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Research Article

The $(G'/G, 1/G)$-Expansion Method and Its Applications to Find the Exact Solutions of Nonlinear PDEs for Nanobiosciences

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The two-variable $(G'/G, 1/G)$-expansion method is employed to construct exact traveling wave solutions with parameters of nanobiosciences partial differential equation. When the parameters are replaced by special values, the solitary wave solutions and the periodic wave solutions of this equation have been obtained from the traveling waves. This method can be thought of as the generalization of well-known original (G'/G)-expansion method proposed by M. Wang et al. It is shown that the two-variable $(G'/G, 1/G)$-expansion method provides a more powerful mathematical tool for solving many other nonlinear PDEs in mathematical physics. Comparison between our results and the well-known results is given.

1. Introduction

In recent years, investigations of exact solutions to nonlinear partial differential equations (PDEs) play an important role in the study of nonlinear physical phenomena. Many powerful methods for finding these exact solutions have been presented, such as the inverse scattering method [1], the Hirota bilinear transform method [2], the truncated Painleve expansion method [3–6], the Backlund transform method [7, 8], the exp-function method [9–13], the tanh-function method [14–17], the Jacobi elliptic function expansion method [18–20], the (G'/G)-expansion method [21–30], the modified (G'/G)-expansion method [31], the $(G'/G, 1/G)$-expansion method [32–35], the modified simple equation method [36], the multiple exp-function algorithm method [37], the transformed rational function method [38], the local fractional variation iteration method [39], and the local fractional series expansion method [40]. Further exact solutions to some real-life physical problems were already given in the recent articles [41–44]. The key idea of the one-variable (G'/G)-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in one variable (G'/G) in which $G = G(\xi)$ satisfies the second-order linear ODE $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are constants and $' = d/d\xi$. The key idea of the two variable $(G'/G, 1/G)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$ in which $G = G(\xi)$ satisfies the second-order linear ODE $G''(\xi) + \lambda G(\xi) = \mu$, where λ and μ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear PDEs. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulting from the process of using this method. Recently, Li et al. [32] have applied the $(G'/G, 1/G)$-expansion method and determined the exact solutions of Zakharov equations, while Zayed et al. [33–35] have used this method to find the exact solutions of the combined KdV-mKdV equation, the Kadomtsev-Petviashvili equation, and the potential YTSF equation, respectively.

The objective of this paper is to apply the two-variable $(G'/G, 1/G)$-expansion method to find the exact traveling

wave solutions of the following nonlinear partial differential equation of nanobiosciences [45, 46]:

$$\frac{\ell^2}{3}u_{xxx} + \frac{Z^{3/2}}{\ell}\left(\chi G_0 - 2\delta C_0\right)uu_t + 2u_x + \frac{ZC_0}{\ell}u_t + \frac{1}{\ell}\left(RZ^{-1} - G_0 Z\right)u = 0, \tag{1}$$

where $R = 0.34 \times 10^9\,\Omega$ is the resistance of the ER with length $\ell = 8 \times 10^{-9}$ m, $C_0 = 1.8 \times 10^{-15}$ F is total maximal capacitance of the ER, $G_0 = 1.1 \times 10^{-13}$ Si is conductance of pertaining NPs, and $Z = 5.56 \times 10^{10}\,\Omega$ is the characteristic impedance of our system. Parameters δ and χ describe nonlinearity of ER capacitor and conductance of NPs in RE, respectively.

The rest of this paper is organized as follows: In Section 2, we give the description of the two-variable $(G'/G, 1/G)$-expansion method. In Section 3, we apply this method to solve (1). In Section 4, some conclusions are given.

2. Description of the Two-Variable $(G'/G, 1/G)$-Expansion Method

Before we describe the main steps of this method, we need the following remarks (see [32–35]).

Remark 1. If we consider the second-order linear ODE

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{2}$$

and set $\phi = G'/G$, $\psi = 1/G$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi, \tag{3}$$

where λ and μ are constants.

Remark 2. If $\lambda < 0$, then the general solution of (2) has the form

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{4}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma_1 + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{5}$$

where $\sigma_1 = A_1^2 - A_2^2$.

