Research Article

Robust H_∞ Control for Linear Stochastic Partial Differential Systems with Time Delay

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This paper investigates the problems of robust stochastic mean square exponential stabilization and robust H_∞ for stochastic partial differential time delay systems. Sufficient conditions for the existence of state feedback controllers are proposed, which ensure mean square exponential stability of the resulting closed-loop system and reduce the effect of the disturbance input on the controlled output to a prescribed level of H_∞ performance. A linear matrix inequality approach is employed to design the desired state feedback controllers. An illustrative example is provided to show the usefulness of the proposed technique.

1. Introduction

The H_∞ control, since it was first formulated by [1], has been extensively studied in the past years, and a great number of results on this subject have been reported in the literature; see, for example, [2–7] and the references therein. Recently, a great deal of attention has been paid regarding the study of partial differential systems (PDSs) [8–15]. Many phenomena in science and engineering have been modeled by deterministic partial differential systems, such as the control for elastic oscillating systems, the control for temperature field [13], the control for nuclear reactor, the robot with flexible connecting rod [15], population dynamics [16], neurophysiology, and biodynamics [17]. On the other hand, since most of the phenomena have spatiotemporal uncertainties due to the existence of different stochastic fluctuations, for a more accurate representation of the behavior, a stochastic partial differential system is an ideal model [16–21]. The control problem for SPDSs has been widely studied, including stability [16, 20, 21], stabilization [22], boundary and point adaptive control [23], optimal control [24–26], and parameter estimation [27]. However, it should be noted that up to now, there is little corresponding work on the robust H_∞ control for SPDSs.

In this paper, we focus on the robust H_∞ control problem of linear SPDSs with time delay. For robust H_∞ control of deterministic PDSs, the main research method is operator semigroup (see [8, 10, 12]), which is associated with solving operator equation or linear operator inequality [14]. However, general methods for solving linear operator inequality have not been developed yet, which makes most existing results difficult to be applied in practice. Later, the linear matrix inequality (LMI) is extended to uncertain distributed parameter systems [28, 29], and are used in stability analysis and H_∞ control [13, 14], respectively. Very recently, [17] studied robust filter for SPDSs by using LMI, which presented an explicit expression for the robust H_∞ filter. Motivated by these facts, our main purpose in this paper is to examine stochastic exponential stabilization and robust H_∞ control for linear SPDSs with time delay under Dirichlet boundary and Robin boundary conditions, respectively. The time delay is assumed to be unknown but bounded. First, we consider the problem of stochastic exponential stabilization for which a state feedback controller is designed such that the resulting closed-loop system is mean-square exponentially stable. Then, the problem of robust H_∞ control is addressed for which a state feedback controller is designed, for which not only is the resulting closed-loop system mean-square exponentially stable, but also is a prescribed H_∞ performance

level satisfied. In terms of LMIs, sufficient conditions for the solvability of the above problems are obtained and explicit expressions of the desired state feedback controllers are presented. Here, the main method is constructing Lyapunov functional and using linear matrix inequality. Finally, an example is given to demonstrate the applicability and effectiveness of the developed theoretic results.

For convenience, we adopt the following basic notations in this paper. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n -dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript “ T ” denotes the transpose and the notation $X \geq Y$ (resp., $X > Y$) where X and Y are symmetric matrices, meaning that $X - Y$ is positive semidefinite (resp., positive definite). $\lambda_{\max}(P)$, $\lambda_{\min}(P)$ denote, respectively, maximum and minimum eigenvalue of a real symmetric matrix P . $L^2(\mathcal{O})$ is Lebergue square integrable function space defined on $\mathcal{O}$. For a scalar real value function $h(x) \in L^2(\mathcal{O})$, its $L^2(\mathcal{O})$ norm $\|h\|^2 = \int_{\mathcal{O}} h^2(x) dx$; if $h(x)$ is a vector, that is, $h(x) = (h_1(x), h_2(x), \dots, h_n(x))^T$, then $\|h\|^2 = \int_{\mathcal{O}} h^T(x)h(x) dx$. $\mathcal{L}_2(\mathcal{O} \times [0, \infty); \mathbb{R}^{n_f})$ denotes the family of measurable function $f(x, t) : \mathcal{O} \times [0, T] \rightarrow \mathbb{R}^{n_f}$ such that $\mathbb{E} \int_0^t \int_{\mathcal{O}} f(x, s)^T f(x, s) dx dt := \int_0^t \int_{\mathcal{O}} f(x, s, \omega)^T f(x, s, \omega) dx dt P d\omega < \infty$, where $\mathbb{E}$ represents the mathematical expectation and $\{f(x, t), t \in [0, T]\}$ is stochastic process at the space location $x \in \mathcal{O}$ and a function of three arguments, that is, $f(x, t, \omega)$, $x \in \mathcal{O}$, $t \in [0, T]$, $\omega \in \Omega$.

