Research Article

Almost Sure Asymptotical Adaptive Synchronization for Neutral-Type Neural Networks with Stochastic Perturbation and Markovian Switching

Wuneng Zhou, Xueqing Yang, Jun Yang, Anding Dai, and Huashan Liu

College of Information Sciences and Technology, Donghua University, Shanghai 200051, China

Correspondence should be addressed to Wuneng Zhou; zhouwuneng@163.com

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The problem of almost sure (a.s.) asymptotic adaptive synchronization for neutral-type neural networks with stochastic perturbation and Markovian switching is researched. Firstly, we proposed a new criterion of a.s. asymptotic stability for a general neutral-type stochastic differential equation which extends the existing results. Secondly, based upon this stability criterion, by making use of Lyapunov functional method and designing an adaptive controller, we obtained a condition of a.s. asymptotic adaptive synchronization for neutral-type neural networks with stochastic perturbation and Markovian switching. The synchronization condition is expressed as linear matrix inequality which can be easily solved by Matlab. Finally, we introduced a numerical example to illustrate the effectiveness of the method and result obtained in this paper.

1. Introduction

As it is well known, the stability and synchronization of neural networks can be applied to create chemical and biological systems, secure communication systems, information science, image processing, and so on. In recent years, different control methods are derived to achieve different synchronization, such as randomly occurring control [1], sampled-data control [2, 3], passivity analysis [4], impulsive control [5–8], and adaptive control [9].

By utilizing adaptive control method, the parameters of the system need to be estimated and the control law needs to be updated when the neural networks evolve. In the past decade, much attention has been devoted to the research of the adaptive synchronization for neural networks. In [9, 10], the adaptive lag synchronization of unknown chaotic neural networks is considered. Adaptive synchronization problem of delayed neural networks with stochastic perturbation is studied in [11]. Besides these, there are many literatures to study adaptive synchronization problems (see, e.g., [12, 13] and the references therein).

Recently, the stability and synchronization of neutral-type systems, specially neutral-type neural networks, which depend on the derivative of the state and the delay state have attracted a lot of attention (see, e.g., [14–19] and the references therein) due to the fact that some physical systems in the real world can be described by neutral-type models (see [20]). However, the adaptive control was not investigated in [14–17], and the neutral term of derivative of the delay state was not taken into account in the neural networks proposed in [9–13]. Zhou et al. in [18] did not study the almost sure (a.s.) synchronization for neutral-type neural networks. Zhu et al. in [19] did not research the synchronization problem for neural networks with Markovian switching parameters. From the authors’ best knowledge, so far the almost surely adaptive synchronization problem for neutral-type neural networks with stochastic perturbation and Markovian switching parameters has not been fully investigated yet. This motivates our current work.

In this paper, the problem of almost sure (a.s.) asymptotic adaptive synchronization for neutral-type neural networks with stochastic perturbation and Markovian switching is researched. By making use of Lyapunov functional method and designing an adaptive controller, we obtained a condition of a.s. asymptotic adaptive synchronization for neutral-type...
neural networks with stochastic perturbation and Markovian switching. Finally, we introduced a numerical example to illustrate the effectiveness of the method and result obtained in this paper. The main contributions of this paper are as follows.

(1) A new model for a class of neural-type neural networks with stochastic perturbation and Markovian switching. Finally, we introduced a numerical example to illustrate the effectiveness of the method and result obtained in this paper. The main contributions of this paper are as follows.

(2) A new criterion of a.s. asymptotic stability for a general neural-type stochastic differential equation is proposed which extends the existing results.

The notations are quite standard. Throughout this paper, \( \mathbb{R}_+, \mathbb{R}^n, \) and \( \mathbb{R}^{m \times n} \) denote the set of nonnegative real numbers, \( n \)-dimensional Euclidean space, and the set of all \( n \times n \) real matrices, respectively. The superscript \( T \) denotes matrix transposition, \( \det(\cdot) \) stands for the block diagonal matrix. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (i.e., the filtration contains all \( P \)-null sets and is increasing and right continuous). For \( p > 0 \), denote by \( L^p(\Omega; [−\tau, 0]; \mathbb{R}^n) \) the family of all \( \mathcal{F}_0 \)-measurable, \( C([-\tau, 0]; \mathbb{R}^n) \)-valued random variables \( \xi \) such that the mathematical expectation \( \mathbb{E}[\|\xi\|^p] < \infty \), where \( C([-\tau, 0]; \mathbb{R}^n) \) denotes the family of all \( \mathcal{F}_0 \)-measurable random variables \( \xi \) that are continuous \( \mathbb{R}^n \)-valued functions, and the norm \( \|\xi\| = \sup_{-\tau \leq s \leq 0} |\xi(s)| \).

Denote by \( C_b([-\tau, 0]; \mathbb{R}^n) \) the family of all \( \mathcal{F}_0 \)-measurable, bounded and \( C([-\tau, 0]; \mathbb{R}^n) \)-valued random variables. If \( x(t) \) is a continuous \( \mathbb{R}^n \)-valued stochastic process on \( t \in [-\tau, \infty) \), we let \( x_t = [x(t + \theta) : -\tau \leq \theta \leq 0] \) for \( t \geq 0 \) which is regarded as a \( C([-\tau, 0]; \mathbb{R}^n) \)-valued stochastic process. \( C^2_t(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{F}_t) \) denotes the set of functions from \( \mathbb{R}_+ \times \mathbb{R}^n \) to \( \mathbb{R} \) which are continuously twice differentiable in \( x \in \mathbb{R}^n \) and once differentiable in \( t \in \mathbb{R}_+ \).

### 2. Problem Formulation and Preliminaries

Let \( \{\sigma(t)\}_{t \geq 0} \) be a right-continuous Markov chain on the probability space taking values in a finite state space \( S = \{1, 2, \ldots, N\} \) with generator \( \Gamma = (\gamma_{ij})_{N \times N} \) given by

\[
\mathbb{P}[r(t + \delta) = j | r(t) = i] = \begin{cases} 
\gamma_{ij} \delta + o(\delta), & \text{if } i \neq j, \\
1 + \gamma_{ij} \delta + o(\delta), & \text{if } i = j,
\end{cases}
\]

(1)

where \( \delta > 0 \) and \( \gamma_{ij} \geq 0 \) is the transition rate from \( i \) to \( j \) if \( i \neq j \) while

\[
\gamma_{ii} = - \sum_{j=1, j \neq i}^{N} \gamma_{ij}
\]

(2)