Remark 3. If $\lambda > 0$, then the general solution of (2) has the form

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{6}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma_2 - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{7}$$

where $\sigma_2 = A_1^2 + A_2^2$.

Remark 4. If $\lambda = 0$, then the general solution of (2) has the form

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{8}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right). \tag{9}$$

Suppose we have the following nonlinear evolution equation:

$$F\left(u, u_t, u_x, u_{xx}, \ldots\right) = 0, \tag{10}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives. In the following, we give the main steps of the $(G'/G, 1/G)$-expansion method [32–35].

Step 1. The traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{11}$$

where c is a nonzero constant, reduces (10) to an ODE in the form

$$P\left(u, u', u'', \ldots\right) = 0, \tag{12}$$

where P is a polynomial of $u(\xi)$ and its total derivatives with respect to ξ.

Step 2. Assume that the solution of (12) can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{13}$$

where a_i $(i = 0, 1, 2, \ldots, N)$ and b_i $(i = 1, 2, \ldots, N)$ are constants to be determined later.

Step 3. Determine the positive integer N in (13) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in (12).

Step 4. Substitute (13) into (12) along with (3) and (5); the left-hand side of (12) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than one. Equating each coefficient of this polynomial to zero yields a system of algebraic equations which can be solved by using the Maple or Mathematica to get the values of a_i, b_i, c, μ, A_1, A_2, and λ, where $\lambda < 0$.

Step 5. Similar to Step 4, substituting (13) into (12) along with (3) and (7) for $\lambda > 0$, (or (3) and (9) for $\lambda = 0$), we obtain the exact solutions of (12) expressed by trigonometric functions (or by rational functions), respectively.

3. An Application

In this section, we will apply the method described in Section 2 to find the exact traveling wave solutions of (1)

which describe models of microtubules (MTs) as nonlinear transmission lines. The physical details of derivation of (1) describing its ionic currents are elaborated in [45].

In order to solve (1) by the two-variable $(G'/G, 1/G)$-expansion method, we see that the traveling wave transformation (11) with $\xi = (1/\ell)x - c(t/\tau)$ and the characteristic time of charging ER capacitor is $\tau = RC_0 = 0.6\times 10^{-6}$ s. Thus (1) takes the form of ODE

$$u''' + Acuu' + (6 - Bc)\,u' + Cu = 0, \tag{14}$$

where $A = 3(Z^{3/2}/\tau)(2\delta C_0 - \chi G_0)$, $B = 3(ZC_0/\tau)$, $C = 3(RZ^{-1} - G_0 Z)$.

By balancing u''' with uu' in (14), we get $N = 2$. Consequently, we get

$$\begin{aligned} u(\xi) = {} & a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) \\ & + b_1\psi(\xi) + b_2\phi(\xi)\psi(\xi), \end{aligned} \tag{15}$$

where a_0, a_1, a_2, b_1, and b_2 are constants to be determined later. There are three cases to be discussed as follows.

Case 1 (hyperbolic function solutions ($\lambda < 0$)). If $\lambda < 0$, substituting (15) into (14) and using (3) and (5), the left-hand side of (14) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in $a_0, a_1, a_2, b_1, b_2, \mu, \lambda$, and c as follows:

$$\phi^5: \frac{2Ac\lambda b_2^2}{\sigma_1\lambda^2+\mu^2} - 2Aca_2^2 - 24a_2 = 0,$$

$$\begin{aligned} \phi^4: {} & \frac{3Ac\lambda b_1 b_2}{\sigma_1\lambda^2+\mu^2} - 3Aca_1a_2 - \frac{36\lambda\mu b_2}{\sigma_1\lambda^2+\mu^2} \\ & - 6a_1 - \frac{3Ac\lambda\mu a_2 b_2}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\phi^4\psi: -24b_2 - 4Aca_2b_2 = 0,$$