2. Problem Statement and Preliminaries

Consider the following linear stochastic partial differential system with time delay:

$$\begin{aligned} dy(x, t) = & [D\Delta y(x, t) + Ay(x, t) + A_1y(x, t - \tau) \\ & + Bu(x, t) + B_vv(x, t)] dt \\ & + [Cy(x, t) + C_1y(x, t - \tau) + C_vv(x, t)] dW \\ & \times(x, t), \\ z(x, t) = & Ly(x, t), \end{aligned} \quad (1)$$

where $(x, t) \in \mathcal{O} \times \mathbb{R}^+$ and $\mathcal{O} = \{x, 0 \leq |x| \leq l < +\infty\} \subset \mathbb{R}^m$ is the bounded domain with smooth boundary $\partial\mathcal{O}$. $D > 0$ is constant. The symbol $\Delta = \sum_{i=1}^m (\partial^2 / \partial x_i^2)$ is Laplace operator defined on $\mathcal{O}$, $y(x, t) = [y_1(x, t), y_2(x, t), \dots, y_n(x, t)]^T$ is the systems state variable, and x and t are the space and time variables, respectively. $u(x, t) \in \mathcal{L}_2(\mathcal{O} \times [0, \infty); \mathbb{R}^{n_u})$ is admissible control and $z(x, t) \in \mathcal{L}_2(\mathcal{O} \times [0, \infty); \mathbb{R}^{n_z})$ is measured output. $v(x, t) \in \mathcal{L}_2(\mathcal{O} \times [0, \infty); \mathbb{R}^{n_v})$ is the vector of the random external disturbance and $B_u, C_v \in \mathbb{R}^{n \times n_v}$ are the disturbance influence matrix. A, A_1, C, C_1, L are known real constant matrix of appropriate dimension. The scalar $\tau > 0$ is an unknown but bounded time delay of the system. $W(x, t)$ is Wiener random field (see [18]) with covariance operator $\mathcal{R}$ in $L^2(\mathcal{O})$, that is, $W(x, t) = \sum_{i=1}^{\infty} \sqrt{\mu_i} w_i(t) e_i(x)$, where $\{w_i(t)\}$ is a sequence of independent, identically

distributed standard Brownian motions defining a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and the set $\{e_i(x)\}$ is a complete orthonormal basis on $L^2(\mathcal{O})$. Then $\mathbb{E}W(x, t) = 0$, $\mathbb{E}[W(x_1, s)W(x_2, t)] = r(x_1, x_2)(t \wedge s)$, where $r(x_1, x_2)$ is symmetric kernel of operator $\mathcal{R}$. For a continuous adapted random process $\sigma(x, t)$, $\mathbb{E}[\int_0^t \int_{\mathcal{O}} \sigma(x, s) dW_s dx]^2 = \mathbb{E} \int_0^t \int_{\mathcal{O}} r(x, x) \sigma(x, s)^2 dx dt$. In this paper, we assume covariance function $r(x_1, x_2)$ is bounded, that is, $r(x_1, x_2) \leq r_0$, $\forall x_1, x_2 \in \mathcal{O}$.

Initial value and boundary value conditions of (1) satisfied

$$y(x, s) = \phi(x, s), \quad s \in [-\tau, 0], \quad (2)$$

$$y(x, t) = 0, \quad (x, t) \in \partial\mathcal{O} \times [-h, \infty) \quad (3)$$

or

$$\frac{\partial y(x, t)}{\partial \nu} + Ny(x, t) = 0, \quad (x, t) \in \partial\mathcal{O} \times [-h, \infty), \quad (4)$$

where ν is the unit outward normal vector of $\partial\mathcal{O}$ and N is positive constant. $\phi(x, s)$ is continuous adapted random process and $\mathbb{E}\|\phi\|_C^2 = \mathbb{E}[\sup_{-\tau \leq s \leq 0} \|\phi(\cdot, s)\|^2] < \infty$.

Definition 1. The equilibrium point $y(0, t) \equiv 0$ of the system (1) is said to be mean-square exponentially stable with a decay rate $\delta > 0$ if there exist positive constants c such that $\mathbb{E}\|y(\cdot, t)\|^2 \leq ce^{-2\delta t} \mathbb{E}\|\phi\|_C^2, t \geq 0$.