Consider the following neutral-type neural networks called drive system and represented by the compact form as follows:

\[
d [x(t) - D(r(t)) x(t - \tau)] = -C(r(t)) x(t) + A(r(t)) f(x(t)) + B(r(t))
\]

\[
\times f(x(t - \tau)) + E(r(t))
\]

\[
\times \int_{t-\tau}^{t} f(x(s)) ds + J(r(t))\ dt,
\]

(3)

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) is the state vector associated with \( n \) neurons, \( f(\cdot) \) denotes the neuron activation functions, and \( r \) represents the transmission delay. For \( t \geq 0 \), we denote \( r(t) = i, A(r(t)) = A^i, B(r(t)) = B^i, C(r(t)) = C^i, D(r(t)) = D^i, E(r(t)) = E^i, \) and \( J(r(t)) = J^i \), respectively. In neutral network (3), \( \forall i \in S, A^i, B^i, \) and \( E^i \) are the connection weight, the discrete delay connection weight, and distributed delay connection weight matrix, respectively; \( C^i = diag(c^i_1, c^i_2, \ldots, c^i_n) \) is a positive diagonal matrix; \( D^i \) is called the neutral-type parameter matrix; \( f^i = [f^i_1, f^i_2, \ldots, f^i_n]^T \in \mathbb{R}^n \) is the constant external input vector.

The initial condition of system (3) is given in the following form:

\[
x(s) = \xi_x(s), \quad s \in [-\tau, 0], \quad r(0) = i_0
\]

(4)

for any \( \xi_x \in L_2(\mathbb{R}; [-\tau, 0]; \mathbb{R}^n) \).

For the drive system (3), the response system is

\[
d [y(t) - D(r(t)) y(t - \tau)]
\]

\[
= -C(r(t)) y(t) + A(r(t)) f(y(t))
\]

\[
+ B(r(t)) f(y(t - \tau)) + E(r(t))
\]

\[
\times \int_{t-\tau}^{t} f(y(s)) ds + J(r(t)) + U(r(t))\ dt,
\]

(5)

where \( y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n \) is the state vector of the response system (5), \( U(r(t)) = [U_1^i, U_2^i, \ldots, U_n^i]^T \in \mathbb{R}^n \) is a control input vector, \( \omega(t) = [\omega_1(t), \omega_2(t), \ldots, \omega_n(t)]^T \) is an \( n \)-dimensional Brownian motion defined on the complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) (i.e., \( \mathcal{F}_t = \sigma(\omega(s) : 0 \leq s \leq t) \) is a \( \sigma \)-algebra) and is independent of the Markovian process \( \{r(t)\}_{t \geq 0} \) and \( \sigma : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{2m} \) is the noise intensity matrix. It is known that external random fluctuation and other probabilistic causes often lead to this type of stochastic perturbations.
The initial condition of system (5) is given in the following form:

\[ y(s) = \xi_y(s), \quad s \in [-\tau, 0], \quad r(0) = i_0 \]  

for any \( \xi_y \in L^2_{\mathbb{P}_0}([-\tau, 0]; \mathbb{R}^n) \).

Let \( e(t) = y(t) - x(t) \) be the synchronization error vector. From the drive system and the response system, the error system can be written as follows:

\[
d[ e(t) - D(r(t)) e_t ] \]

\[ = \left[ -C(r(t)) e(t) + A(r(t)) g(e(t)) + B(r(t)) g(e_t) \right. \]

\[ + E(r(t)) \int_{t-\tau}^t g(e(s)) ds + U(r(t)) \right] dt \]

\[ \left. + \sigma(t, r(t), e(t), e_t) dw(t), \right. \]

where \( e_t = e(t - \tau), \quad g(e(t)) = f(x(t) + e(t)) - f(x(t)) \).

The initial condition of system (7) is given in the following form:

\[ e(s) = \xi(s) = \xi_y(s) - \xi_x(s), \quad s \in [-\tau, 0], \quad r(0) = i_0, \]

with \( e(0) = 0 \).

The following concept is necessary in this paper.

**Definition 4** (see [21]). The trivial solution \( e(t; \xi, i_0) \) of the error system (7) is said to be almost surely asymptotically stable if

\[
P \left( \lim_{t \to \infty} |e(t; \xi, i_0)| = 0 \right) = 1
\]

for any initial value \( \xi \in C([-\tau, 0]; \mathbb{R}^n) \).

If the error system (7) is almost surely asymptotically stable, then the drive system (3) and the response system (5) are said to be almost surely asymptotically synchronized.

Consider the more general neutral-type stochastic delay differential equation (NSDDE) with Markovian switching:

\[
d [ x(t) - \overline{D}(x(t - \tau), r(t)) ] \]

\[ = \overline{F}(t, r(t), x(t), x(t - \tau)) dt \]

\[ + \overline{G}(t, r(t), x(t), x(t - \tau)) dB(t), \]

where \( B(t) \) is an \( n \)-dimensional Brownian motion defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) but independent of the Markov chain \( \{r(t)\}_{t \geq 0} \) and

\[ \overline{D} : \mathbb{R}^n \times S \to \mathbb{R}^n, \]

\[ \overline{F} : \mathbb{R}_+ \times S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \]

\[ \overline{G} : \mathbb{R}_+ \times S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \]

are all Borel-measurable functions.

For NSDDE (13), the following hypothesis is needed.

**(H1)** Both \( \overline{F} \) and \( \overline{G} \) satisfy the local Lipschitz condition. That is, for each \( h > 0 \), there is an \( L_h > 0 \) such that

\[
\left| \overline{F}(t, i, x, y) - \overline{F}(t, i, \overline{x}, \overline{y}) \right| \]

\[
\leq \left( \overline{F}(t, i, x, y) \right) \quad \forall x, y \in \mathbb{R}^n.
\]

**(H1)** Given any initial data \( \{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C_b([-\tau, 0]; \mathbb{R}^n), (13) has a unique solution denoted by \( x(t; \xi, i_0) \) on \( t \geq 0. \) Moreover, both \( \overline{F}(t, r(t), x(t), y(t)) \) and \( \overline{G}(t, r(t), x(t), y(t)) \) are locally bounded in \( (x, y) \) while uniformly bounded in \( (t, r(t)) \). That is, for any \( h > 0 \), there is a \( K_h > 0 \), such that

\[ \left| \overline{F}(t, r(t), x(t), y(t)) \right| \leq K_h, \quad \forall t \geq 0, r(t) \in S, \quad x, y \in \mathbb{R}^n \]
(H2) For each \(i \in S\), there is a constant \(\kappa_i \in (0,1)\) such that
\[
|\overline{D}(x,i) - \overline{D}(\bar{x},i)| \leq \kappa_i |x - \bar{x}| \quad \forall x, \bar{x} \in \mathbb{R}^n. \tag{17}
\]

(H3) For each \((t,i) \in \mathbb{R}_+ \times S\),
\[
\overline{F}(t,i,0,0) = 0, \quad \overline{G}(t,i,0,0) = 0, \quad \overline{D}(0,i) = 0. \tag{18}
\]

Then, we present some preliminary lemmas which play an important role in the proof of the main results.