$$\begin{aligned} \phi^3: {} & 2Bca_2 - 40\lambda a_2 - 12a_2 - Aca_1^2 \\ & + \frac{24\lambda\mu^2 a_2}{\sigma_1\lambda^2+\mu^2} - 2Aca_0a_2 - \frac{6\lambda\mu b_1}{\sigma_1\lambda^2+\mu^2} - 2Ac\lambda a_2^2 \\ & + \frac{3Ac\lambda^2 b_2^2}{\sigma_1\lambda^2+\mu^2} + \frac{Ac\lambda b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2Ac\lambda^2\mu^2 b_2^2}{(\mu^2+\lambda^2\sigma_1)^2} \\ & - \frac{2Ac\lambda\mu a_1 b_2}{\sigma_1\lambda^2+\mu^2} - \frac{2Ac\lambda\mu a_2 b_1}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\begin{aligned} \phi^3\psi: {} & 54\mu a_2 - 6b_1 - 3Aca_1b_2 - 3Aca_2b_1 \\ & + 2Ac\mu a_2^2 - \frac{5Ac\lambda\mu b_2^2}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\begin{aligned} \phi^2: {} & Ca_2 - 6a_1 - 8\lambda a_1 + Bca_1 \\ & + \frac{3\lambda\mu^2 a_1}{\sigma_1\lambda^2+\mu^2} - \frac{47\lambda^2\mu b_2}{\sigma_1\lambda^2+\mu^2} - Aca_0a_1 - \frac{6\lambda\mu b_2}{\sigma_1\lambda^2+\mu^2} \\ & + \frac{12\lambda^2\mu^3 b_2}{(\sigma_1\lambda^2+\mu^2)^2} - 3Ac\lambda a_1a_2 + \frac{4Ac\lambda^2 b_1b_2}{\sigma_1\lambda^2+\mu^2} \\ & + \frac{Bc\lambda\mu b_2}{\sigma_1\lambda^2+\mu^2} - \frac{2Ac\lambda^2\mu^2 b_1b_2}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{3Ac\lambda^2\mu a_2b_2}{\sigma_1\lambda^2+\mu^2} \\ & - \frac{Ac\lambda\mu a_0b_2}{\sigma_1\lambda^2+\mu^2} - \frac{Ac\lambda\mu a_1b_1}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\begin{aligned} \phi^2\psi: {} & 12\mu a_1 - 12b_2 - 28\lambda b_2 + 2Bcb_2 \\ & + \frac{78\lambda\mu^2 b_2}{\sigma_1\lambda^2+\mu^2} - 2Aca_0b_2 - 2Aca_1b_1 + 3Ac\mu a_1a_2 \\ & - 3Ac\lambda a_2b_2 + \frac{6Ac\lambda\mu^2 a_2b_2}{\sigma_1\lambda^2+\mu^2} - \frac{7Ac\lambda\mu b_1b_2}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\begin{aligned} \phi: {} & Ca_1 - 12\lambda a_2 - 16\lambda^2 a_2 - \frac{6\lambda^2\mu b_1}{\sigma_1\lambda^2+\mu^2} \\ & + 2Bc\lambda a_2 + \frac{24\lambda^2\mu^2 a_2}{\sigma_1\lambda^2+\mu^2} - Ac\lambda a_1^2 + \frac{Ac\lambda^2 b_1^2}{\sigma_1\lambda^2+\mu^2} \\ & + \frac{Ac\lambda^3 b_2^2}{\sigma_1\lambda^2+\mu^2} - 2Ac\lambda a_0a_2 - \frac{2Ac\lambda^3\mu^2 b_2^2}{(\sigma_1\lambda^2+\mu^2)^2} \\ & - \frac{2Ac\lambda^2\mu a_1b_2}{\sigma_1\lambda^2+\mu^2} - \frac{2Ac\lambda^2\mu a_2b_1}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\begin{aligned} \phi\psi: {} & Cb_2 - 6b_1 - 5\lambda b_1 + 12\mu a_2 + Bcb_1 \\ & + 40\lambda\mu a_2 + \frac{12\lambda\mu^2 b_1}{\sigma_1\lambda^2+\mu^2} - \frac{48\lambda\mu^3 a_2}{\sigma_1\lambda^2+\mu^2} - 2Bc\mu a_2 \\ & - Aca_0b_1 + Ac\mu a_1^2 + 2Ac\mu a_0a_2 - 2Ac\lambda a_1b_2 \\ & - 2Ac\lambda a_2b_1 - \frac{2Ac\lambda\mu b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{4Ac\lambda^2\mu^3 b_2^2}{(\sigma_1\lambda^2+\mu^2)^2} \\ & - \frac{3Ac\lambda^2\mu b_2^2}{\sigma_1\lambda^2+\mu^2} + \frac{4Ac\lambda\mu^2 a_1b_2}{\sigma_1\lambda^2+\mu^2} + \frac{4Ac\lambda\mu^2 a_2b_1}{\sigma_1\lambda^2+\mu^2} = 0, \end{aligned}$$