In this paper, our aim is to develop techniques of robust stochastic stabilization and robust H_{∞} control for stochastic partial differential time delay systems (1). More specifically, we are concerned with the following two problems.

- (1) Stochastic exponential stabilization problem: design a state feedback controller

$$u(x, t) = Ky(x, t) \quad (5)$$

for systems (1) with initial boundary value condition (2) and (3) or (4) with $v(x, t) = 0$ such that the resulting closed-loop system is mean-square exponentially stable.

- (2) Robust H_{∞} control problem: given a constant scalar $\gamma > 0$, design a state feedback controller in the form of (5) such that the resulting closed-loop system is mean square exponentially stable, and for any nonzero $v(x, t) \in \mathcal{L}_2(\mathcal{O} \times [0, \infty); \mathbb{R}^{n_v})$, $\phi(x, s) = 0$, $s \in [-\tau, 0]$, and we have

$$\begin{aligned} & \mathbb{E} \int_0^{\infty} \int_{\mathcal{O}} z^T(x, t) z(x, t) dx dt \\ & \leq \gamma^2 \int_0^{\infty} \int_{\mathcal{O}} v^T(x, t) v(x, t) dx dt. \end{aligned} \quad (6)$$

We conclude this section by recalling the following lemmas which will be used in the proof of our main results.

Lemma 2 (Schur complement [29]). *Given constant matrices S_1, S_2, S_3 where $S_1 = S_1^T$, and $0 < S_2 = S_2^T$, then $S_1 + S_3^T S_2^{-1} S_3 < 0$ if and only if*

$$\begin{bmatrix} S_1 & S_3^T \\ S_3 & -S_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -S_2 & S_3 \\ S_3^T & S_1 \end{bmatrix} < 0. \quad (7)$$

3. Mean Square Exponential Stabilization

In this section, an LMI approach is developed to solve the problem of exponential stabilization formulated in the previous section. The main result is given in the following theorem.

Theorem 3. *Consider the stochastic time-delay partial differential system*

$$\begin{aligned} dy(x, t) &= [D\Delta y(x, t) + Ay(x, t) \\ &\quad + A_1 y(x, t - \tau) + Bu(x, t)] dt \\ &\quad + [Cy(x, t) + C_1 y(x, t - \tau)] dW(x, t), \end{aligned} \quad (8)$$

with (2) and (3) being its initial and boundary value conditions, respectively. Then system (8) is exponential stabilizable in mean square with decay rate δ if there exist matrices $Y \in \mathbb{R}^{n_u \times n}$ and $X > 0, Z > 0$, such that the following LMI holds:

$$\Gamma = \begin{bmatrix} \Gamma_{11} & A_1 Z & \sqrt{r_0} X C^T & X \\ * & \Gamma_{22} & \sqrt{r_0} Z C_1^T & 0 \\ * & * & -X & 0 \\ * & * & * & -Z \end{bmatrix} < 0, \quad (9)$$

where

$$\begin{aligned} \Gamma_{11} &= 2\delta X + AX + XA^T + BY + Y^T B^T, \\ \Gamma_{22} &= \Pi_{22} = -e^{-2\delta\tau} Z. \end{aligned} \quad (10)$$

In this case, a stabilizing state feedback controller can be chosen by $u(x, t) = YX^{-1}y(x, t)$.

Proof. Applying the controller in (5) to system (8), we obtain the closed-loop system as

$$\begin{aligned} dy(x, t) &= [D\Delta y(x, t) + Ay(x, t) + A_1 y(x, t - \tau) + BKy(x, t)] dt \\ &\quad + [Cy(x, t) + C_1 y(x, t - \tau)] dW(x, t). \end{aligned} \quad (11)$$

For given decay rate δ , choose a Lyapunov functional candidate for system (11) as

$$\begin{aligned} V(t, y) &= V_1 + V_2 \\ &= e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) P y(x, t) dx \\ &\quad + \int_{t-\tau}^t \int_{\mathcal{O}} e^{2\delta s} y^T(x, s) Q y(x, s) dx ds, \end{aligned} \quad (12)$$

where P, Q are a pair of positive symmetric matrices. Then, by Itô's formula, the stochastic differential $dV(t, y)$ along (11) can be obtained as (see, e.g., [16, 18, 20])

$$\begin{aligned} dV(t, y) &= LV(t, y) dt \\ &\quad + 2 \int_{\mathcal{O}} y^T(x, t) \\ &\quad \times [Cy(x, t) + C_1 y(x, t - \tau)] dW(x, t), \end{aligned} \quad (13)$$

where $LV(t, y) = LV_1(t, y) + LV_2(t, y)$.