**Lemma 5** (see [16]). Let \(x, y \in \mathbb{R}^n\). Then
\[
x^T y + y^T x \leq e x^T x + e^{-1} y^T y \tag{19}
\]
for any \(e > 0\).

**Lemma 6** (the generalized Ito formula, see [17]). Let \(V \in C^{2,1}(\mathbb{R}_+ \times S \times \mathbb{R}^n; \mathbb{R})\) and \(x(t)\) be a solution of neutral-type stochastic delay differential equation (13).

Then for any stopping times \(0 \leq \rho_1 \leq \rho_2 < \infty\) a.s.
\[
EV(\rho_2, r(\rho_2), x(\rho_2) - \overline{D}(x(\rho_2 - \tau), r(\rho_2)))
= EV(\rho_1, r(\rho_1), x(\rho_1) - \overline{D}(x(\rho_1 - \tau), r(\rho_1)))
+ E \int_{\rho_1}^{\rho_2} \mathcal{L}V(s, r(s), x(s), x(s-\tau)) ds \tag{20}
\]
holds provided that \(V(t, r(t), x(t) - \overline{D}(x(t), r(t)))\) and \(\mathcal{L}V(t, r(t), x(t), x(t-\tau))\) are bounded on \(t \in [\rho_1, \rho_2]\) with probability 1, where the operator \(\mathcal{L}V: \mathbb{R}_+ \times S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) is defined by
\[
\mathcal{L}V(t, i, x, y)
= V_t(t, i, x - \overline{D}(y, i)) + V_x(t, i, x - \overline{D}(y, i)) \overline{F}(t,i,x,y)
+ \frac{1}{2} \text{trace} \left[ G^T(t, i, x, y) V_{xx}(t, i, x - \overline{D}(y, i)) 
\times \overline{G}(t,i,x,y) \right]
+ \sum_{j=1}^N \gamma_j V(t, j, x - \overline{D}(y, i)). \tag{21}
\]

Now we cite the convergence theorem of nonnegative semimartingales (see [22], Theorem 7 on page 139) which is a useful lemma.

**Lemma 7** (the convergence theorem of nonnegative semimartingales). Let \(A_1(t)\) and \(A_2(t)\) be two continuous adapted increasing processes on \(t \geq 0\) with \(A_1(0) = A_2(0) = 0\) a.s. Let \(M(t)\) be a real-valued continuous local martingale with \(M(0) = 0\) a.s. Let \(\xi\) be a nonnegative \(\mathcal{F}_0\)-measurable random variable such that \(E\xi < \infty\). Define
\[
X(t) = \xi + A_1(t) - A_2(t) + M(t), \quad t \geq 0. \tag{22}
\]

If \(X(t)\) is nonnegative, then
\[
\left\{ \lim_{t \to \infty} A_1(t) < \infty \right\} \subset \left\{ \lim_{t \to \infty} X(t) < \infty \right\} \tag{23}
\]
and
\[
\left\{ \lim_{t \to \infty} A_2(t) < \infty \right\} \subset \left\{ \lim_{t \to \infty} X(t) < \infty \right\} \\text{a.s.}, \tag{24}
\]
where \(C \subset D\) a.s. means \(P(C \cap D^c) = 0\). In particular, if \(\lim_{t \to \infty} A_1(t) < \infty\) a.s. then, with probability one, we have
\[
\lim_{t \to \infty} X(t) < \infty, \quad \lim_{t \to \infty} A_2(t) < \infty, \quad -\infty < \lim_{t \to \infty} M(t) < \infty.
\]
That is, all of the three processes \(X(t), A_2(t),\) and \(M(t)\) converge to finite random variables.

**Lemma 8** (Hölder inequality; see [21]). Let \(a_i \in \mathbb{R}, k, p \in \mathbb{Z},\) and \(p \geq 1\). Then
\[
\left| \sum_{i=1}^k a_i \right|^p \leq k^{p-1} \sum_{i=1}^k |a_i|^p. \tag{25}
\]

**Lemma 9** (Doob martingale inequality; see [21]). Let \(\{M_t\}_{t \geq 0}\) be an \(\mathbb{R}^n\)-martingale. Let \([a,b]\) be a bounded interval in \(\mathbb{R}_+. If\ p > 1, then \(M_t \in L^p(\Omega; \mathbb{R}^n)\) (the family of \(\mathbb{R}^n\)-valued random variables \(X \in \mathbb{E}[|X|^p] < \infty\)), then
\[
E \left( \sup_{a \leq t \leq b} |M_t|^p \right) \leq \left( \frac{p}{p-1} \right)^p E[M_b]^p. \tag{26}
\]

**Lemma 10** (Chebyshev’s inequality; see [21]). If \(c > 0, p > 0, X \in L^p(\Omega; \mathbb{R}^n), then
\[
P\{ \omega : |X(\omega)| \geq c \} \leq \frac{1}{c^p} E[|X|^p]. \tag{27}
\]

3. **Main Results**

In this section, we give some criteria of adaptive synchronization for the drive system (3) and the response system (5). First, we establish a general result which can be applied widely.

3.1. **Almost Surely Asymptotically Stable**

**Theorem 11.** Let (H1), (H2), and (H3) hold. Assume that there are functions \(V \in C^{3,1}(\mathbb{R}_+ \times S \times \mathbb{R}^n; \mathbb{R}), y \in L^1(\mathbb{R}_+; \mathbb{R}_+),\) and \(W_1, W_2, W_3 \in C(\mathbb{R}^n; \mathbb{R}_+),\) such that
\( |x| \to \infty, \quad i \in S, \quad 0 \leq t < \infty, \quad (28) \)

\( \mathcal{V}(t, i, x, y) \leq \gamma(t) - \mathcal{W}_1(x) + \mathcal{W}_2(y) - \mathcal{W}_3 \ldots (r(t)), \quad (39) \)

(31)

Then for any initial data \( x(t) : -\tau \leq \theta \leq 0 = \xi \in C_{\mathbb{F}}^b([-\tau, 0]; \mathbb{R}^n) \) and \( r(0) = i_0 \in S, \) one has the following.

(R1) Equation (13) has a unique global solution which is denoted by \( x(t; \xi, i_0). \)

(R2) Assume that \( W_1(x) = 0 \) if and only if \( x = 0. \) The solution \( x(t; \xi, i_0) \) obeys that

\( \lim_{t \to \infty} x(t; \xi, i_0) = 0 \quad \text{a.s.} \quad (32) \)

That is, \( x(t; \xi, i_0) \) is almost surely asymptotically stable.