$$\begin{aligned} \psi: {} & Cb_1 + 6\mu a_1 - 6\lambda b_2 - 5\lambda^2 b_2 + 5\lambda\mu a_1 \\ & - \frac{6\lambda\mu^3 a_1}{\sigma_1\lambda^2+\mu^2} + \frac{12\lambda\mu^2 b_2}{\sigma_1\lambda^2+\mu^2} - Bc\mu a_1 + Bc\lambda b_2 \\ & + \frac{28\lambda^2\mu^2 b_2}{\sigma_1\lambda^2+\mu^2} - \frac{24\lambda^2\mu^4 b_2}{(\sigma_1\lambda^2+\mu^2)^2} + Ac\mu a_0a_1 - Ac\lambda a_0b_2 \\ & - Ac\lambda a_1b_1 - \frac{2Bc\lambda\mu^2 b_2}{\sigma_1\lambda^2+\mu^2} + \frac{4Ac\lambda^2\mu^3 b_1b_2}{(\sigma_1\lambda^2+\mu^2)^2} \end{aligned}$$

$$+\frac{2Ac\lambda\mu^2a_0b_2}{\sigma_1\lambda^2+\mu^2}+\frac{2Ac\lambda\mu^2a_1b_1}{\sigma_1\lambda^2+\mu^2}-\frac{3Ac\lambda^2\mu b_1b_2}{\sigma_1\lambda^2+\mu^2}=0,$$

$$\begin{aligned}\phi^0:\ & Ca_0-6\lambda a_1-2\lambda^2a_1-\frac{6\lambda^2\mu b_2}{\sigma_1\lambda^2+\mu^2}\\ &-\frac{11\lambda^3\mu b_2}{\sigma_1\lambda^2+\mu^2}+Bc\lambda a_1+\frac{3\lambda^2\mu^2a_1}{\sigma_1\lambda^2+\mu^2}+\frac{12\lambda^3\mu^3b_2}{(\sigma_1\lambda^2+\mu^2)^2}\\ &-Ac\lambda a_0a_1+\frac{Bc\lambda^2\mu b_2}{\sigma_1\lambda^2+\mu^2}+\frac{Ac\lambda^3b_1b_2}{\sigma_1\lambda^2+\mu^2}\\ &-\frac{2Ac\lambda^3\mu^2b_1b_2}{(\sigma_1\lambda^2+\mu^2)^2}-\frac{Ac\lambda^2\mu a_0b_2}{\sigma_1\lambda^2+\mu^2}=0.\end{aligned}\tag{16}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Result 1. Consider

$$\begin{gathered}a_0=\frac{B}{A}\pm\frac{b_2(5\lambda+6)\sqrt{-\lambda}}{6\sqrt{\lambda^2\sigma_1+\mu^2}},\\ a_1=0,\qquad a_2=\pm\frac{b_2\sqrt{-\lambda}}{\sqrt{\lambda^2\sigma_1+\mu^2}},\qquad b_1=\pm\frac{\mu b_2\sqrt{-\lambda}}{\sqrt{\lambda^2\sigma_1+\mu^2}},\\ c=\pm\frac{6}{b_2A}\sqrt{\frac{\lambda^2\sigma_1+\mu^2}{-\lambda}},\qquad b_2=b_2,\qquad \mu=\mu,\qquad C=0.\end{gathered}\tag{17}$$