We can deduce that

$$\begin{aligned} LV_1(t, y) &\leq 2\delta e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) P y(x, t) dx \\ &\quad + e^{2\delta t} \int_{\mathcal{O}} [D\Delta y(x, t) + Ay(x, t) + A_1 y(x, t_h) \\ &\quad + BKy(x, t)]^T P y(x, t) dx \\ &\quad + e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) P [D\Delta y(x, t) + Ay(x, t) \\ &\quad + A_1 y(x, t_h) \\ &\quad + BKy(x, t)] dx \\ &\quad + r_0 e^{2\delta t} \int_{\mathcal{O}} [Cy(x, t) + C_1 y(x, t - \tau)]^T \\ &\quad \times P [Cy(x, t) + C_1 y(x, t_h)] dx \\ &= 2e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) P y(x, t) dx \\ &\quad + e^{2\delta t} \int_{\mathcal{O}} [(\Delta y(x, t))^T \cdot DP y(x, t) \\ &\quad + y^T(x, t) P D\Delta y(x, t)] dx \\ &\quad + e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) [A^T P + PA + (BK)^T P + PBK \\ &\quad + r_0 C^T P C] y(x, t) dx \\ &\quad + e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) (A_1^T P + PA_1 \\ &\quad + r_0 C^T P C_1 + r_0 C_1^T P C) \\ &\quad \times y(x, t - \tau) dx \\ &\quad + r_0 e^{2\delta t} \int_{\mathcal{O}} y^T(x, t - \tau) C_1^T P C_1 y(x, t - \tau) dx, \end{aligned} \quad (14)$$

$$\begin{aligned} LV_2(t, y) &= e^{2\delta t} \int_{\mathcal{O}} y^T(x, t) Q y(x, t) dx \\ &\quad - e^{2\delta(t-\tau)} \int_{\mathcal{O}} y^T(x, t - \tau) Q y(x, t_h) dx \\ &= e^{2\delta t} \left(\int_{\mathcal{O}} y^T(x, t) Q y(x, t) dx \right. \\ &\quad \left. - e^{-2\delta\tau} \int_{\mathcal{O}} y^T(x, t - \tau) Q y(x, t - \tau) dx \right). \end{aligned} \quad (15)$$

Considering Dirichlet boundary condition (3) and using Green formula, we have

$$\begin{aligned} &\int_{\mathcal{O}} (\Delta y(x, t))^T DP y(x, t) dx \\ &= \int_{\partial\mathcal{O}} y(x, t)^T DP \frac{\partial y(x, t)}{\partial \nu} dx \\ &\quad - \int_{\mathcal{O}} (\nabla y(x, t))^T DP \nabla y(x, t) dx \\ &= -D \int_{\mathcal{O}} (\nabla y(x, t))^T P \nabla y(x, t) dx. \end{aligned} \quad (16)$$

Because $D > 0$ and P is positive definite matrix, then by (16) we have

$$e^{2\delta t} \int_{\mathcal{O}} \left[(\Delta y(x, t))^T D P y(x, t) + y^T(x, t) P D \Delta y(x, t) \right] dx \leq 0 \quad (17)$$

which together with (14)~(17) yields

$$LV(t, y) \leq e^{2\delta t} \int_{\mathcal{O}} \eta(x, t)^T \bar{\Gamma} \eta(x, t) dx, \quad (18)$$

where $\eta^T(x, t) = [y^T(x, t), y^T(x, t_h)]$,

$$\begin{aligned} \bar{\Gamma} &= \begin{bmatrix} \bar{\Gamma}_{11} & \bar{\Gamma}_{12} \\ * & \bar{\Gamma}_{22} \end{bmatrix}, \\ \bar{\Gamma}_{11} &= 2\delta P + PA + A^T P + PBK \\ &\quad + (BK)^T P + Q + r_0 C^T P C, \\ \bar{\Gamma}_{12} &= PA_1 + r_0 C^T P C_1, \\ \bar{\Gamma}_{22} &= -e^{-2\delta\tau} Q + r_0 C_1^T P C_1. \end{aligned} \quad (19)$$