The proof of this theorem is given in the Appendix.

Remark 12. Theorem II is an extension of Theorem I in [16]; that is, when we take \( W_1(x) = W_2(x) \) in our theorem, then Theorem II is coincident with Theorem 3.1 in [16]. Moreover, Theorem II is also an extension of Theorem 2.1 in [23] when we take \( W_1(x) = 0 \) with \( D = 0. \)

Remark 13. From the proof of Theorem II, we can see that if condition (H1) is substituted by (H1)', then the conclusion (R2) is also true.

3.2. Almost Sure Asymptotical Synchronization. In this subsection, we give a criterion of adaptive almost sure asymptotical synchronization for the drive system (3) and the response system (5).

Theorem 14. For systems (3) and (5), let Assumptions 1–3 hold, and the error system (7) has a unique solution denoted by \( e(t; \xi, i_0) \) on \( t \geq 0 \) for any initial data \( e(\theta) : -\tau \leq \theta \leq 0 = \xi \in C_{\mathbb{F}}^b([-\tau, 0]; \mathbb{R}^n) \) with \( e(0) = 0. \)

Assume also that there exist symmetric matrix \( Q_1 > 0, \) diagonal matrix \( P \mathbf{r} > 0 \) \( (i = 1, \ldots, N), \) and positive scalars \( \rho, \rho_1, \rho_2, \epsilon_1 \) \( (i = 1, 2, 3, 4), \) such that

\( \rho_2 I < Q_1 < \rho_1 I, \quad (33) \)

\( P \mathbf{r} < \rho_1 I, \quad (34) \)

\[ \begin{bmatrix}
  (L_2^2 \rho_1 + H_1 \rho) - 2P^2C_i & C_i P_i & L_2 I & \tau L_2 I \\
  * & -\epsilon_1 I & 0 & 0 \\
  * & 0 & -\epsilon_2 I & 0 \\
  * & 0 & 0 & -\epsilon_4 I \\
\end{bmatrix} < 0, \quad (35) \]

\[ \begin{bmatrix}
  \sum_{k=1}^{N} \nu_k P^k + 2P^2 K^* \epsilon_2 P^2 A_i^* \epsilon_3 P^2 B_i^* \epsilon_4 P^2 E_i \\
  * & -\epsilon_2 I & 0 & 0 \\
  * & 0 & -\epsilon_4 I & 0 \\
  * & 0 & 0 & -\epsilon_4 I \\
\end{bmatrix} < 0, \quad (37) \]

\[ \begin{bmatrix}
  \Xi_{11} C_i P_i & L_2 I & L_2 I & \tau L_2 I \\
  * & -\epsilon_1 I & 0 & 0 \\
  * & 0 & -\epsilon_2 I & 0 \\
  * & 0 & 0 & -\epsilon_4 I \\
\end{bmatrix} < 0, \quad (38) \]

where \( i = 1, 2, \ldots, N, \) \( K^* = \text{diag}(k_1^*, k_2^*, \ldots, k_n^*) \) with \( k_i^* \) being arbitrary negative constants to be chosen, and \( \Xi_{11} = (L_2^2 \rho_1 + H_1 \rho - \rho_2 L_1^2 + H_2 \rho) I - 2P^2C_i + \epsilon_4 D_i^T D_i. \)

We choose the feedback control \( U^* \) with the update law as \( U^* = \text{diag}(k_1, k_2, \ldots, k_n)(e - D_i e_i) \) and \( k_j = -\beta_j P_i^j (e - D_i e_i)^j, \) where \( \beta_j > 0 \) \( (j = 1, 2, \ldots, n) \) are arbitrary constants, \( p_i^j \) is the \( j \)th diagonal entry of matrix \( P^i, \) and \( (e - D_i e_i)^j \) is the \( j \)th element of \( e - D_i e_i. \) Then the error system (7) is almost surely asymptotically stable. Therefore, the drive system (3) and the response system (5) are adaptive synchronized a.s.

Proof. Under Assumptions 1–3 and the existence of \( e(t; \xi, i_0), \) it can be seen that \( \overline{F}(t, r(t), e(t), e_i(t)), \overline{G}(t, r(t), e(t), e_i(t)), \) and \( \overline{D}(e_i(t), r(t)) \) satisfy (H1)', (H2), and (H3), where

\[ \overline{F}(t, r(t), e(t), e_i(t)) = -C(r(t)) e(t) + A(r(t)) g(e(t)) + B(r(t)) g(e_i(t)) + E(r(t)) \int_{t-\tau}^{t} g(e(s)) \, ds + U(r(t)), \]

\[ \overline{G}(t, r(t), e(t), e_i(t)) = \sigma(t, r(t), e(t), e_i(t)), \]

\[ \overline{D}(e_i(t), r(t)) = D(r(t)) e_i(t). \]
For each $i \in S$, choosing a nonnegative function
\[
V(t, i, e) = x^T P_i x + \int_{t-\tau}^t e^T (\theta) Q_x e(\theta) d\theta ds \\
+ \int_{t-\tau}^t e^T (\theta) Q_e e(\theta) d\theta ds \\
+ \sum_{j=1}^n \frac{1}{\beta_j} (k_j - k_j^*)^2,
\]
where $Q_x = \epsilon^{-1} \tau L_2^2 I$, and computing $\mathcal{L}V(t, i, e, e)$ along the trajectory of error system (7), we have
\[
\mathcal{L}V(t, i, e, e) = V_t(t, i, e - D' e_e) \\
+ V_x(t, i, e - D' e_e) + \sum_{j=1}^n \frac{2}{\beta_j} (k_j - k_j^*) k_j \\
+ \sum_{j=1}^n \sum_{k=1}^n \gamma_{jk} V(t, k, e - D' e_e) \\
\leq \frac{1}{2} \text{trace} \left[ \sigma^T(t, i, e, e) V_{xx}(t, i, e - D' e_e) \sigma(t, i, e, e) \right] \\
+ \sum_{k=1}^n \gamma_{kk} V(t, k, e - D' e_e). 
\]

