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Research Article

The Dual Triple I Methods of FMT and IFMT

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The Triple I method for the model of intuitionistic fuzzy modus tollens (IFMT) satisfies the local reductivity instead of the reductivity. In order to improve the quality of the Triple I method for lack of reductivity, the paper is intended to present a new approximate reasoning method for IFMT problem. First, the concept of intuitionistic fuzzy difference operator is proposed and its properties on the lattice structure of intuitionistic fuzzy sets are studied. Then, the dual Triple I method for FMT based on residual fuzzy difference operator is presented and the dual Triple I method is generated for IFMT. Moreover, a decomposition method of IFMT is provided. Furthermore, the reductivity of methods is investigated. Finally, α-dual Triple I method of IFMT is proposed.

1. Introduction

The real world is too complicated to be described precisely and it is full of uncertainty. It seems that humans have a remarkable capability to deal with the uncertain information. We need a theory to formulate human knowledge representation. The theory of fuzzy sets introduced by Zadeh [1] has been found to be useful to deal with uncertainty, imprecision, and vagueness of information. It is well known that the fuzzy logic and approximate reasoning are significant parts of the theory of fuzzy sets. To provide foundations for approximate reasoning with fuzzy propositions, the basic models of deductive processes with fuzzy sets, which were called the fuzzy modus pones (FMP) and the fuzzy modus tollens (FMT), were proposed in the seminal paper of Zadeh [2]. The basic reasoning principle was the composition rule of inference (CRI). Based on the CRI method, fuzzy reasoning has been successfully applied to a wide variety of fields [3–6].

In [7], Wang pointed out that the composition seems to lack logic foundation. For trying to provide a logic foundation for fuzzy reasoning, Wang proposed the full implication Triple I method (Triple I method for short). Triple I method has attracted the attention of many scholars and many results have been reported [7–14]. Pei [10] comprehensively investigated the method based on a class residual fuzzy implication derived from left-continuous t-norms. Liu and Wang [8] obtained a class restriction Triple I solution for FMP and FMT. For the formalization of the Triple I fuzzy reasoning, Wang [13] and Pei [11] considered the propositional logic system and the first order logic system, respectively.

From a knowledge representation point of view, however, the role of fuzziness is not always to capture uncertainty [15, 16]. The intuitionistic fuzzy sets introduced by Atanassov [17] is a pair of fuzzy sets, namely, a membership and a nonmembership function, which represent positive and negative aspects of the given information. It constitutes an appropriate knowledge representation framework.

The intuitionistic fuzzy set theory has been widely applied in many fields such as pattern recognition, machine learning, decision making, market prediction, and image processing [18–23]. In order to construct the theoretical foundation of intuitionistic fuzzy reasoning, Cornelis et al. [24–26] have made fruitful pioneering work. They presented the intuitionistic fuzzy t-norms and t-conorms [25]. In [24, 26], they concentrated on the intuitionistic fuzzy implication operator theory, and the CRI method of the intuitionistic fuzzy reasoning is discussed in [24]. However, because restrictions on the intuitionistic fuzzy implications are much more complicated than that of fuzzy implication operator, the Triple I method of the intuitionistic fuzzy reasoning has not been paid enough

attention to. Zheng et al. [27] presented the Triple I method of intuitionistic fuzzy reasoning, proved the reductivity of the Triple I method for IFMP, and showed that the Triple I method of IFMT satisfied the local reductivity instead of the reductivity. In order to improve the quality of the Triple I method for lack of reductivity, we intend to propose the dual Triple I method of approximate reasoning for IFMT. First of all, we introduce the concepts of the intuitionistic fuzzy difference operators and coadjoint pair and provide the unified form of the residual intuitionistic fuzzy difference operators adjoint to intuitionistic t-conorms derived from the left-continuous t-norms. Secondly, we present the dual Triple I method of fuzzy reasoning for FMT and give the equivalence conditions between dual Triple I method and Triple I method. Lastly, we propose the dual Triple I method and the decomposition method of approximate reasoning for IFMT and discuss the reductivity of methods. Furthermore, α-Triple I method of IFMT is proposed.

Throughout this paper, we denote $L = [0,1]$, $a \vee b = \max\{a,b\}$, $a \wedge b = \min\{a,b\}$, $a^c = 1-a$, $a, b \in [0,1]$, and $L^* = \{(u,v) \in [0,1]^2 \mid u+v \leq 1\}$. I stands for any index set. We denote by FS$s(X)$, IFS$s(X)$ the set of all fuzzy sets in X and the set of all intuitionistic fuzzy sets, respectively.

2. Preliminaries

In this section, we recall some basic concepts and results, which we will need in the subsequent sections.

Definition 1 (see [28]). A triangular norm (t-norm for short) is a binary operation $\otimes$ on L satisfying commutativity, associativity, monotonicity, and boundary condition $a \otimes 1 = a$, $\forall a \in L$. A triangular conorm (t-conorm for short) is a binary operation $\oplus$ on L satisfying commutativity, associativity, monotonicity, and boundary condition $a \oplus 0 = a$, $\forall a \in L$.

The t-conorm $\oplus$ is called the dual t-conorm of the norm $\otimes$ if $a \oplus b = 1-(1-a) \otimes (1-b)$, $\forall a, b \in L$ and, analogously, the t-norm $\otimes$ is called the dual t-norm of the conorm $\oplus$ if $a \otimes b = 1-(1-a) \oplus (1-b)$, $\forall a, b \in L$.

Definition 2 (see [28]). A t-norm $\otimes$ is a left-continuous t-norm if for all $a_i, b \in L$ and $\otimes$ satisfies $(\vee_{i\in I} a_i) \otimes b = \vee_{i \in I}(a_i \otimes b)$. A t-conorm $\oplus$ is a right-continuous t-conorm if for all $a_i, b \in L$, $\oplus$ satisfies $(\wedge_{i\in I} a_i) \oplus b = \wedge_{i\in I}(a_i \oplus b)$.

Proposition 3 (see [28]). *A t-norm is left-continuous if and only if the dual i-conorm of the t-norm is right-continuous.*

Proposition 4 (see [29]). *If $\otimes$ is a left-continuous t-norm, then there exists a binary operation $\rightarrow$ (the $\otimes$-residuum) on L such that $(\otimes, \rightarrow)$ satisfy the residual principle; that is, $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$, where $\rightarrow$ is given by*

$$a \longrightarrow b = \vee \{c \in L \mid c \otimes a \leq b\} \tag{1}$$

and is called residual implication derived from $\otimes$.