So if $\bar{\Gamma} < 0$, then (18) implies $LV(t, y) < 0$. By Lemma 2, $\bar{\Gamma} < 0$ is equivalent to

$$\tilde{\Gamma} = \begin{bmatrix} \tilde{\Gamma}_{11} & PA_1 & \sqrt{r_0} C^T P \\ * & \tilde{\Gamma}_{22} & \sqrt{r_0} C_1^T P \\ * & * & -P \end{bmatrix} < 0, \quad (20)$$

where

$$\begin{aligned} \tilde{\Gamma}_{11} &= 2\delta P + PA + A^T P + PBK + (BK)^T P + Q, \\ \tilde{\Gamma}_{22} &= -e^{-2\delta\tau} Q. \end{aligned} \quad (21)$$

Then, pre- and postmultiplying the LMI in (20) by $\text{diag}\{P^{-1}, Q^{-1}, P^{-1}\}$ and let $X = P^{-1}$, $Z = Q^{-1}$, $Y = KX$, we have

$$\hat{\Gamma} = \begin{bmatrix} \hat{\Gamma}_{11} & A_1 Z & \sqrt{r_0} X C^T \\ * & \hat{\Gamma}_{22} & \sqrt{r_0} Z C_1^T \\ * & * & -X \end{bmatrix} < 0, \quad (22)$$

where

$$\begin{aligned} \hat{\Gamma}_{11} &= 2\delta X + AX + XA^T + BY + Y^T B^T + XZ^{-1}X, \\ \hat{\Gamma}_{22} &= \Pi_{22} = -e^{-2\delta\tau} Z. \end{aligned} \quad (23)$$

Therefore, by Lemma 2 again, if the matrix inequality in (9) holds, then the inequality (22) or, equivalently, (20) holds, which leads to $\bar{\Gamma} < 0$. Hence $LV(t, y) < 0$.

Now we prove system (8) to be mean square exponentially stable. Integrating both sides of (13) and taking expectation, we have

$$\begin{aligned} \mathbb{E}V(t, y(x, t)) &= \mathbb{E}V(0, y(x, 0)) \\ &\quad + \mathbb{E} \left\{ \int_0^t LV(s, y(x, s)) ds \right. \\ &\quad \left. + 2 \int_0^t \int_{\mathcal{O}} y(x, s)^T P \right. \\ &\quad \left. \times (Cy(x, s) + C_1 y(x, s_h)) dW_s dx \right\} \\ &\leq \mathbb{E}V(0, y(x, 0)) + \mathbb{E} \int_0^t LV(s, y(x, s)) ds \leq \mathbb{E}V(0, y_0), \end{aligned} \quad (24)$$

$$\begin{aligned} \mathbb{E}V(0, y(x, 0)) &= \mathbb{E} \int_{\mathcal{O}} y(x, 0)^T P y(x, 0) dx \\ &\quad + \mathbb{E} \int_{-\tau}^0 \int_{\mathcal{O}} e^{2\delta s} y(x, s)^T Q y(x, s) dx ds \\ &\leq \lambda_{\max}(P) \mathbb{E} \|\phi\|_C^2 + \lambda_{\max}(Q) \mathbb{E} \|\phi\|_C^2 \int_{-\tau}^0 e^{2\delta s} ds \leq c_1 \mathbb{E} \|\phi\|_C^2, \end{aligned} \quad (25)$$

where constant $c_1 = [\lambda_{\max}(P) + \lambda_{\max}(Q)((1 - e^{-2\delta\tau})/2\delta)]$.
On the other hand, by (12), we have

$$V(t, y(x, t)) \geq e^{2\delta t} \int_{\mathcal{O}} y(x, t)^T P y(x, t) dx, \quad (26)$$

and, hence, (24) and (25) yield

$$\begin{aligned} \lambda_{\min}(P) e^{2\delta t} \mathbb{E} \int_{\mathcal{O}} y^T(x, t) y(x, t) dx &\leq e^{2\delta t} \mathbb{E} \int_{\mathcal{O}} y^T(x, t) P y(x, t) dx \leq \mathbb{E}V(t, y(x, t)) \\ &\leq \mathbb{E}V(0, y_0). \end{aligned} \quad (27)$$

Then from (25) and (27), we have

$$\mathbb{E} \|y(\cdot, t)\|^2 = \mathbb{E} \int_{\mathcal{O}} y(x, t)^T y(x, t) dx \leq c \mathbb{E} \|\phi\|_C^2 e^{-2\delta t}, \quad (28)$$

where $c = c_1/\lambda_{\min}(P)$ therefore, by Definition 1, system (8) is mean square exponentially stable with decay rate δ . The proof of Theorem 3 is complete. $\square$