While
\[
V_t(t, i, e - D' e_e) \\
= g^T(e(t)) Q_i g(e(t)) - g^T(e(t - \tau)) Q_i g(e(t - \tau)) \\
+ \tau e^T(t) Q_e e(t) - \int_{t-\tau}^t e^T(s) Q_e e(s) ds \\
+ \sum_{j=1}^n \frac{2}{\beta_j} (k_j - k_j^*) k_j \\
= g^T(e(t)) Q_i g(e(t)) - g^T(e_e) Q_i g(e_e) \\
+ \tau e^T(t) Q_e e(t) - \int_{t-\tau}^t e^T(s) Q_e e(s) ds \\
- \sum_{j=1}^n \frac{2}{\beta_j} (k_j + k_j^*) p_j^* (e - D' e_e)^2
\]
so
\[
\mathcal{L}V(t, i, e, e) \\
\leq g^T(e) Q_1 g(e) - g^T(e_e) Q_1 g(e_e) + \tau e^T Q_2 e \\
- \int_{t-\tau}^t e^T(s) Q_e e(s) ds - \sum_{j=1}^n (k_j - k_j^*) p_j^* (e - D' e_e)^2
\]
\[
\times p_j^* (e - D' e_e)^2 + 2(e - D' e_e)^T \\
\times p^* \left[ -C^T e + A^T g(e) + B^T g(e_e) \right] \\
+ E^T \int_{t-\tau}^t g(e(s)) ds \\
+ \text{trace} \left[ \sigma^T(t, i, e, e) P^* \sigma(t, i, e, e) \right] \\
+ \sum_{k=1}^n \gamma_{kk} (e - D' e_e)^T P^* (e - D' e_e). 
\]

It is easy to get that
\[
2(e - D' e_e)^T P^* \text{diag} \{k_1, \ldots, k_n\} (e - D' e_e) \\
= 2 \sum_{j=1}^n k_j p_j^* (e - D' e_e)^2. 
\]

By (44), we have
\[
2(e - D' e_e)^T P^* \text{diag} \{k_1, \ldots, k_n\} (e - D' e_e) \\
- 2 \sum_{j=1}^n (k_j - k_j^*) p_j^* (e - D' e_e)^2 \\
= 2 \sum_{j=1}^n k_j p_j^* (e - D' e_e)^2 \\
- 2(e - D' e_e)^T P^* K^* (e - D' e_e). 
\]

According to Assumption 1 and Lemma 5, we have that
\[
g^T(e) Q_1 g(e) \leq \lambda_{\max}(Q_1) g^T(e) g(e) \leq \rho_1 L_2^2 |e|^2, \\
- g^T(e_e) Q_1 g(e_e) \leq -\lambda_{\min}(Q_1) g^T(e_e) g(e_e) \leq -\rho_2 L_2^2 |e_e|^2, \\
2e^T D^T P^T C^T e \leq e^T e^T C^T P^T P^T C e, \\
2(e - D' e_e)^T P^* A^T g(e) \\
\leq \epsilon_1 (e - D' e_e)^T P^* A^T P^* (e - D' e_e) + \epsilon_2^{-1} g^T(e) g(e) \\
\leq \epsilon_1 (e - D' e_e)^T P^* A^T P^* (e - D' e_e) + \epsilon_2^{-1} L_2^2 e e,
\[ 2(e - D'\epsilon_t)^T P B g(\epsilon_t) \]
\[ \leq e_3(e - D'\epsilon_t)^T P^T B' B^T P^i (e - D'\epsilon_t) + e_3^{-1} g^T (e) g(\epsilon_t) \]
\[ \leq e_3(e - D'\epsilon_t)^T P^T B' B^T P^i (e - D'\epsilon_t) + e_3^{-1} L_2^2 e_t e_t, \]
\[ 2(e - D'\epsilon_t)^T P B \int_{t-\tau}^{t} g(e(s)) ds \]
\[ \leq e_4(e - D'\epsilon_t)^T P^T P E^T E^T P^i (e - D'\epsilon_t) \]
\[ + e_4^{-1} \left( \int_{t-\tau}^{t} g(e(s)) ds \right)^T \left( \int_{t-\tau}^{t} g(e(s)) ds \right) \]
\[ \leq e_4(e - D'\epsilon_t)^T P^T P E^T E^T P^i (e - D'\epsilon_t) \]
\[ + e_4^{-1} \int_{t-\tau}^{t} g(e(s)) g(e(s)) ds \]
\[ \leq e_4(e - D'\epsilon_t)^T P^T P E^T E^T P^i (e - D'\epsilon_t) \]
\[ + e_4^{-1} \tau L_2^2 \int_{t-\tau}^{t} e_t e(s) e(s) ds, \]
\[ \text{trace}(\sigma^T (t, i, e, \epsilon_t) P^T \sigma(t, i, e, \epsilon_t)) \]
\[ \leq \rho \left( H_1 e^T e + H_2 e_t e_t \right). \]
\[ (46) \]

Substituting (45)-(46) into (43) yields
\[ \dot{V}(t, i, e, \epsilon_t) \]
\[ \leq e^T \left[ \rho_1 L_2^2 I - 2P^i C^i + e_1^{-1} C^T T P^i P^i \right] e \]
\[ + e_4^{-1} L_2^2 I + \rho H_1 I + e_4^{-1} e_t^T e_t I \]
\[ + e_t^T \left[ -\rho_2 L_3^2 I + e_1 D^T D^i + e_3^{-1} L_3^2 I \right] e_t \]
\[ + \rho H_2 I \] \[ (47) \]
\[ + \rho \left( H_1 e^T e + H_2 e_t e_t \right). \]

Therefore,
\[ \dot{V}(t, i, e, \epsilon_t) \]
\[ \leq -W_1(e) + W_2(e_t) - W_3(e - D'\epsilon_t), \]
\[ (48) \]

where
\[ W_1(e) = e^T W_1 e, \]
\[ W_2(e_t) = e_t^T W_2 e_t, \]
\[ W_3(e - D'\epsilon_t) = (e - D'\epsilon_t)^T W_3 (e - D'\epsilon_t), \]
\[ (49) \]

Thus, the conditions (C1), (C2), and (C3) in Theorem 11 are all satisfied. So by Theorem II, the error system (7) is almost surely asymptotically stable. And hence the drive system (3) and the response system (5) are adaptive synchronized a.s. The proof of Theorem 14 is completed.

Remark 15. In this section, a numerical example will be given to support the main results obtained in this paper.

4. Numerical Examples

In this section, a numerical example will be given to support the main results obtained in this paper.