Proposition 5 (see [30]). *If $\oplus$ is a right-continuous t-conorm then there exists a binary operation $\ominus$(the $\oplus$ -coresiduum) on L such that $(\oplus, \ominus)$ forms a coadjoint pair; that is, $a \leq b \oplus c$ if and only if $a \ominus c \leq b$, and $\ominus$ is given by*

$$a \ominus b = \wedge \{c \in L \mid a \leq c \oplus b\}. \tag{2}$$

The $\oplus$-coresiduum $\ominus$ is called fuzzy difference operator derived from $\oplus$.

Definition 6 (see [27]). $\rightarrow$, $\oplus$, and $\ominus$ are called associated operators of $\otimes$ if $(\otimes, \rightarrow)$ is a adjoint pair, $(\oplus, \ominus)$ is a coadjoint pair, and $\oplus$ is the dual t-conorm of the norm $\otimes$.

Proposition 7 (see [27]). *If $\rightarrow$, $\oplus$, and $\ominus$ are associated operators of $\otimes$, then $a \ominus b = 1-(1-b) \rightarrow (1-a)$.*

Example 8. The following are four important t-norms. The first three are all continuous t-norms, but the last one is left-continuous:

(1) Gödel t-norm $a \otimes_G b = a \wedge b$,

(2) Lukasiewicz t-norm $a \otimes_{Lu} b = (a+b-1) \vee 0$,

(3) product t-norm $a \otimes_\pi b = ab$,

(4) R_0 t-norm.

Consider

$$a \otimes_0 b = \begin{cases} 0, & a+b \leq 1; \\ a \wedge b, & a+b > 1. \end{cases} \tag{3}$$

The associated operators of the above four t-norms are as follows, respectively:

(1′)

$$\begin{aligned} a \longrightarrow_G b &= \begin{cases} 1, & a \leq b; \\ b, & a > b, \end{cases} \\ a \oplus_G b &= a \vee b, \\ b \ominus_G a &= \begin{cases} 0, & b \leq a; \\ b, & b > a, \end{cases} \end{aligned} \tag{4}$$

(2′)

$$\begin{aligned} a \longrightarrow_{Lu} b &= (1-a+b) \wedge 1, \\ a \oplus_{Lu} b &= (a+b) \wedge 1, \\ b \ominus_{Lu} a &= (b-a) \vee 0, \end{aligned} \tag{5}$$

(3′)

$$\begin{aligned} a \longrightarrow_\pi b &= \begin{cases} 1, & a \leq b; \\ \dfrac{b}{a}, & a > b, \end{cases} \\ a \oplus_\pi b &= a+b-ab, \\ b \ominus_\pi a &= \begin{cases} 0, & b \leq a; \\ \dfrac{b-a}{1-a}, & b > a, \end{cases} \end{aligned} \tag{6}$$

(4′)

$$a \longrightarrow_0 b = \begin{cases} 1, & a \le b; \\ (1-a) \vee b, & a > b, \end{cases}$$

$$a \oplus_0 b = \begin{cases} 1, & (a+b) \ge 1; \\ a \vee b, & a+b < 1, \end{cases} \tag{7}$$

$$b \ominus_0 a = \begin{cases} 0, & b \le a; \\ b \wedge (1-a), & b > a. \end{cases}$$

Lemma 9. *If $a \le b$, $a, b \in L$, then $b \ominus_{Lu}(b \ominus_{Lu} a) = a$.*

Lemma 10. *If $1/2 \le a \le b$, $a, b \in L$, then $b \ominus_0 (b \ominus_0 a) = a$.*

Definition 11 (see [17]). An intuitionistic fuzzy set on the nonempty universe of discourse X is given by $A = \{\langle x, A_t(x), A_f(x)\rangle \mid x \in X\}$, where

$$A_t : X \longrightarrow [0,1], \qquad A_f : X \longrightarrow [0,1] \tag{8}$$

with the condition

$$0 \le A_t(x) + A_f(x) \le 1, \quad \forall x \in X. \tag{9}$$

$A_t(x)$ and $A_f(x)$ denote a membership function and a nonmembership function of x to A, respectively. It is clear that the intuitionistic fuzzy set A in X can be written as

$$A(x) = (u, v), \quad 0 \le u + v \le 1, \qquad u, v \in [0,1], \quad \forall x \in X. \tag{10}$$

As a generalization of fuzzy sets, intuitionistic fuzzy sets extend the character value from $[0,1]$ to the triangle domain $L^* = \{(u,v) \in [0,1]^2 \mid u + v \le 1\}$.

If $A_t(x) = 1 - A_f(x)$, $\forall x \in X$, then intuitionistic fuzzy sets degenerate into fuzzy sets. We denote by IFS$s(X)$ the set of all intuitionistic fuzzy sets in X.

We can define a partial order on L^* as follows:

$$\alpha, \beta \in L^*, \quad \alpha = (a_1, a_2), \quad \beta = (b_1, b_2), \qquad \alpha \le \beta \quad \text{iff } a_1 \le b_1, \quad a_2 \ge b_2. \tag{11}$$

Obviously, $\alpha \wedge \beta = (a_1 \wedge b_1, a_2 \vee b_2)$, $\alpha \vee \beta = (a_1 \vee b_1, a_2 \wedge b_2)$, $0^* = (0,1)$ and $1^* = (1,0)$, are the smallest element and the greatest element of L^*, respectively. It is easy to verify the fact that $(L^*, \le)$ is a complete lattice.

Definition 12 (see [27]). $\otimes_{L^*}$ is called an intuitionistic t-norm derived from t-norm $\otimes$ if

$$\alpha \otimes_{L^*} \beta = (a_1 \otimes b_1, a_2 \oplus b_2) \tag{12}$$

and $\oplus_{L^*}$ is called an intuitionistic t-conorm derived from t-norm if

$$\alpha \oplus_{L^*} \beta = (a_1 \oplus b_1, a_2 \otimes b_2), \tag{13}$$

where $\oplus$ is the dual t-conorm of the t-norm $\otimes$.

