{\xi_2} + b_1 b_2 A_{12} e^{\xi_1 + \xi_2})^{-2} \right), \quad (23)$$

where $\xi_1 = w_1 x - (w_1(2\beta k^3 + \alpha w_1^2)/(1 - \beta k^2 w_1^2))t + \lambda_1$; $\xi_2 = w_2 x - (w_2(2\beta k^3 + \alpha w_2^2)/(1 - \beta k^2 w_2^2))t + \lambda_2$; and w_1 , w_2 , λ_1 , λ_2 , b_1 , and b_2 are arbitrary constants. The two-soliton solution (18) is shown in Figure 2.

From Figure 2 we can find the wave chase phenomenon. The speed of the first wave $w_1 = 1$; the speed of the second wave $w_2 = 1.1$. As time increases, the second wave exceeds the first one.

In what follows, we now suppose that the three-soliton solution of (13) can be expressed as follows:

$$q(x, t) = (a_1 e^{\xi_1} + a_2 e^{\xi_2} + a_3 e^{\xi_3} + a_4 e^{\xi_1 + \xi_2} + a_5 e^{\xi_1 + \xi_3} + a_6 e^{\xi_2 + \xi_3} + a_7 e^{\xi_1 + \xi_2 + \xi_3}) \times (1 + b_1 e^{\xi_1} + b_2 e^{\xi_2} + b_3 e^{\xi_3} + b_4 e^{\xi_1 + \xi_2} + b_5 e^{\xi_1 + \xi_3} + b_6 e^{\xi_2 + \xi_3} + b_7 e^{\xi_1 + \xi_2 + \xi_3})^{-1}, \quad (24)$$

FIGURE 3: Three-soliton waves and with $\alpha = 1$, $\beta = -1$, $w_1 = 1$, $w_2 = 1.1$, $w_3 = 0.9$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $k = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$. (a) Spatial plots in the intervals $x \in [-10, 20]$ and $t \in [-15, 15]$; (b)–(d) plan plots and with $t = -15, 0, 15$, $x \in [-5, 30]$, $x \in [-10, 20]$, $x \in [-20, 15]$.

where, $\xi_1 = w_1 x - c_1 t + \lambda_1$; $\xi_2 = w_2 x - c_2 t + \lambda_2$; $\xi_3 = w_3 x - c_3 t + \lambda_3$; a_i, b_i ($i = 1, 2, \dots, 7$), w_1, w_2 , and w_3 are constants to be determined; and λ_1, λ_2 , and λ_3 are arbitrary constants. Obviously, (24) has the same form as (11). Substituting (24) into (13) and using manipulations similar to those illustrated above, we obtain

$$\begin{aligned} a_1 &= -2w_1 b_1, & a_2 &= -2w_2 b_2, \\ a_3 &= -2w_3 b_3, & a_4 &= -2b_1 b_2 (w_1 + w_2) A_{12}, \\ a_5 &= -2b_1 b_3 (w_1 + w_3) A_{13}, & a_6 &= -2b_2 b_3 (w_2 + w_3) A_{23}, \\ a_7 &= -2b_1 b_2 b_3 (w_1 + w_2 + w_3) A_{12} A_{13} A_{23}, \\ b_4 &= b_1 b_2 A_{12}, & b_5 &= b_1 b_3 A_{13}, \end{aligned}$$

$$\begin{aligned} b_6 &= b_2 b_3 A_{23}, & b_7 &= b_1 b_2 b_3 A_{12} A_{13} A_{23}, \\ c_1 &= \frac{w_1 (2\beta k^2 + \alpha w_1^2)}{1 - \beta k^2 w_1^2}, & c_2 &= \frac{w_2 (2\beta k^2 + \alpha w_2^2)}{1 - \beta k^2 w_2^2}, \\ c_3 &= \frac{w_3 (2\beta k^2 + \alpha w_3^2)}{1 - \beta k^2 w_3^2}, \end{aligned} \quad (25)$$

where

$$A_{ij} = \frac{(w_i - w_j)^2}{(w_i + w_j)^2}, \quad 1 \leq i < j \leq 3. \quad (26)$$

Substituting (25) into (24) and using potential (12), we can obtain the three-soliton solution of (6). The expression is so complicated; therefore we do not give out it in detail. The three-soliton solution is shown in Figure 3, from which wave chase phenomenon also can be found.