If the boundary value condition of system (8) is replaced by Robin boundary condition (4), then

$$\begin{aligned}
& \int_{\mathcal{O}} (\Delta y(x, t))^T DP y(x, t) dx \\
&= \int_{\partial \mathcal{O}} (y(x, t))^T DP \frac{\partial y(x, t)}{\partial \nu} dx \\
&\quad - \int_{\mathcal{O}} (\nabla y(x, t))^T DP \nabla y(x, t) dx \quad (29) \\
&\leq \int_{\partial \mathcal{O}} (y(x, t))^T DP \frac{\partial y(x, t)}{\partial \nu} dx \\
&= -D \int_{\partial \mathcal{O}} y^T(x, t) PN y(x, t) dx \leq 0.
\end{aligned}$$

Substituting (29) into (14), similar to the proof of Theorem 3, we can obtain the following.

Theorem 4. Consider the stochastic partial differential system (8) whose initial condition is (2) and boundary value (4). Then the system is mean square exponentially stabilizable if there exist matrices $Y \in \mathbb{R}^{n_u \times n}$ and $X > 0$, $Z > 0$, such that the following LMI holds:

$$\Phi = \begin{bmatrix} \Phi_{11} & A_1 Z & \sqrt{r_0} X C^T & X \\ * & \Phi_{22} & \sqrt{r_0} Z C_1^T & 0 \\ * & * & -X & 0 \\ * & * & * & -Z \end{bmatrix} < 0, \quad (30)$$

where

$$\begin{aligned}
\Phi_{11} &= 2\delta X + AX + XA^T + BY + Y^T B^T, \\
\Phi_{22} &= \Pi_{22} = -e^{-2\delta\tau} Z. \quad (31)
\end{aligned}$$

In this case, a stabilizing state feedback controller can be chosen by $u(x, t) = YX^{-1}y(x, t)$.

4. Robust H_∞ Control

In this section, a sufficient condition for the solvability of the robust H_∞ control problem is proposed and an LMI approach for designing the desired state feedback controllers is developed. Now, we are ready to give our main result in this paper as follows.

Theorem 5. Given a scalar $\gamma > 0$, then under initial boundary value conditions (2) and (3), the stochastic partial differential system (1) is robust mean-square exponentially stabilizable with disturbance attenuation $\gamma > 0$ if there exist matrices $Y \in \mathbb{R}^{n_u \times n}$ and $X > 0$, $Z > 0$, such that the following LMI holds:

$$Y = \begin{bmatrix} Y_{11} & A_1 Z & B_v & \sqrt{r_0} X C^T & X L^T & X \\ * & Y_{22} & 0 & \sqrt{r_0} Z C_1^T & 0 & 0 \\ * & * & -\gamma^2 I & \sqrt{r_0} C_v^T & 0 & 0 \\ * & * & * & -X & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -Z \end{bmatrix} < 0, \quad (32)$$

where

$$\begin{aligned}
Y_{11} &= 2\delta X + AX + XA^T + BY + Y^T B^T, \\
Y_{22} &= -e^{-2\delta\tau} Z. \quad (33)
\end{aligned}$$

Then a suitable robust H_∞ controller can be chosen by $u(x, t) = YX^{-1}y(x, t)$.

Proof. Obviously, by Lemma 2, if (32) holds, then (9) also holds. Therefore, by Theorem 3, system (1) is mean square exponentially stabilizable if $v(x, t) = 0$. Next, we shall show that under the zero-initial condition, (6) holds for nonzero $v(x, t) \in \mathcal{L}_2(\mathcal{O} \times [0, \infty); \mathbb{R}^{n_v})$.

We consider the Lyapunov functional $V(t, y)$ in (12), by Itô's formula [16, 18, 20],

$$\begin{aligned}
dV(t, y) &= L_v V(t, y) dt \\
&\quad + 2 \int_{\mathcal{O}} y^T(x, t) [C y(x, t) + C_1 y(x, t_h) \\
&\quad \quad + C_v v(x, t)] dW(x, t). \quad (34)
\end{aligned}$$