Proposition 13 (see [27]). *$(L^*, \otimes_{L^*}, 1^*)$ is a commutative monoid and $\otimes_{L^*}$ is isotone; $(L^*, \oplus_{L^*}, 0^*)$ is a commutative monoid and $\oplus_{L^*}$ is isotone.*

Proposition 14 (see [27]). *Let $\otimes$ be a left-continuous t-norm; then*

(1) *$\otimes_{L^*}$ derived from $\otimes$ is a left-continuous intuitionistic t-norm on L^*; that is, $(\vee_{i\in I}\alpha_i) \otimes_{L^*} \gamma = \vee_{i \in I}(\alpha_i \otimes_{L^*} \gamma)$;*

(2) *$\oplus_{L^*}$ derived from $\otimes$ is a right-continuous intuitionistic t-conorm on L^*; that is, $(\wedge_{i\in I}\alpha_i) \oplus_{L^*} \gamma = \wedge_{i \in I}(\alpha_i \oplus_{L^*} \gamma)$.*

Theorem 15 (see [27]). *Let $\otimes_{L^*}$ be an intuitionistic t-norm derived from a left-continuous t-norm $\otimes$; then there exists a binary operation $\rightarrow_{L^*}$ on L^* such that*

$$\gamma \otimes_{L^*} \alpha \le \beta \quad \text{iff } \gamma \le \alpha \longrightarrow_{L^*} \beta \tag{14}$$

and $\rightarrow_{L^*}$ is given by

$$\alpha \rightarrow_{L^*} \beta = \vee \{\eta \in L^* \mid \eta \otimes_{L^*} \alpha \le \beta\}. \tag{15}$$

Definition 16 (see [27]). $(\otimes_{L^*}, \rightarrow_{L^*})$ is called an intuitionistic adjoint pair if $(\otimes_{L^*}, \rightarrow_{L^*})$ satisfy the residual principle (14), and $\rightarrow_{L^*}$ is called a residual intuitionistic implication derived from a left-continuous $\otimes$ if $\otimes_{L^*}$ is an intuitionistic t-norm derived from a left-continuous $\otimes$.

Theorem 17 (see [27]). *Let $\alpha, \beta \in L^*$, $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$, and $\rightarrow_{L^*}$ be a residual intuitionistic implication derived from a left-continuous t-norm $\otimes$; then*

$$\alpha \longrightarrow_{L^*} \beta = ((a_1 \rightarrow b_1) \wedge (1 - (b_2 \ominus a_2)), b_2 \ominus a_2). \tag{16}$$

3. The Triple I Method of IFMT

As one of the basic inference models of fuzzy reasoning, FMT has the following form:

$$\begin{array}{llll} \text{Suppose that} & A(x) \longrightarrow B(y) & \cdots & \text{major premise} \\ \text{and given} & B^*(y) & \cdots & \text{minor premise} \\ \hline \text{caculate} & A^*(x) & \cdots & \text{conclution} \end{array}, \tag{17}$$

where $A(x)$, $A^*(x)$ are the fuzzy sets on the nonempty universe of discourse X and $B(y)$, $B^*(y)$ are the fuzzy sets on the nonempty universe of discourse Y.

The Triple I principle is as follows:

$A^*(x)$ should be the biggest fuzzy set on X satisfying

$$(A(x) \longrightarrow B(y)) \longrightarrow (A^*(x) \longrightarrow B^*(y)) = 1. \tag{18}$$

Theorem 18 (see [10]). *Let $\rightarrow$ be a residual implication derived from a left-continuous t-norm; the expression of the Triple I solution A^* for FMT problem* (17) *is as follows:*

$$A^*(x) = \wedge_{y \in Y} \{(A(x) \longrightarrow B(y)) \longrightarrow B^*(y)\}. \tag{19}$$

Theorem 19 (see [10, 14]). *Let $\rightarrow$ be a residual implication derived from a left-continuous t-norm and satisfy contrapositive symmetry; that is, $a \rightarrow b = b^c \rightarrow a^c$; then the expression of the Triple I solution A^* for FMT problem* (17) *becomes*

$$A^*(x) = \wedge_{y \in Y} \{B^*(y) \oplus (A(x) \longrightarrow B(y))^c\}. \tag{20}$$

The Triple I method of approximate reasoning was extended from FMT to IFMT in [27]. IFMT has the same form as FMT as follows:

$$\begin{array}{lll} \text{Suppose that } A(x) \longrightarrow_{L^*} B(y) & \cdots & \text{major premise} \\ \text{and given} \qquad\qquad\qquad B^*(y) & \cdots & \text{minor premise} \\ \hline \text{caculate} \qquad A^*(x) & \cdots & \text{conclution} \end{array}, \tag{21}$$

where $A(x)$, $A^*(x)$ are the intuitionistic fuzzy sets on the nonempty universe of discourse X; $B(y)$, $B^*(y)$ are the intuitionistic fuzzy sets on the nonempty universe of discourse Y; and $\rightarrow_{L^*}$ is a residual intuitionistic fuzzy implication on L^*. We denote $A(x) = (A_t(x), A_f(x))$, $B(y) = (B_t(y), B_f(y))$, $B^*(y) = (B_t^*(y), B_f^*(y))$, $A_{-f}(x) = 1 - A_f(x)$, $B_{-f}(y) = 1 - B_f(y)$, and $B_{-f}^*(y) = 1 - B_f^*(y)$. Clearly, A_t, A_f, and A_{-f} are the fuzzy sets on X, respectively, and B_t, B_f, B_t^*, B_f^*, and B_{-f} are the fuzzy sets on Y, respectively.

Because $\rightarrow_{L^*}$ is the residual intuitionistic fuzzy implication on L^*, the extension of the Triple I principle is as follows:

$A^*(x)$ should be the biggest intuitionistic fuzzy set on X satisfying

$$(A(x) \longrightarrow B(y)) \longrightarrow (A^*(x) \longrightarrow B^*(y)) = 1^* \tag{22}$$

under the order of L^*.

Theorem 20 (see [27]). *Let the implication $\rightarrow_{L^*}$ in IFMT be the residual implication derived from a left-continuous t-norm $\otimes$; then the expression of the Triple I solution A^* for IFMT problem* (21) *is as follows:*

$$A^*(x) = \wedge_{y\in Y}\{(A(x) \longrightarrow_{L^*} B(y)) \longrightarrow_{L^*} B^*(y)\}, \quad x \in X. \tag{23}$$

We know that the Triple I method of FMT possesses virtue of reductivity if $B(y)$ satisfies the condition $\exists y_0 \in Y$ such that $B(y_0) = 0$ (see [10]). Unfortunately, the Triple I method of IFMT only possess the local reductivity instead of the reductivity (see [27]).