Letting \( \Gamma = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \), which means \( N = 2 \), we give the parameters concerning the drive system (3), the response system (5), and the error system (7) as follows:

\[
D(1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix},
\]
\[
C(1) = \begin{bmatrix} 6 & 1 \\ 1 & 7 \end{bmatrix}, \quad C(2) = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix},
\]
\[
A(1) = \begin{bmatrix} -4 & 2 \\ -6 & 2 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -3 & 2 \\ -3 & 1 \end{bmatrix},
\]
\[
B(1) = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad B(2) = \begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix},
\]
\[
E(1) = \begin{bmatrix} -5 & 2 \\ 2 & -3 \end{bmatrix}, \quad E(2) = \begin{bmatrix} -4 & -2 \\ -2 & -3 \end{bmatrix},
\]
\[
J(1) = [1, 0]^T, \quad J(2) = [-1, 1]^T.
\]

We further set \( \tau = 1, f(\cdot) = \tanh(\cdot), \sigma(\cdot) = e(t) + \epsilon_t(t) \). Then we can confirm that Assumptions 1–3 are satisfied with \( L_1 = 0, \)
\( L_2 = 1, H_1 = H_2 = 2, \) and \( k_1 = k_2 = \kappa = 0.3. \)
Letting $K^* = \begin{bmatrix} 6 & 0 \\ 0 & 8 \end{bmatrix}$ and using LMI toolbox in Matlab, we solve matrix inequalities (33)-(38) and obtain the following results:

$$Q_1 = \begin{bmatrix} 0.3196 & 0 \\ 0 & 0.3196 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.3899 & 0 \\ 0 & 0.3899 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.4690 & 0 \\ 0 & 0.4690 \end{bmatrix}, \quad \rho = 0.5077,$$

$$\rho_1 = 0.4498, \quad \rho_2 = 0.1078, \quad \epsilon_1 = 9.9260,$$

$$\epsilon_2 = 178.9801, \quad \epsilon_3 = 194.4959, \quad \epsilon_4 = 226.2334.$$  

(54)

So from Theorem 14, the drive system (3) and the response system (5) are adaptive synchronized a.s. when the error system (7) has a unique solution.

To illustrate the effectiveness of the result in this paper, we depict the evolution figures of the systems as Figures 1, 2, 3, and 4. Figure 1 shows the two-state Markov chain in the systems. Figure 2 shows that the drive system (3) synchronizes the response system (5) from the moment of $t = 7$. It can be seen from Figure 3 that the state of the error system (7) tends to zero from $t = 7$, which also describes the synchronization of the drive system (3) and the response system (5). The update law of the adaptive control gain $K(t)$ is depicted in Figure 4. Figure 4 shows us that the update law of the control gain $K(t)$ no longer varies after the response system (5) synchronizes with the drive system (3).

5. Conclusions

In this paper, we have proposed a new criterion of a.s. asymptotic stability for a general neutral-type stochastic differential equation which extends the existing results. Based upon this new stability criterion, we have obtained a condition of a.s. asymptotic adaptive synchronization for neutral-type neural networks with stochastic perturbation and Markovian switching by making use of Lyapunov functional method and designing an adaptive controller. The synchronization condition is expressed as linear matrix inequality which can be easily solved by Matlab. Finally, we have employed a numerical example to illustrate the effectiveness of the method and result obtained in this paper. In the future, we will consider the condition of a.s. asymptotic adaptive synchronization for neutral-type neural networks with time-varying delay by making use of M-matrix method.
Appendix

Proof. The proof of (R1) is the same as [16] and is omitted here. To prove (R2), we will divide it into five steps. We change \( \tilde{D} \) into \( D \) in subsequence for simplicity.

Step 1. We prove that the solution \( x(t, i_0, \xi) \) of the system obeys

\[
\limsup_{t \to \infty} V(t, r(t), x(t) - D(r(t), x(t - \tau))) < \infty \quad \text{a.s.} \tag{A.1}
\]

In fact, let

\[
M(t) = \int_0^t V_x(s, r(s), x(s) - D(x(s - \tau), r(s))) dB(s)
\]

+ \[ \int_0^t \int_R \left( V \left( s, i_0 + \tilde{h}(r(s), x(s) - D(x(s - \tau), r(s))) \right) - D(x(s - \tau), r(s)) \right) \mu(ds, dl) \]

which is a continuous local martingale with \( M(0) = 0 \), a.s. By generalized Ito formula (Lemma 6), we have

\[
V(t, i, x(t) - D(i, x(t - \tau)))
\]

\[
\leq V(0, i_0, x(0) - D(i, x(-\tau)))
\]

\[
+ \int_0^t \mathcal{L}V(s, r(s), x(s), x(s) - D(r(s), x(s - \tau))) \, ds 
\]

+ \( M(t) \)

\[
\leq V(0, i_0, x(0) - D(i, x(-\tau)))
\]

+ \[ \int_0^t (\gamma(s) - W_1(x(s)) + W_2(x(s - \tau) - D(x(s - \tau), r(s)))) \, ds 
\]

\[
= V(0, i_0, x(0) - D(i, x(\xi - \tau)))
\]

+ \[ \int_0^t \gamma(s) \, ds - \int_0^t W_1(x(s)) \, ds + \int_{-\tau}^t W_2(x(s)) \, ds 
\]

\[
- \int_0^t W_3(x(s) - D(r(s), x(s - \tau))) \, ds + M(t) \]

\[
\leq V(0, i_0, x(0) - D(i, x(\xi - \tau))) + \int_0^t W_2(x(s)) \, ds 
\]

\[
+ \int_0^t \gamma(s) \, ds - \int_0^t W_1(x(s)) \, ds + \int_{-\tau}^t W_2(x(s)) \, ds 
\]

\[
- \int_0^t W_3(x(s) - D(r(s), x(s - \tau))) \, ds + M(t) \]  

\[
= V(0, i_0, x(0) - D(i, x(t - \tau)))
\]

+ \[ \int_0^t \gamma(s) \, ds - \int_0^t W_1(x(s)) \, ds + \int_{-\tau}^t W_2(x(s)) \, ds 
\]

\[
- \int_0^t W_3(x(s) - D(r(s), x(s - \tau))) \, ds + M(t) \]  

\[
\leq V(0, i_0, x(0) - D(i, x(t - \tau))) + \int_0^t W_2(x(s)) \, ds 
\]

\[
+ \int_0^t \gamma(s) \, ds - \int_0^t W_1(x(s)) \, ds + \int_{-\tau}^t W_2(x(s)) \, ds 
\]

\[
- \int_0^t W_3(x(s) - D(r(s), x(s - \tau))) \, ds + M(t) \]  

\[
\leq V(0, i_0, x(0) - D(i, x(t - \tau))) + \int_0^t W_2(x(s)) \, ds 
\]

\[
+ \int_0^t \gamma(s) \, ds - \int_0^t W_1(x(s)) \, ds + \int_{-\tau}^t W_2(x(s)) \, ds 
\]

\[
- \int_0^t W_3(x(s) - D(r(s), x(s - \tau))) \, ds + M(t) \]  

By the convergence theorem of nonnegative semimartingales (Lemma 7), we have (A.1).