From (4), (15), and (17), we deduce the traveling wave solution of (1) as follows:

$$\begin{aligned}u(\xi)=&\frac{B}{A}\pm\frac{b_2(5\lambda+6)\sqrt{-\lambda}}{6\sqrt{\lambda^2\sigma_1+\mu^2}}\pm\frac{b_2\lambda\sqrt{-\lambda}}{\sqrt{\lambda^2\sigma_1+\mu^2}}\\ &\times\left[\frac{A_1\cosh\left(\xi\sqrt{-\lambda}\right)+A_2\sinh\left(\xi\sqrt{-\lambda}\right)}{A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\mu/\lambda}\right]^2\\ &+\left[\frac{\sqrt{-\lambda}b_2}{A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\mu/\lambda}\right]\\ &\times\left[\frac{A_1\cosh\left(\xi\sqrt{-\lambda}\right)+A_2\sinh\left(\xi\sqrt{-\lambda}\right)}{A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\mu/\lambda}\right.\\ &\left.\pm\frac{\mu}{\sqrt{\lambda^2\sigma_1+\mu^2}}\right],\end{aligned}\tag{18}$$

where b_2 is a nonzero constant, $\xi=(1/\ell)x\pm(6/b_2A)\sqrt{(\lambda^2\sigma_1+\mu^2)/-\lambda}(t/\tau)$.

In particular, by setting $A_1=0$, $A_2\neq0$, and $\mu=0$ in (18), we have the solitary wave solution

$$\begin{aligned}u(\xi)=&\frac{B}{A}\pm i\frac{b_2}{A_2}\left(\frac{5\lambda+6}{6\sqrt{-\lambda}}\right)+\frac{b_2\sqrt{-\lambda}}{A_2}\\ &\times\tanh\left(\xi\sqrt{-\lambda}\right)\left[\operatorname{sech}\left(\xi\sqrt{-\lambda}\right)\pm i\tanh\left(\xi\sqrt{-\lambda}\right)\right],\\ &i=\sqrt{-1},\end{aligned}\tag{19}$$

while if $A_1\neq0$, $A_2=0$, and $\mu=0$, then we have the solitary wave solution

$$\begin{aligned}u(\xi)=&\frac{B}{A}\pm\frac{b_2}{A_1}\left(\frac{5\lambda+6}{6\sqrt{-\lambda}}\right)+\frac{b_2\sqrt{-\lambda}}{A_1}\\ &\times\coth\left(\xi\sqrt{-\lambda}\right)\left[\operatorname{csch}\left(\xi\sqrt{-\lambda}\right)\pm\coth\left(\xi\sqrt{-\lambda}\right)\right].\end{aligned}\tag{20}$$

Result 2. Consider

$$\begin{gathered}a_0=a_0,\qquad a_1=0,\qquad a_2=\frac{6(Aa_0-B)}{A(3+4\lambda)},\\ b_1=0,\qquad b_2=0,\qquad c=-\frac{2(3+4\lambda)}{Aa_0-B},\\ \mu=0,\qquad C=0.\end{gathered}\tag{21}$$

In this result, we deduce the traveling wave solution of (1) as follows:

$$\begin{aligned}u(\xi)=&a_0-\frac{6\lambda(Aa_0-B)}{A(3+4\lambda)}\\ &\times\left[\frac{A_1\cosh\left(\xi\sqrt{-\lambda}\right)+A_2\sinh\left(\xi\sqrt{-\lambda}\right)}{A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)}\right]^2,\end{aligned}\tag{22}$$

where a_0 is an arbitrary constant and $\xi=(1/\ell)x+(2(3+4\lambda)/(Aa_0-B))(t/\tau)$.

In particular, by setting $A_1=0$ and $A_2\neq0$ in (22), we have the solitary wave solution (Figure 1)

$$u(\xi)=a_0-\frac{6\lambda(Aa_0-B)}{A(3+4\lambda)}\tanh^2\left(\xi\sqrt{-\lambda}\right),\tag{23}$$

while if $A_1\neq0$ and $A_2=0$, then we have the solitary wave solution

$$u(\xi)=a_0-\frac{6\lambda(Aa_0-B)}{A(3+4\lambda)}\coth^2\left(\xi\sqrt{-\lambda}\right).\tag{24}$$

Note that our results (23) and (24) are in agreement with the results u_i $(i=1,2)$ obtained in page 1251 of [47] with interchanges $\lambda\leftrightarrow qr$ and with the results (28) and (29) in page 1544 of [48] with the interchanges $\lambda\leftrightarrow\mu$.

We close this case with the remark that our results (18)–(20) and (22) are new and are not reported elsewhere.