Theorem 21 (see [27]). *Let the implication $\rightarrow_{L^*}$ in IFMT be the residual implication derived from a left-continuous t-norm $\otimes$ satisfying $1 \ominus (1 \ominus a) = a$; then the Triple I method is local reductive; that is, $A_f^* = A_f$ whenever $B^* = B$ satisfying $\exists y_0 \in Y$ such that $B(y_0) = 0^*$.*

Corollary 22 (see [27]). *Let the implication $\rightarrow_{L^*}$ in IFMT be the residual implication derived from Lukasiewicz t-norm or R_0 t-norm; then the Triple I method is local reductive; that is, $A_f^* = A_f$ whenever $B^* = B$ satisfying $\exists y_0 \in Y$, $B(y_0) = 0^*$.*

Theorem 23 (see [27]). *Let the implication $\rightarrow_{L^*}$ in IFMT be the residual implication derived from Lukasiewicz t-norm; then the Triple I method is local reductive, that is, $A_f^* = A_f$ whenever $B_f^* = B_f$ satisfying $\forall x \in X$, $\exists y_0 \in Y$ such that $A_f(x) \le B_f(y_0)$.*

Theorem 24 (see [27]). *Let the implication $\rightarrow_{L^*}$ in IFMT be the residual implication derived from R_0 t-norm; then the Triple I method is local reductive; that is, $A_f^* = A_f$ whenever $B_f^* = B_f$ satisfying $\forall x \in X$, $\exists y_0 \in Y$ such that $1/2 \le A_f(x) \le B_f(y_0)$.*

4. Intuitionistic Fuzzy Difference Operator

In this section, we give the unified form of the adjoint operator for the intuitionistic t-conorm derived from a left-continuous t-norm.

Theorem 25. *Let $\oplus_{L^*}$ be an intuitionistic t-conorm derived from a left-continuous t-norm $\otimes$; then there exists a binary operation $\ominus_{L^*}$ on L^* such that*

$$\alpha \le \gamma \oplus_{L^*} \beta \quad iff \ \alpha \ominus_{L^*} \beta \le \gamma, \tag{24}$$

and $\ominus_{L^*}$ is given by

$$\alpha \ominus_{L^*} \beta = \wedge\{\eta \in L^* \mid \alpha \le \eta \oplus_{L^*} \beta\}. \tag{25}$$

Proof. By (25), if $\alpha \le \gamma \oplus_{L^*} \beta$, then $\alpha \ominus_{L^*} \beta \le \gamma$. Conversely, if $\alpha \ominus_{L^*} \beta \le \gamma$, then $\wedge\{\eta \mid \alpha \le \eta \oplus_{L^*} \beta\} \le \gamma$. From the monotonicity of $\oplus_{L^*}$, $(\wedge\{\eta \mid \alpha \le \eta \oplus_{L^*} \beta\}) \oplus_{L^*} \beta \le \gamma \oplus_{L^*} \beta$. According to the left-continuity of $\oplus_{L^*}$, $\wedge\{\eta \oplus_{L^*} \beta \mid \alpha \le \eta \oplus_{L^*} \beta\} \le \gamma \oplus_{L^*} \beta$. Thus $\alpha \le \gamma \oplus_{L^*} \beta$. □

Definition 26. $(\oplus_{L^*}, \ominus_{L^*})$ is called an intuitionistic coadjoint pair if $(\oplus_{L^*}, \ominus_{L^*})$ satisfy the residual principle (24), and $\ominus_{L^*}$ is called a residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm $\otimes$ if $\oplus_{L^*}$ is an intuitionistic t-conorm derived from a left-continuous t-norm $\otimes$.

Proposition 27. *Suppose that $\ominus_{L^*}$ is a residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm and $(\oplus_{L^*}, \ominus_{L^*})$ is an intuitionistic coadjoint pair; then*

(1) $\alpha = \alpha \ominus_{L^*} 0^*$;

(2) $\alpha \ominus_{L^*} \beta = 0^* iff \ \alpha \le \beta$;

(3) $\alpha \ominus_{L^*} \beta \le \gamma iff \ \alpha \ominus_{L^*} \gamma \le \beta$;

(4) $\alpha \ominus_{L^*} (\beta \oplus_{L^*} \gamma) = (\alpha \ominus_{L^*} \beta) \ominus_{L^*} \gamma = (\alpha \ominus_{L^*} \gamma) \ominus_{L^*} \beta$;

(5) $(\alpha \oplus_{L^*} \beta) \ominus_{L^*} \beta \le \alpha \le (\alpha \ominus_{L^*} \beta) \oplus_{L^*} \beta$;

(6) $((\alpha \oplus_{L^*} \beta) \ominus_{L^*} \beta) \ominus_{L^*} \alpha = ((\alpha \oplus_{L^*} \beta) \ominus_{L^*} \alpha) \ominus_{L^*} \beta = 0$;

(7) $(\alpha \oplus_{L^*} \gamma) \ominus_{L^*} (\gamma \oplus_{L^*} \beta) \le \alpha \ominus_{L^*} \beta \le (\alpha \ominus_{L^*} \gamma) \oplus_{L^*} (\gamma \ominus_{L^*} \beta)$;

(8) $\alpha \ominus_{L^*} (\wedge_{i\in I} \beta_i) = \vee_{i\in I} (\alpha \ominus_{L^*} \beta_i)$;

(9) $(\vee_{i\in I} \alpha_i) \ominus_{L^*} \beta = \vee_{i\in I} (\alpha_i \ominus_{L^*} \beta)$;

(10) *$\alpha \ominus_{L^*} \beta$ is isotone in the first variable and antitone in the second variable.*

Theorem 28. *Suppose that $\alpha, \beta \in L^*$, $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$, and $\ominus_{L^*}$ is a residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm $\otimes$; then*

$$\alpha \ominus_{L^*} \beta = (a_1 \ominus b_1, (b_2 \longrightarrow a_2) \wedge (1 - a_1 \ominus b_1)). \tag{26}$$

Proof. Let $\gamma = (u, v) = \alpha \ominus_{L^*} \beta$ and $\gamma_i = (u_i, v_i) \in L^*$.

By Theorem 25,

$$\begin{aligned}\gamma = (u,v) &= \alpha \ominus_{L^*} \beta = \wedge\{\gamma_i \in L^* \mid \alpha \le \gamma_i \oplus_{L^*} \beta\} \\ &= \wedge\{(u_i,v_i) \mid (a_1,a_2) \le (u_i,v_i) \oplus_{L^*} (b_1,b_2)\} \\ &= \wedge\{(u_i,v_i) \mid a_1 \le u_i \oplus b_1, a_2 \le v_i \otimes b_2, u_i + v_i \le 1\} \\ &= (\wedge u_i, \vee v_i).\end{aligned} \tag{27}$$

For the first argument u, it follows from Proposition 13 that $u = \wedge\{u_i \mid a_1 \le u_i \oplus b_1\} = a_1 \ominus b_1$.