Step 2. We prove

\[
\sup_{0 \leq t < \infty} |x(t)| < \infty \quad \text{a.s.} \tag{A.4}
\]

Indeed, from (A.1), we have

\[
\sup_{0 \leq t < \infty} V(t, r(t), x(t) - D(r(t), x(t - \tau))) < \infty \quad \text{a.s.} \tag{A.5}
\]

which together with (C1) yields

\[
\sup_{0 \leq t < \infty} |x(t) - D(r(t), x(t - \tau))| < \infty \quad \text{a.s.} \tag{A.6}
\]

Now, for any \( T > 0 \), by (H2), we have that if \( 0 \leq t \leq T \), then

\[
|x(t)|
\]

\[
\leq |D(r(t), x(t - \tau))| + |x(t) - D(r(t), x(t - \tau))|
\]

\[
\leq \kappa |x(t - \tau)| + |x(t) - D(r(t), x(t - \tau))|. \tag{A.7}
\]
where $\kappa = \max_{i \in S} \kappa_i < 1$. This implies
\[
\sup_{0 \leq t \leq T} |x(t)| \leq \kappa \sup_{0 \leq t \leq T} |x(t) - D(r(t), x(t - \tau))| + \kappa \beta \sup_{0 \leq t \leq T} |x(t)| + \kappa \beta < 1.
\]
This implies
\[
\sup_{0 \leq t \leq T} |x(t)| \leq \kappa \sup_{0 \leq t \leq T} |x(t - \tau)| + \kappa \beta \sup_{0 \leq t \leq T} |x(t)| + \kappa \beta < 1.
\]
where $\beta$ is the bound for the initial data $\xi$. Hence
\[
\sup_{0 \leq t \leq T} |x(t)| \leq \frac{1}{1 - \kappa} \left( \kappa \beta + \kappa \sup_{0 \leq t \leq T} |x(t)| + \kappa \beta \right).
\]
Letting $T \to \infty$ and using (A.6), we obtain (A.4).

Step 3. We prove
\[
\lim_{t \to \infty} W_3(x(t)) = 0 \quad \text{a.s.}
\]
In fact, taking the expectations on both sides of (A.3) and letting $t \to \infty$, we obtain that
\[
\mathbb{E} \int_0^\infty W(s) ds < \infty, \quad (A.11)
\]
where $W(s) = W_1(x(s)) - W_2(x(s)) + W_3(z(s))$, $z(s) = x(s) - D(r(s), x(s - \tau))$.

This implies
\[
\int_0^\infty W(s) ds < \infty \quad \text{a.s.} \quad (A.12)
\]
or equivalently
\[
\int_0^\infty \left( W_1(x(s)) - W_2(x(s)) \right) ds < \infty \quad \text{a.s.}
\]
\[
\int_0^\infty W_3(z(s)) ds < \infty \quad \text{a.s.} \quad (A.13)
\]
From (A.12), we have
\[
\lim_{t \to \infty} \inf W(t) = 0 \quad \text{a.s.} 
\]
or equivalently
\[
\lim_{t \to \infty} \inf (W_1(x(t)) - W_2(x(t))) = 0 \quad \text{a.s.}
\]
\[
\lim_{t \to \infty} \inf W_3(z(t)) = 0 \quad \text{a.s.} \quad (A.15)
\]
Now we will prove (A.10): $\lim_{t \to \infty} W_3(z(t)) = 0$ a.s. In fact, if (A.10) is false, then
\[
\mathbb{P} \left( \lim_{t \to \infty} \sup_{0 \leq t \leq T} W_3(z(t)) > 0 \right) > 0.
\]
Hence there is a number $\varepsilon > 0$ such that
\[
\mathbb{P} \left( \Omega_1 \right) \geq 3 \varepsilon
\]
where $\Omega_1 = \{ \limsup_{T \to \infty} W_3(z(t)) > 2\varepsilon \}$.

Recalling (A.4), as well as the boundedness of the initial data $\xi$, we can find a positive number $h$, which depends on $\varepsilon$, sufficiently large for
\[
\mathbb{P} \left( \Omega_2 \right) \geq 1 - \varepsilon,
\]
where $\Omega_2 = \{ \sup_{0 \leq t \leq \infty} |z(t)| < h \}$.

It is easy to see from (A.17) and (A.18) that
\[
\mathbb{P} \left( \Omega_1 \cap \Omega_2 \right) \geq 2 \varepsilon.
\]

We now define a sequence of stopping times as follows:
\[
\tau_h = \inf \{ t \geq 0 : |x(t)| \wedge |z(t)| \geq h \}
\]
\[
\sigma_1 = \inf \{ t \geq 0 : W_3(z(t)) \geq 2 \varepsilon \}
\]
\[
\sigma_{2k} = \{ t \geq \sigma_{2k-1} : W_3(z(t)) < \varepsilon \}, \quad k = 1, 2, \ldots
\]
\[
\sigma_{2k+1} = \{ t \geq \sigma_{2k} : W_3(z(t)) \geq 2 \varepsilon \}, \quad k = 1, 2, \ldots
\]
where, throughout this paper, we set $\inf \emptyset = \infty$.

From (A.14) and the definition of $\Omega_1$ and $\Omega_2$, we observe that if $\omega \in \Omega_1 \cap \Omega_2$, then
\[
\tau_h = \infty, \quad \sigma_k < \infty \quad \forall k \geq 1.
\]

Let $I_\omega$ denote the indication function of set $\omega$. Noting the fact that $\sigma_{2k} < \infty$, whenever $\sigma_{2k-1} < \infty$, we can derive from (A.11) that
\[
\mathbb{E} \int_0^\infty W_3(z(t)) dt \geq \varepsilon \mathbb{E} \int_0^\infty I_{\sigma_{2k-1} < \infty, \sigma_{2k} = \infty} \int_{\sigma_{2k-1}}^{\sigma_{2k}} W_3(z(t)) dt
\]
\[
\geq \varepsilon \mathbb{E} \left[ \int_{\sigma_{2k-1} < \infty, \tau_h = \infty} (\sigma_{2k} - \sigma_{2k-1}) \right].
\]

On the other hand, by (H1), there exists a constant $K_h > 0$, such that
\[
|f(t, i, x, y)|^2 \vee |g(t, i, x, y)|^2 \leq K_h^2
\]
whenever $|x| \vee |y| < h$ and $(t, i) \in R_h \times S$.