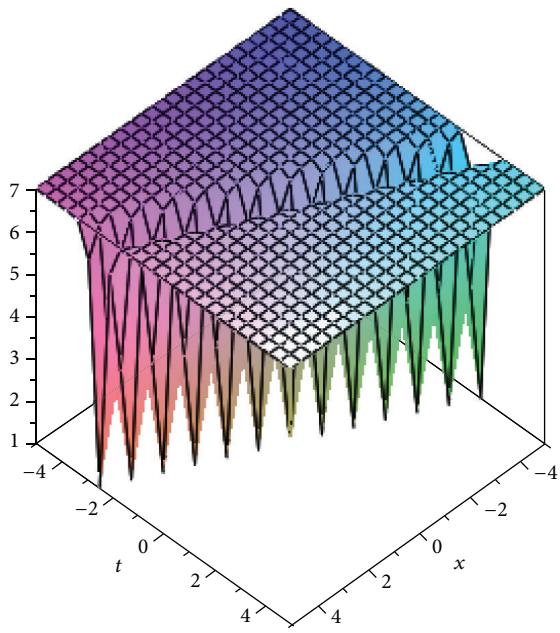

FIGURE 1: The plot of solution (23) when $\lambda = -1$, $A = 1$, $B = 2$, $a_0 = 1$, $\tau = \ell = 1/2$.

Case 2 (trigonometric function solution ($\lambda > 0$)). If $\lambda > 0$, substituting (15) into (14) and using (3) and (7), the left-hand side of (14) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, a_2, b_1, b_2, μ, λ, and c as follows:

$$\phi^5: -24a_2 + \frac{2Ac\lambda b_2^2}{\mu^2 - \sigma_2\lambda^2} - 2Aca_2^2 = 0,$$

$$\phi^4: -6a_1 - \frac{36\lambda\mu b_2}{\mu^2 - \sigma_2\lambda^2} + \frac{3Ac\lambda b_1 b_2}{\mu^2 - \sigma_2\lambda^2} - 3Aca_1a_2 - \frac{3Ac\lambda\mu a_2 b_2}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\phi^4\psi: -24b_2 - 4Aca_2b_2 = 0,$$