For the second argument v,

$$\begin{aligned}v &= \vee\{v_i \mid v_i \otimes b_2 \le a_2, v_i \le 1 - u_i, a_1 \le v_i \oplus b_1\} \\ &\le (\vee\{v_i \mid v_i \otimes b_2 \le a_2\}) \\ &\quad \wedge (\vee\{v_i \mid v_i \le 1 - u_i, a_1 \le u_i \oplus b_1\}) \\ &\le (\vee\{v_i \mid v_i \otimes b_2 \le a_2\}) \wedge (1 - \wedge\{u_i \mid a_1 \le u_i \oplus b_1\}) \\ &= (b_2 \longrightarrow a_2) \wedge (1 - a_1 \ominus b_1),\end{aligned} \tag{28}$$

so $\eta = (u,v) \le ((a_1 \to b_1) \wedge (1 - b_2 \ominus a_2), b_2 \ominus a_2)$. Moreover,

$$\begin{aligned}&(a_1 \ominus b_1, (b_2 \longrightarrow a_2) \wedge (1 - a_1 \ominus b_1)) \oplus_{L^*} \beta \\ &\quad = (a_1 \ominus b_1, (b_2 \longrightarrow a_2) \wedge (1 - a_1 \ominus b_1)) \oplus_{L^*} (b_1, b_2) \\ &\quad \ge ((a_1 \ominus b_1) \oplus b_1, (b_2 \longrightarrow a_2) \otimes b_2) \ge (a_1, a_2) = \alpha.\end{aligned} \tag{29}$$

According to Theorem 25

$$\gamma = (u,v) = \wedge\{(u_i,v_i) \mid \alpha \le (u_i,v_i) \oplus_{L^*} \beta\}. \tag{30}$$

Therefore, $\gamma = (u,v) = (a_1 \ominus b_1, (b_2 \to a_2) \wedge (1 - a_1 \ominus b_1))$. □

Example 29. The residual intuitionistic fuzzy difference operators derived from Gödel t-norm, Lukasiewicz t-norm, product t-norm, and R_0 t-norm are as follows, respectively:

(1)

$$\alpha \ominus_{L^*} \beta = \begin{cases} (0,1), & a_1 \le b_1, \quad b_2 \le a_2; \\ (0,a_2), & a_1 \le b_1, \quad b_2 > a_2; \\ (a_1, 1 - a_1), & a_1 > b_1, \quad b_2 \le a_2; \\ (a_1, a_2), & a_1 > b_1, \quad b_2 > a_2, \end{cases} \tag{31}$$

(2)

$$\alpha \ominus_{L^*} \beta = ((a_1 - b_1) \vee 0, (1 - b_2 + a_2) \wedge (1 - a_1 + b_1) \wedge 1), \tag{32}$$

(3)

$$\alpha \ominus_{L^*} \beta = \begin{cases} (0,1), & a_1 \le b_1, \quad b_2 \le a_2; \\ \left(0, \dfrac{a_2}{b_2}\right), & a_1 \le b_1, \quad b_2 > a_2; \\ \left(\dfrac{a_1 - b_1}{1 - b_1}, \dfrac{1 - a_1}{1 - b_1}\right), & a_1 > b_1, \quad b_2 \le a_2; \\ \left(\dfrac{a_1 - b_1}{1 - b_1}, \dfrac{1 - a_1}{1 - b_1} \wedge \dfrac{a_2}{b_2}\right), & a_1 > b_1, \quad b_2 > a_2, \end{cases} \tag{33}$$

(4)

$$\alpha \ominus_{L^*} \beta = \begin{cases} (0,1), \\ \quad a_1 \le b_1, \quad b_2 \le a_2; \\ (0, (1 - b_2) \vee a_2), \\ \quad a_1 \le b_1, \quad b_2 > a_2; \\ (a_1 \wedge (1 - b_1), (1 - a_1) \vee b_1), \\ \quad a_1 > b_1, \quad b_2 \le a_2; \\ (a_1 \wedge (1 - b_1), \\ ((1 - a_1) \vee b_1) \wedge ((1 - b_2) \vee a_2)), \\ \quad a_1 > b_1, \quad b_2 > a_2. \end{cases} \tag{34}$$

5. The Dual Triple I Method of FMT and IFMT

If we take the fuzzy difference operator instead of the fuzzy implication, then the model FMT has the following form:

$$\begin{array}{lll} \text{Suppose that } A(x) \ominus B(y) & \cdots & \text{major premise} \\ \text{and given } \qquad\qquad B^*(y) & \cdots & \text{minor premise} \\ \hline \text{caculate } \quad A^*(x) & \cdots & \text{conclution} \end{array}, \tag{35}$$

where $A(x)$, $A^*(x)$ are the fuzzy sets on the nonempty universe of discourse X; $B(y)$, $B^*(y)$ are the fuzzy sets on the nonempty universe of discourse Y; and $\ominus$ is a fuzzy difference operator.

The dual Triple I principle is as follows:

$A^*(x)$ should be the biggest fuzzy set on X satisfying

$$(A^*(x) \ominus B^*(y)) \ominus (A(x) \ominus B(y)) = 0. \tag{36}$$

Remark 30. Because there are three fuzzy difference operators in formula (36), the dual Triple I method could be called Triple D method.