Integrating $dV(t, y)$ from 0 to t and taking expectation, we can obtain that

$$\mathbb{E}V(t, y) = \mathbb{E} \int_0^t L_v V(s, y) ds. \quad (35)$$

We can calculate that

$$\begin{aligned}
L_v V(t, y) &\leq LV_1(t, y) + LV_2(t, y) \\
&\quad + r_0 e^{2\delta t} \int_{\mathcal{O}} y(x, t)^T (PB_v + B_v^T P) v(x, t) dx \\
&\quad + r_0 e^{2\delta t} \int_{\mathcal{O}} y(x, t)^T (C^T PC_v + C_v^T PC) \\
&\quad \quad \times v(x, t) dx \\
&\quad + r_0 e^{2\delta t} \int_{\mathcal{O}} y(x, t_h)^T (C_1^T PC_v + C_v^T PC_1) \\
&\quad \quad \times v(x, t) dx \\
&\quad + r_0 e^{2\delta t} \int_{\mathcal{O}} v(x, t)^T C_v^T PC_v v(x, t) dx, \quad (36)
\end{aligned}$$

where $LV_1(t, y)$ and $LV_2(t, y)$ satisfy (14) and (15), respectively. Then

$$L_v V(t, y) \leq e^{2\delta t} \int_{\mathcal{O}} \zeta(x, t)^T \bar{Y} \zeta(x, t) dx, \quad (37)$$

where $\zeta^T(x, t) = [y^T(x, t) \ y^T(x, t_h) \ v^T(x, t)]$, and

$$\bar{Y} = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} & r_0 (PB_v + C^T PC_v) \\ * & \bar{Y}_{22} & r_0 C_1^T PC_v \\ * & * & r_0 C_v^T PC_v \end{bmatrix}, \quad (38)$$

$$\bar{Y}_{ij} = \bar{\Gamma}_{ij}, \quad i, j = 1, 2. \quad (39)$$

Let

$$J = \mathbb{E} \left\{ \int_0^t \int_{\mathcal{O}} \left(z^T(x, t) z(x, t) - \gamma^2 v^T(x, t) v(x, t) \right) dx dt \right\}, \quad (40)$$

and then (6) is equivalent to $J < 0$. Moreover, by (35)~(38), it follows that

$$\begin{aligned} J &= \mathbb{E} \left\{ \int_0^t \int_{\mathcal{O}} \left(z^T(x, t) z(x, t) - \gamma^2 v^T(x, t) v(x, t) \right) dx dt \right\} \\ &\leq \mathbb{E} \left\{ \int_0^t \int_{\mathcal{O}} \left(z^T(x, t) z(x, t) - \gamma^2 v^T(x, t) v(x, t) \right) dx dt \right. \\ &\quad \left. + e^{-2\delta t} \left(\mathbb{E} \int_0^t LV_v(s, y) ds - \mathbb{E} V(t, y) \right) \right\} \quad (41) \\ &\leq \mathbb{E} \left\{ \int_0^t \int_{\mathcal{O}} \left(z^T(x, t) z(x, t) - \gamma^2 v^T(x, t) v(x, t) \right) dx dt \right. \\ &\quad \left. + e^{-2\delta t} \mathbb{E} \int_0^t LV_v(s, y) ds \right\} \\ &\leq \mathbb{E} \left\{ \int_0^t \int_{\mathcal{O}} \zeta(x, t)^T \Upsilon \zeta(x, t) dx \right\}, \end{aligned}$$

where

$$\hat{\Upsilon} = \begin{bmatrix} \hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} & r_0 PB_v + r_0 C_1^T PC_v \\ * & \hat{\Upsilon}_{22} & r_0 C_1^T PC_v \\ * & * & -\gamma^2 I + r_0 C_v^T PC_v \end{bmatrix} \quad (42)$$

with $\hat{\Upsilon}_{ij} = \bar{\Upsilon}_{ij} = \bar{\Gamma}_{ij}$, $i, j = 1, 2$.

According to Lemma 2, $\hat{\Upsilon} < 0$ is equivalent to

$$\tilde{\Upsilon} = \begin{bmatrix} \tilde{\Upsilon}_{11} & PA_1 & PB_v & \sqrt{r_0} C_1^T P \\ * & \tilde{\Upsilon}_{22} & 0 & \sqrt{r_0} C_1^T P \\ * & * & -\gamma^2 I & \sqrt{r_0} C_v^T P \\ * & * & * & -P \end{bmatrix} < 0, \quad (43)$$

where

$$\begin{aligned} \tilde{\Upsilon}_{11} &= 2\delta P + PA + A^T P \\ &\quad + PBK + (BK)^T P + Q + L^T L, \quad (44) \\ \tilde{\Upsilon}_{22} &= -e^{-2\delta\tau} Q. \end{aligned}$$

In (43), pre- and postmultiplying the LMI by $\text{diag}\{P^{-1}, Q^{-1}, I, P^{-1}\}$ and letting $X = P^{-1}$, $Z = Q^{-1}$, $Y = KX^{-1}$ and following the same line as in the proof of Theorem 3, we can deduce that (32) is equivalent to (43) and $\hat{\Upsilon} < 0$, which together with (41) implies that $J < 0$. Therefore, the inequality (6) holds. This completes the proof. $\square$

Remark 6. Similar to the proof of Theorem 3, in order to calculate the $L_v V$, the proof of Theorem 5 has been used in Itô's formula of infinite dimensional version (see [18]).