$$\phi^3: 2Bca_2 - 40\lambda a_2 - 12a_2 + \frac{24\lambda\mu^2 a_2}{\mu^2 - \sigma_2\lambda^2} - \frac{6\lambda\mu b_1}{\mu^2 - \sigma_2\lambda^2} + \frac{3Ac\lambda^2 b_2^2}{\mu^2 - \sigma_2\lambda^2} - Aca_1^2 - 2Aca_0a_2 - 2Ac\lambda a_2^2 + \frac{Ac\lambda b_1^2}{\mu^2 - \sigma_2\lambda^2} - \frac{2Ac\lambda^2\mu^2 b_2^2}{(\sigma_2\lambda^2 - \mu^2)^2} - \frac{2Ac\lambda\mu a_1 b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{2Ac\lambda\mu a_2 b_1}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\phi^3\psi: -6b_1 + 54\mu a_2 + 2Ac\mu a_2^2 - 3Aca_2b_1 - 3Aca_1b_2 - \frac{5Ac\lambda\mu b_2^2}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\phi^2: -8\lambda a_1 - 6a_1 + Bca_1 + \frac{3\lambda\mu^2 a_1}{\mu^2 - \sigma_2\lambda^2} - \frac{47\lambda^2\mu b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{6\lambda\mu b_2}{\mu^2 - \sigma_2\lambda^2} + \frac{12\lambda^2\mu^3 b_2}{(\sigma_2\lambda^2 - \mu^2)^2} + \frac{Bc\lambda\mu b_2}{\mu^2 - \sigma_2\lambda^2} + Ca_2 - Aca_0a_1 - 3Ac\lambda a_1a_2 + \frac{4Ac\lambda^2 b_1b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{2Ac\lambda^2\mu^2 b_1b_2}{(\sigma_2\lambda^2 - \mu^2)^2} - \frac{3Ac\lambda^2\mu a_2b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{Ac\lambda\mu a_0b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{Ac\lambda\mu a_1b_1}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\phi^2\psi: 12\mu a_1 - 12b_2 - 28\lambda b_2 + 2Bcb_2 + \frac{78\lambda\mu^2 b_2}{\mu^2 - \sigma_2\lambda^2} + 3Ac\mu a_1a_2 - 2Aca_1b_1 - 2Aca_0b_2 - 3Ac\lambda a_2b_2 + \frac{6Ac\lambda\mu^2 a_2b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{7Ac\lambda\mu b_1b_2}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\phi: 2Bc\lambda a_2 - 12\lambda a_2 - 16\lambda^2 a_2 - \frac{6\lambda^2\mu b_1}{\mu^2 - \sigma_2\lambda^2} + \frac{24\lambda^2\mu^2 a_2}{\mu^2 - \sigma_2\lambda^2} + Ca_1 - Ac\lambda a_1^2 + \frac{Ac\lambda^2 b_1^2}{\mu^2 - \sigma_2\lambda^2} + \frac{Ac\lambda^3 b_2^2}{\mu^2 - \sigma_2\lambda^2} - 2Ac\lambda a_0a_2 - \frac{2Ac\lambda^3\mu^2 b_2^2}{(\sigma_2\lambda^2 - \mu^2)^2} - \frac{2Ac\lambda^2\mu a_1b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{2Ac\lambda^2\mu a_2b_1}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\phi\psi: 12\mu a_2 - 5\lambda b_1 - 6b_1 + Bcb_1 + 40\lambda\mu a_2 + \frac{12\lambda\mu^2 b_1}{\mu^2 - \sigma_2\lambda^2} - \frac{48\lambda\mu^3 a_2}{\mu^2 - \sigma_2\lambda^2} - 2Bc\mu a_2 + Cb_2 - Aca_0b_1 + Ac\mu a_1^2 + 2Ac\mu a_0a_2 - 2Ac\lambda a_1b_2 - 2Ac\lambda a_2b_1 - \frac{2Ac\lambda\mu b_1^2}{\mu^2 - \sigma_2\lambda^2} + \frac{4Ac\lambda^2\mu^3 b_2^2}{(\sigma_2\lambda^2 - \mu^2)^2} - \frac{3Ac\lambda^2\mu b_2^2}{\mu^2 - \sigma_2\lambda^2} + \frac{4Ac\lambda\mu^2 a_1b_2}{\mu^2 - \sigma_2\lambda^2} + \frac{4Ac\lambda\mu^2 a_2b_1}{\mu^2 - \sigma_2\lambda^2} = 0,$$

$$\psi: 6\mu a_1 - 6\lambda b_2 - 5\lambda^2 b_2 + 5\lambda\mu a_1 - \frac{6\lambda\mu^3 a_1}{\mu^2 - \sigma_2\lambda^2} + \frac{12\lambda\mu^2 b_2}{\mu^2 - \sigma_2\lambda^2} - Bc\mu a_1 + Bc\lambda b_2 + \frac{28\lambda^2\mu^2 b_2}{\mu^2 - \sigma_2\lambda^2} - \frac{24\lambda^2\mu^4 b_2}{(\sigma_2\lambda^2 - \mu^2)^2} - \frac{2Bc\lambda\mu^2 b_2}{\mu^2 - \sigma_2\lambda^2} + Cb_1$$

$$+ Ac\mu a_0 a_1 - Ac\lambda a_0 b_2 - Ac\lambda a_1 b_1 + \frac{4Ac\lambda^2\mu^3 b_1 b_2}{(\sigma_2\lambda^2-\mu^2)^2}$$
$$+ \frac{2Ac\lambda\mu^2 a_0 b_2}{\mu^2-\sigma_2\lambda^2} + \frac{2Ac\lambda\mu^2 a_1 b_1}{\mu^2-\sigma_2\lambda^2} - \frac{3Ac\lambda^2\mu b_1 b_2}{\mu^2-\sigma_2\lambda^2} = 0,$$
$$\phi^0: Bc\lambda a_1 - 6\lambda a_1 - 2\lambda^2 a_1 - \frac{6\lambda^2\mu b_2}{\mu^2-\sigma_2\lambda^2} - \frac{11\lambda^3\mu b_2}{\mu^2-\sigma_2\lambda^2}$$
$$+ \frac{3\lambda^2\mu^2 a_1}{\mu^2-\sigma_2\lambda^2} + \frac{12\lambda^3\mu^3 b_2}{(\sigma_2\lambda^2-\mu^2)^2} + \frac{Bc\lambda^2\mu b_2}{\mu^2-\sigma_2\lambda^2}$$
$$+ Ca_0 - Ac\lambda a_0 a_1 + \frac{Ac\lambda^3 b_1 b_2}{\mu^2-\sigma_2\lambda^2} - \frac{2Ac\lambda^3\mu^2 b_1 b_2}{(\sigma_2\lambda^2-\mu^2)^2}$$
$$- \frac{Ac\lambda^2\mu a_0 b_2}{\mu^2-\sigma_2\lambda^2} - \frac{Ac\lambda^2\mu a_1 b_1}{\mu^2-\sigma_2\lambda^2} = 0. \tag{25}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Result 1. Consider