Theorem 31. *Let $\ominus$ be the fuzzy difference operator derived from a right-continuous t-conorm $\oplus$; the expression of the Triple D solution A^* of FMT problem* (35) *is as follows:*

$$A^*(x) = \wedge_{y \in Y}\{(A(x) \ominus B(y)) \oplus B^*(y)\}. \tag{37}$$

Proof. It follows from formula (37) that

$$\forall y \in Y, \quad A^*(x) \le (A(x) \ominus B(y)) \oplus B^*(y), \quad x \in X. \tag{38}$$

Since $\ominus$ is the residual fuzzy difference operator, then

$$\forall y \in Y, \quad A^*(x) \ominus B^*(y) \le A(x) \ominus B(y), \quad x \in X. \tag{39}$$

That is,

$$\forall y \in Y, \quad (A^*(x) \ominus B^*(y)) \ominus (A(x) \ominus B(y)) = 0, \quad x \in X. \tag{40}$$

Suppose that $C(x)$ is a fuzzy set on X such that

$$(C(x) \ominus B^*(y)) \ominus (A(x) \ominus B(y)) = 0, \quad x \in X, \ y \in Y. \tag{41}$$

Since $\ominus$ is the residual fuzzy difference operator, then

$$\forall y \in Y, \quad C(x) \ominus B^*(y) \le A(x) \ominus B(y). \tag{42}$$

That is,

$$\forall y \in Y, \quad C(x) \le (A(x) \ominus B(y)) \oplus B^*(y). \tag{43}$$

It follows from formula (37) that $C(x) \le A^*(x)$. Therefore, A^* given by the formula (37) is the Triple D solution of IFMT. □

Theorem 32. *Let $\ominus$ be the fuzzy difference operator derived from a right-continuous t-conorm $\oplus$; then the Triple D solution A^* of FMT given by* (37) *is reductive; that is, $A^*(x) = A(x)$ whenever $B^*(y) = B(y)$ satisfying $\exists y_0 \in Y$ such that $B(y_0) = 0$.*

Proof. It follows from (37) that if $B^*(y) = B(y)$ then $A^*(x) = \wedge_{y\in Y}\{(A(x)\ominus B(y))\oplus B(y)\}$. Because $A(x)\ominus B(y) \le A(x)\ominus B(y)$ and $(\oplus, \ominus)$ is a coadjoint pair, then $A(x) \le (A(x) \ominus B(y)) \oplus B(y)$. Thus it follows from (37) that $A(x) \le A^*(x)$, $x \in X$. Moreover, if there exists $y_0 \in Y$ such that $B(y_0) = 0$, then it follows from formula (37) that $A^*(x) \le (A(x)\ominus 0)\oplus 0 = A(x)$. Therefore, $A^* = A$. □

Theorem 33. *Suppose that $\rightarrow$, $\oplus$, and $\ominus$ are associated operators of $\otimes$; if $\rightarrow$ satisfies contrapositive symmetry, then the Triple I solution A^* is equivalent to the Triple D solution A^*.*

Proof. Since $\rightarrow$ satisfies contrapositive symmetry, then by Theorem 19 and Proposition 7

$$\begin{aligned} A^*(x) &= \wedge_{y\in Y}\left\{B^*(y) \oplus (A(x) \longrightarrow B(y))^c\right\} \\ &= \wedge_{y\in Y}\left\{B^*(y) \oplus \left((B(y))^c \longrightarrow (A(x))^c\right)^c\right\} \\ &= \wedge_{y\in Y}\{B^*(y) \oplus (A(x) \ominus B(y))\} \\ &= \wedge_{y\in Y}\{(A(x) \ominus B(y)) \oplus B^*(y)\}. \end{aligned} \tag{44}$$

Therefore, the Triple I solution A^* is equivalent to the Triple D solution A^*. □

It is natural that the model IFMT can be transformed to the following form:

$$\begin{array}{llll} \text{Suppose that} & A(x)\ominus_{L^*} & B(y) & \cdots \text{ major premise} \\ \text{and given} & & B^*(y) & \cdots \text{ minor premise} \\ \hline \text{caculate} & A^*(x) & & \cdots \text{ conclution} \end{array}, \tag{45}$$

where $A(x)$, $A^*(x)$ are the intuitionistic fuzzy sets on the nonempty universe of discourse X; $B(y), B^*(y)$ are the intuitionistic fuzzy sets on the nonempty universe of discourse Y; and $\ominus_{L^*}$ is a residual intuitionistic fuzzy difference operator on L^*. We denote $A(x) = (A_t(x), A_f(x))$, $B(y) = (B_t(y), B_f(y))$, $B^*(y) = (B_t^*(y), B_f^*(y))$, $A_{-f}(x) = 1 - A_f(x)$, $B_{-f}(y) = 1 - B_f(y)$, and $B_{-f}^*(y) = 1 - B_f^*(y)$. Clearly, A_t, A_f, and A_{-f} are the fuzzy sets on X, respectively, and B_t, B_f, B_t^*, B_f^*, B_{-f}, and B_{-f}^* are the fuzzy sets on Y, respectively.

The Triple D principle of IFMT is as follows:

$A^*(x)$ should be the biggest intuitionistic fuzzy set on X satisfying

$$(A^*(x) \ominus_{L^*} B^*(y)) \ominus_{L^*} (A(x) \ominus_{L^*} B(y)) = 0^* \tag{46}$$

under the order of L^*.

Theorem 34. *Let $\ominus_{L^*}$ be the residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm; the expression of the Triple D solution A^* for IFMT problem* (45) *is as follows:*

$$A^*(x) = \wedge_{y\in Y}\{(A(x) \ominus_{L^*} B(y)) \oplus_{L^*} B^*(y)\}. \tag{47}$$

Proof. It is similar to the proof of Theorem 31. □

Corollary 35. *Suppose that $\ominus_{L^*}$ is the intuitionistic fuzzy difference operator derived from a left-continuous t-norm; then the Triple D solution A^* of IFMT is given by the following formula:*

$$A^*(x) = \left(A_t^*(x), A_f^*(x)\right), \tag{48}$$

where

$$\begin{aligned} A_t^*(x) &= \wedge_{y\in Y}\{(A_t(x) \ominus B_t(y)) \oplus B_t^*(y)\} \\ A_f^*(x) &= \vee_{y\in Y}\left\{\left(\left(B_f(y) \longrightarrow A_f(x)\right)\right.\right. \\ &\qquad \left.\left.\wedge (1 - A_t(x) \ominus B_t(y))\right) \otimes B_f^*(y)\right\}. \end{aligned} \tag{49}$$

Proof. The proof is trivial by Theorems 28 and 34. □

According to Corollary 35, we consider the Triple D solutions of the following two FMT problems and the Triple I solution of the following FMP problem:

$$\begin{array}{llll} \text{Suppose that} & A(x)\ominus & B_t(y) & \cdots \text{ major premise} \\ \text{and given} & & B_t^*(y) & \cdots \text{ minor premise} \\ \hline \text{caculate} & \widetilde{A}_t^*(x) & & \cdots \text{ conclution} \end{array}, \tag{50}$$

$$\begin{array}{llll} \text{Suppose that} & A_t(x)\ominus & B_t(y) & \cdots \text{ major premise} \\ \text{and given} & & B_{-f}^*(y) & \cdots \text{ minor premise} \\ \hline \text{caculate} & \overline{A}_t^*(x) & & \cdots \text{ conclution} \end{array}, \tag{51}$$

$$\begin{array}{llll} \text{Suppose that} & B_f(y) \longrightarrow & A_f(x) & \cdots \text{ major premise} \\ \text{and given} & B_f^*(y) & & \cdots \text{ minor premise} \\ \hline \text{caculate} & & \widetilde{A}_f^*(x) & \cdots \text{ conclution} \end{array}. \tag{52}$$