When $N \geq 4$, similar computational work becomes more and more complicated since the coefficients of the exponential functions become a highly nonlinear system as shown in [23]. Fortunately, we can find a uniform formula of the N -soliton solutions by analyzing the obtained solutions (17), (22), and (24). We rewrite solutions (17), (22), and (24) in an alternative form:

$$u(x, t) = -2 \left[\ln \left(1 + b_1 e^{\xi_1} \right) \right]_{xx}, \quad (27)$$

where $\xi_1 = w_1 x - c_1 t + \lambda_1$;

$$u(x, t) = -2 \left[\ln \left(1 + b_1 e^{\xi_1} + b_2 e^{\xi_2} + b_1 b_2 A_{12} e^{\xi_1 + \xi_2} \right) \right]_{xx}, \quad (28)$$

where $\xi_i = w_i x - c_i t + \lambda_i$ ($i = 1, 2$), $A_{12} = (w_1 - w_2)^2 / (w_1 + w_2)^2$; and

$$\begin{aligned} u(x, t) = -2 \left[\ln \left(1 + b_1 e^{\xi_1} + b_2 e^{\xi_2} + b_3 e^{\xi_3} \right. \right. \\ \left. \left. + b_1 b_2 A_{12} e^{\xi_1 + \xi_2} + b_1 b_3 A_{13} e^{\xi_1 + \xi_3} \right. \right. \\ \left. \left. + b_2 b_3 A_{23} e^{\xi_2 + \xi_3} \right. \right. \\ \left. \left. + b_1 b_2 b_3 A_{12} A_{13} A_{23} e^{\xi_1 + \xi_2 + \xi_3} \right) \right]_{xx}, \end{aligned} \quad (29)$$

where $\xi_i = w_i x - c_i t + \lambda_i$ ($i = 1, 2, 3$), $A_{ij} = ((w_i - w_j)^2 / (w_i + w_j)^2)$ ($1 \leq i < j \leq 3$).

The uniform formula of the N -soliton solutions can be constructed as follows:

$$\begin{aligned} u(x, t) = -2 \left[\ln \left(1 + \sum_{i=1}^N b_i e^{\xi_i} + \sum_{1 \leq i < j \leq N} b_i b_j A_{ij} e^{\xi_i + \xi_j} \right. \right. \\ \left. \left. + \cdots + \prod_{i=1}^N b_i \prod_{1 \leq i < j \leq N} A_{ij} e^{\xi_i + \xi_j + \cdots + \xi_N} \right) \right]_{xx}, \end{aligned} \quad (30)$$

where $\xi_i = w_i x - c_i t + \lambda_i$ ($i = 1, 2, \dots, N$) and $A_{ij} = ((w_i - w_j)^2 / (w_i + w_j)^2)$ ($1 \leq i < j \leq N$).

4. Conclusions

In this paper, one-soliton, two-soliton, and three-soliton solutions of the ACH-KdV equation have been successfully obtained, from which the uniform formula of N -soliton solutions is derived. This is due to the generalization of the Exp-function method. Figures 1–3 imply that these obtained solutions have rich local structures. It may be important to explain some physical phenomena. The method with the help of mathematical software for generating 1-soliton, 2-soliton, and 3-soliton solutions is more simple and straightforward

than Hirota bilinear method, without employing the bilinear operator defined in the Hirota bilinear method. The paper shows that the Exp-function method may provide us with a straightforward and effective mathematical tool for generating N -soliton solutions or testing its existence and can be extended to other NLEEs in mathematical physics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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