If boundary value condition becomes (4), then similar to the proof of Theorem 5, we have the following.

Theorem 7. *Given a scalar $\gamma > 0$, then under initial boundary value conditions (2) and (4), the stochastic partial differential system (1) is robust stabilizable with disturbance attenuation γ if there exist matrices $Y \in \mathbb{R}^{n_u \times n}$ and $X > 0$, $Z > 0$, such that the following LMI holds:*

$$\Lambda = \begin{bmatrix} \Lambda_{11} & A_1 Z & B_v & \sqrt{r_0} X C^T & X L^T & X \\ * & \Lambda_{22} & 0 & \sqrt{r_0} Z C_1^T & 0 & 0 \\ * & * & -\gamma^2 I & \sqrt{r_0} C_v^T & 0 & 0 \\ * & * & * & -X & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -Z \end{bmatrix} < 0, \quad (45)$$

where

$$\begin{aligned} \Lambda_{11} &= 2\delta X + AX + XA^T + BY + Y^T B^T, \\ \Lambda_{22} &= -e^{-2\delta\tau} Z. \end{aligned} \quad (46)$$

Then a suitable robust H_∞ controller can be chosen by $u(x, t) = YX^{-1}y(x, t)$.

5. An Illustrative Example

In this section, we provide an illustrative example to demonstrate the effectiveness of the proposed method.

Consider the stochastic partial differential system with time delay in (1) under initial boundary value conditions (2) and (3). We let $\mathcal{O} = [0, 1]$, that is, $0 \leq x \leq 1$, $D = 1$, $y(x, t) = [y_1(x, t), y_2(x, t)]^T$. Then $l = 1$, $m = 1$, $n = 2$, $d = 1$. Let time delay $\tau = 0.1$, decay rate $\delta = 0.1$, and the upper bound of covariance function is $r_0 = 1$,

$$\begin{aligned} A &= \begin{bmatrix} -0.8 & 0.5 \\ 0.2 & -0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & -0.6 \end{bmatrix}, \\ B &= \begin{bmatrix} -0.5 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}, \quad B_v = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & -0.5 \end{bmatrix}, \\ C &= \begin{bmatrix} 0.6 & 0.4 \\ 0.1 & -0.5 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.5 & -0.2 \\ 0 & 0.4 \end{bmatrix}, \\ L &= \begin{bmatrix} -0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C_v = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}. \end{aligned} \quad (47)$$

In this example, attention is focused on the design of a state feedback controller, the resulting closed-loop system is

robustly stochastically mean square exponential stable with disturbance attenuation $\gamma = 0.8$. For this purpose, we use the Matlab LMI Control Toolbox to solve the LMI (32) and obtain the solution as follows:

$$\begin{aligned} X &= \begin{bmatrix} 2.8437 & -0.4194 \\ -0.4194 & 2.7987 \end{bmatrix}, \\ Y &= \begin{bmatrix} -8.6142 & 77.4013 \\ -94.8986 & 0.9777 \end{bmatrix}, \\ Z &= \begin{bmatrix} 1.2837 & -0.4050 \\ -0.4050 & 1.4857 \end{bmatrix}. \end{aligned} \quad (48)$$

Therefore, by Theorem 5, it can be seen that the robust H_∞ control problem is solvable and a desired state feedback control law can be chosen as

$$u(x, t) = \begin{bmatrix} -7.2557 & -28.7426 \\ -34.0280 & -4.7460 \end{bmatrix} y(x, t). \quad (49)$$

6. Conclusions

In this paper, the problems of robust stochastic exponential stabilization and robust H_∞ control for linear stochastic partial differential systems with time delay have been studied under Dirichlet and Robin boundary, respectively. An LMI approach has been developed to design state feedback controllers, which not only guarantees mean square exponential stability of the closed-loop system but also reduces the effect of the disturbance input on the controlled output to a prescribed level. A numerical example has been given to show the effectiveness of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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