Theorem 36. *Suppose that $\widetilde{A}_t^*(x)$ and $\overline{A}_t^*(x)$ are the Triple D solutions of FMT problem* (50) *and* (51)*, respectively, and $\widetilde{A}_f^*(x)$ is the Triple I solution of FMP problem* (52)*; then*

$$\begin{aligned} A_t^*(x) &= \widetilde{A}_t^*(x), \\ A_f^*(x) &\le \left(1 - \overline{A}_t^*(x)\right) \wedge \widetilde{A}_f^*(x) \le 1 - A_t^*(x), \end{aligned} \tag{53}$$

where $A^*(x) = (A_t^*(x), A_f^*(x))$ is the Triple D solution of IFMT problem (45) *given by* (49).

Proof. Obviously, $A_t^*(x) = \widetilde{A}_t^*(x)$ from Theorem 31. It follows from formula (49) that

$$\begin{aligned} A_f^*(x) &= \vee_{y\in Y}\left\{\left(\left(B_f(y) \longrightarrow A_f(x)\right)\right.\right. \\ &\qquad \left.\left.\wedge\left(1 - A_t(x) \ominus B_t(y)\right)\right) \otimes B_f^*(y)\right\} \\ &\le \vee_{y\in Y}\left\{\left(\left(B_f(y) \longrightarrow A_f(x)\right) \otimes B_f^*(y)\right.\right. \\ &\qquad \left.\left.\wedge\left(\left(1 - A_t(x) \ominus B_t(y)\right) \otimes B_f^*(y)\right)\right\} \\ &\le \left(\vee_{y\in Y}\left\{\left(B_f(y) \longrightarrow A_f(x)\right) \otimes B_f^*(y)\right\}\right) \\ &\quad \wedge\left(\vee_{y\in Y}\left\{\left(1 - A_t(x) \ominus B_t(y)\right) \otimes \left(1 - B_{-f}^*(y)\right)\right\}\right) \\ &= \left(\vee_{y\in Y}\left\{\left(B_f(y) \longrightarrow A_f(x)\right) \otimes B_f^*(y)\right\}\right) \\ &\quad \wedge\left(1 - \wedge_{y\in Y}\left\{\left(A_t(x) \ominus B_t(y)\right) \oplus B_{-f}^*(y)\right\}\right) \\ &= \widetilde{A}_f^*(x) \wedge \left(1 - \overline{A}_t^*(x)\right). \end{aligned} \tag{54}$$

Since $\forall y \in Y, B_t^*(y) + B_f^*(y) \le 1$, that is, $\forall y \in Y, B_t^*(y) \le 1 - B_f^*(y) = B_{-f}^*(y)$, then

$$\begin{aligned} &\wedge_{y\in Y}\left\{\left(A_t(x) \ominus B_t(y)\right) \oplus B_t^*(y)\right\} \\ &\qquad \le \wedge_{y\in Y}\left\{\left(A_t(x) \ominus B_t(y)\right) \oplus B_{-f}^*(y)\right\}. \end{aligned} \tag{55}$$

That is, $\widetilde{A}_t^*(x) \le \overline{A}_t^*(x)$. Thus $(1 - \overline{A}_t^*(x)) \wedge \widetilde{A}_f^*(x) \le 1 - \widetilde{A}_t^*(x) = 1 - A_t^*(x)$. The proof is completed. □

Definition 37. $\widetilde{A}^*(x)$ is called the decomposition method solution of IFMT problem (45) if

$$\widetilde{A}^*(x) = \left(\widetilde{A}_t^*(x), \left(1 - \overline{A}_t^*(x)\right) \wedge \widetilde{A}_f^*(x)\right). \tag{56}$$

Do the Triple D solution $A^*(x)$ and the decomposition methods solution $\widetilde{A}^*(x)$ degenerate into the fuzzy sets if the intuitionistic fuzzy sets $A(x)$, $A^*(x)$, and $B(y)$ degenerate into the fuzzy sets? The following theorem answers this question.

Theorem 38. *If the intuitionistic fuzzy sets $A(x)$, $B(y)$, and $B^*(y)$ in IFMT problem (45) degenerate into the fuzzy sets, then the Triple D solution $A^*(x)$ and the decomposition methods solution $\widetilde{A}^*(x)$ accordingly degenerate into the fuzzy sets and coincide with the solution $A^*(x)$ given by Theorem 31.*

Proof. According to Theorem 38, we should only prove that $A_f^*(x) = 1 - A_t^*(x)$. Since $A(x)$, $B(y)$, and $B^*(y)$ degenerate into the fuzzy sets, then $A_f(x) = 1 - A_t(x)$, $B_f(x) = 1 - B_t(x)$, and $B_f^*(x) = 1 - B_t^*(x)$. It follows from Proposition 27 and Corollary 35 that

$$\begin{aligned} A_f^*(x) &= \vee_{y\in Y}\left\{\left(B_f(y) \longrightarrow A_f(x)\right) \otimes B_f^*(y)\right\} \\ &= \vee_{y\in Y}\left\{\left(1 - A_t(x) \ominus B_t(y)\right) \otimes B_f^*(y)\right\} \\ &= \vee_{y\in Y}\left\{\left(1 - A_t(x) \ominus B_t(y)\right) \otimes \left(1 - B_t^*(y)\right)\right\} \\ &= 1 - \wedge_{y\in Y}\left\{\left(A_t(x) \ominus B_t(y)\right) \oplus B_t^*(y)\right\} \\ &= 1 - A_t^*(x). \end{aligned} \tag{57}$$

□

Theorem 39. *Let $\ominus_{L^*}$ be the residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm, then the Triple D solution A^* of IFMT given by (47) is reductive; that is, $A^*(x) = A(x)$ whenever $B^*(y) = B(y)$ satisfying $\exists y_0 \in Y$ such that $B(y_0) = 0^*$.*

Proof. If $B^*(y) = B(y)$, then

$$A^*(x) = \wedge_{y\in Y}\left\{\left(A(x) \ominus_{L^*} B(y)\right) \oplus_{L^*} B(y)\right\}. \tag{58}$$

It follows from Proposition 27 (13) that $(A(x) \le (A(x) \ominus_{L^*} B(y)) \oplus_{L^*} B(y)$, so $A(x) \le A^*(x)$. On the other hand, $A^*(x) \le (A(x) \ominus_{L^*} B(y_0)) \oplus_{L^*} B(y_0) = (A(x) \ominus_{L^*} 0^*) \oplus_{L^*} 0^* = A(x)$. Thus $A^*(x) = A(x)$. □

It indicates that the Triple D method is more meaningful than the Triple I method in point of reductivity for IFMT.