By the H"older inequality (Lemma 8) and the Doob martingale inequality (Lemma 9), we compute that, for any $T > 0$ and $k = 1, 2, \ldots$
\[
\mathbb{E} \left[ I_{\tau_h < \sigma_{2k-1} < \sigma_{2k+1}} \sup_{0 \leq t \leq T} |z(t \wedge \sigma_{2k} + t)| \right. 
\]
\[
\quad \left. - z(t \wedge \sigma_{2k-1}) \right|^2 \leq 2K_h^2 T (T + 4).
\]
Since $W_3(\cdot)$ is continuous in $\mathbb{R}^n$, there exists a closed ball $S_h = \{x \in \mathbb{R}^n : |x| < h\}$ such that $W_3(\cdot)$ is uniformly continuous in $S_h$. We can therefore choose $\delta = \delta(\epsilon) > 0$ so small such that

$$|W_3(x) - W_3(y)| < \frac{\epsilon}{2} \quad \text{whenever } x, y \in S_h, |x - y| < \delta. \quad (A.25)$$

We furthermore choose $T = T(\epsilon, \delta, h) > 0$ sufficiently small for

$$\frac{2K^2T(T + 4)}{\delta^2} < \epsilon. \quad (A.26)$$

It then follows from (A.24) and Chebyshev’s inequality (Lemma 10) that

$$\mathbb{P}\left(\sigma_{2k-1} \land \tau_h < \infty \right) \cap \left\{ \sup_{0 \leq t \leq T} |z(\tau_h \land (\sigma_{2k-1} + t)) - z(\sigma_{2k-1})| \geq \delta \right\} \leq \frac{1}{\delta^2} \left(2K^2T(T + 4)\right) < \epsilon. \quad (A.27)$$

Note that

$$\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \subseteq \{\tau_h \land \sigma_{2k-1} < \infty, \tau < \infty\} \subseteq \{\tau_h \land \sigma_{2k-1} < \infty\}. \quad (A.28)$$

We hence have

$$\mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \left\{ \sup_{0 \leq t \leq T} |z(\sigma_{2k-1} + t) - z(\sigma_{2k-1})| \geq \delta \right\} \right) < \epsilon. \quad (A.29)$$

By (A.19) and (A.21), we further compute

$$\mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \right) \cap \left\{ \sup_{0 \leq t \leq T} |z(\sigma_{2k-1} + t) - z(\sigma_{2k-1})| < \delta \right\} \geq 2\epsilon - \epsilon = \epsilon. \quad (A.30)$$

By (A.25), we hence obtain that

$$\mathbb{P}\left(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \left\{ \sup_{0 \leq t \leq T} |W_3(z(\sigma_{2k-1} + t)) - W_3(z(\sigma_{2k-1}))| \right\} < \epsilon \right) > \epsilon. \quad (A.31)$$

Set

$$\Omega_k = \left\{ \sup_{0 \leq t \leq T} |W_3(z(\sigma_{2k-1} + t)) - W_3(z(\sigma_{2k-1}))| < \epsilon \right\}. \quad (A.32)$$

Note that

$$|\sigma_{2k}(w) - \sigma_{2k-1}(w)| \geq T \quad \text{if } w \in \{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Omega_k. \quad (A.33)$$

We derive from (A.22) and (A.31) that

$$\lim_{t \to \infty} d(x(t; \xi, i_0) - D(x(t-\tau; \xi, i_0), r(t)), \text{Ker}(W_3)) = 0 \quad \text{a.s.} \quad (A.35)$$

By (A.10) and (A.6), we see that there is an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{t \to \infty} W_3(z(t, w)) = 0, \quad \sup_{0 \leq \tau < \infty} |z(t, w)| < \infty, \forall w \in \Omega_0. \quad (A.36)$$

Choose any $w \in \Omega_0$. Then $\{z(t, w)\}_{t \geq 0}$ is bounded in $\mathbb{R}^n$ so there must be an increasing sequence $\{t_k\}_{k \geq 1}$ such that $t_k \to \infty$ and $\{z(t_k, w)\}_{k \geq 1}$ converges to some $\bar{z} \in \mathbb{R}^n$. Thus

$$W_3(\bar{z}) = \lim_{k \to \infty} W_3(z(t_k, w)) = 0 \quad (A.37)$$

which implies that $\bar{z} \in \text{Ker}(W_3)$ whence $\text{Ker}(W_3) \neq \emptyset$. From this, we can show that

$$\lim_{t \to \infty} d(z(t, w), \text{Ker}(W_3)) = 0 \quad \forall w \in \Omega_0. \quad (A.38)$$
If this is false, then there is some $\bar{w} \in \Omega$ such that
\[
\limsup_{t \to \infty} d(z(t, \bar{w}), \text{Ker}(W_3)) > 0.
\] (A.39)

Hence there is a subsequence $\{z(t_k, \bar{w})\}_{k \geq 0}$ of $\{z(t, \bar{w})\}_{t \geq 0}$ such that
\[
\lim_{k \to \infty} d(z(t_k, \bar{w}), \text{Ker}(W_3)) > \bar{\varepsilon}
\] (A.40)

for some $\bar{\varepsilon} > 0$. Since $\{z(t_k, \bar{w})\}_{k \geq 0}$ is bounded, we can find its subsequence $\{z(\bar{t}_k, \bar{w})\}_{k \geq 0}$ which converges to $\bar{z} \in \mathbb{R}^n$. Clearly, $\bar{z} \notin \text{Ker}(W_3)$ so $W_3(\bar{z}) > 0$. But, by (A.36),
\[
W_3(\bar{z}) = \lim_{k \to \infty} W_3(z(\bar{t}_k, \bar{w})) = 0
\] (A.41)
a contradiction. Hence (A.38) must hold and (A.35) holds yet.

\textbf{Step 5.} We prove (R2).

Under the assumption that
\[
W_3(x) = 0 \quad \text{iff} \quad x = 0,
\] (A.42)

we have $\text{Ker}(W_3) = \{0\}$. It then follows from (A.35) that
\[
\lim_{t \to 0} [x(t) - D(x(t-\tau), r(t))] = \lim_{t \to 0} z(t) = 0 \quad \text{a.s.}
\] (A.43)

But, by (H2),
\[
|x(t)| \\
\leq |D(x(t-\tau), r(t))| + |x(t) - D(x(t-\tau), r(t))| \\
\leq \kappa |x(t-\tau)| + |x(t) - D(x(t-\tau), r(t))|,
\] (A.44)

where $\kappa \in (0,1)$ has been defined above. Letting $t \to \infty$, we obtain that
\[
\limsup_{t \to \infty} |x(t)| \leq \kappa \limsup_{t \to \infty} |x(t)| \quad \text{a.s.}
\] (A.45)

This together with (A.4) yields
\[
\lim_{t \to \infty} |x(t)| = 0 \quad \text{a.s.}
\] (A.46)

which is (32) and the proof is therefore completed. \(\Box\)

\textbf{Conflict of Interests}

The authors declare that there is no conflict of interests regarding the publication of this paper.

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