We know that the Triple D method of FMT is reductive; it is easy to prove that the decomposition method of IFMT is reductive.

Theorem 40. *Let $\ominus_{L^*}$ be the residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm; then the decomposition method solution $\widetilde{A}^*(x)$ of IFMT given by (56) is reductive; that is, $A^*(x) = A(x)$ whenever $B^*(y) = B(y)$ satisfying $\exists y_0 \in Y$ such that $B(y_0) = 0^*$.*

Taking into account 0^* being the smallest element of L^* in the Triple D Principle of IFMT, we propose the α-Triple D principle as follows:

$A^*(x)$ should be the biggest intuitionistic fuzzy set on X satisfying

$$\left(A^*(x) \ominus_{L^*} B^*(y)\right) \ominus_{L^*} \left(A(x) \ominus_{L^*} B(y)\right) \le \alpha \tag{59}$$

under the order of L^* where $\alpha \in L^*$.

Theorem 41. *Let $\ominus_{L^*}$ be the residual intuitionistic fuzzy difference operator derived from a left-continuous t-norm; the expression of the α-Triple D solution A^* for IFMT is as follows:*

$$A^*(x) = \wedge_{y\in Y}\left\{\alpha \oplus_{L^*} \left(\left(A(x) \ominus_{L^*} B(y)\right) \oplus_{L^*} B^*(y)\right)\right\}. \tag{60}$$

6. Conclusion

In [27], the Triple I method and the decomposition method of IFMP were first presented and the reductivity of methods

were verified; however, it was confirmed that the Triple I method of IFMT satisfied the local reductivity instead of reductivity. In order to achieve the improvement of reductivity of Triple I method for IFMT, the Triple D method and the decomposition method of IFMT are presented and the reductivity of methods is proved. Moreover, the concepts of the intuitionistic fuzzy difference operators and coadjoint pair are proposed, and the unified form of the residual intuitionistic fuzzy difference operators adjoint to intuitionistic t-conorms derived from left-continuous t-norms is provided.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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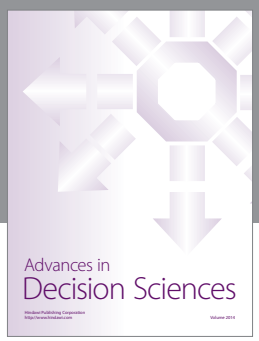

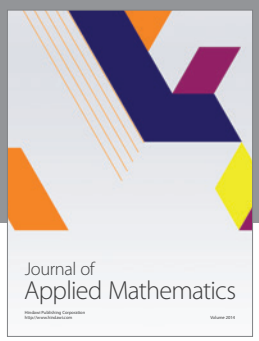

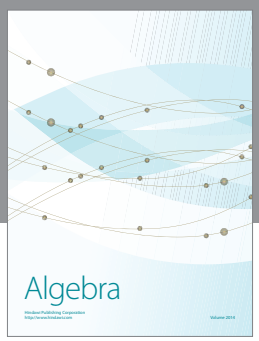

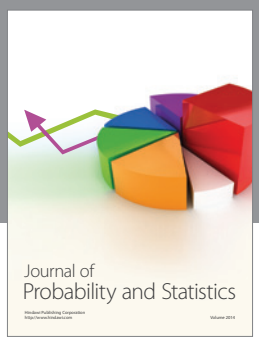

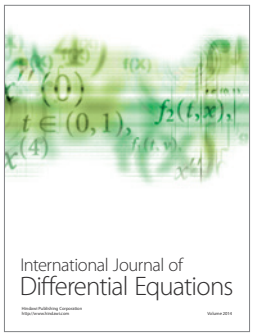

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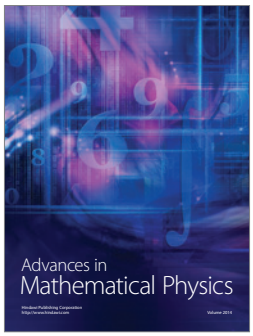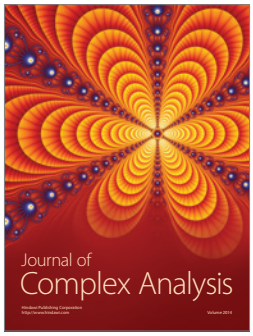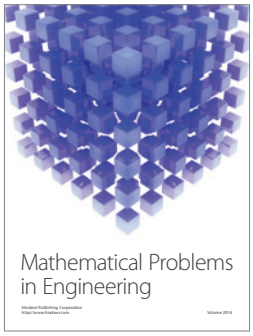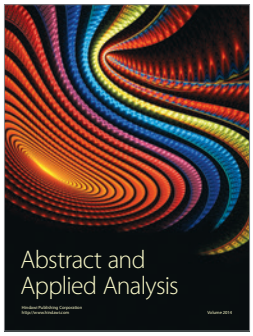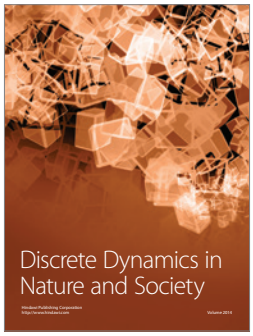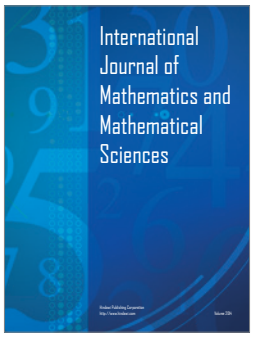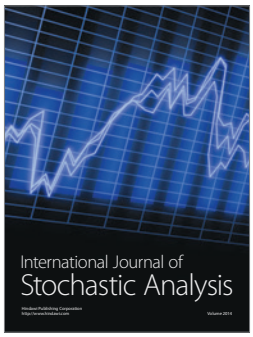
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