$$a_0 = \frac{B}{A} \pm \frac{b_2(5\lambda+6)\sqrt{\lambda}}{6\sqrt{\lambda^2\sigma_2-\mu^2}}, \qquad a_1 = 0, \qquad a_2 = \pm\frac{b_2\sqrt{\lambda}}{\sqrt{\lambda^2\sigma_2-\mu^2}},$$
$$b_1 = \pm\frac{\mu b_2\sqrt{\lambda}}{\sqrt{\lambda^2\sigma_2-\mu^2}}, \qquad c = \pm\frac{6}{b_2 A}\sqrt{\frac{\lambda^2\sigma_2-\mu^2}{\lambda}},$$
$$b_2 = b_2, \qquad \mu = \mu, \qquad C = 0. \tag{26}$$

From (6), (15), and (26), we deduce the traveling wave solution of (1) as follows:

$$u(\xi) = \frac{B}{A} \pm \frac{b_2(5\lambda+6)\sqrt{\lambda}}{6\sqrt{\lambda^2\sigma_2-\mu^2}} \pm \frac{\lambda b_2\sqrt{\lambda}}{\sqrt{\lambda^2\sigma_2-\mu^2}}$$
$$\times\left[\frac{A_1\cos(\xi\sqrt{\lambda}) - A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda}\right]^2$$
$$+\left(\frac{\sqrt{\lambda}b_2}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda}\right) \tag{27}$$
$$\times\left[\frac{A_1\cos(\xi\sqrt{\lambda}) - A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda} \pm \frac{\mu}{\sqrt{\lambda^2\sigma_2-\mu^2}}\right],$$

where b_2 is a nonzero constant, $\xi = (1/\ell)x \pm (6/b_2 A)\sqrt{(\lambda^2\sigma_2-\mu^2)/\lambda}(t/\tau)$.

In particular, by setting $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ in (27), we have the periodic solution

$$u(\xi) = \frac{B}{A} \pm \frac{b_2}{A_2}\left(\frac{5\lambda+6}{6\sqrt{\lambda}}\right) - \frac{\sqrt{\lambda}b_2}{A_2}\tan(\xi\sqrt{\lambda})\left[\sec(\xi\sqrt{\lambda}) \pm \tan(\xi\sqrt{\lambda})\right], \tag{28}$$

while if $A_1 \neq 0$, $A_2 = 0$, and $\mu = 0$, then we have the periodic solution

$$u(\xi) = \frac{B}{A} \pm \frac{b_2}{A_1}\left(\frac{5\lambda+6}{6\sqrt{\lambda}}\right) + \frac{\sqrt{\lambda}b_2}{A_1}\cot(\xi\sqrt{\lambda})\left[\csc(\xi\sqrt{\lambda}) \pm \cot(\xi\sqrt{\lambda})\right]. \tag{29}$$

Result 2. Consider

$$a_0 = a_0, \qquad a_1 = 0, \qquad a_2 = \frac{6(Aa_0 - B)}{A(3+4\lambda)},$$
$$b_1 = 0, \qquad b_2 = 0, \qquad \mu = 0, \tag{30}$$
$$c = -\frac{2(3+4\lambda)}{Aa_0 - B}, \qquad C = 0.$$

In this result, we deduce the traveling wave solution of (1) as follows:

$$u(\xi) = a_0 + \frac{6\lambda(Aa_0 - B)}{A(3+4\lambda)} \times\left[\frac{A_1\cos(\xi\sqrt{\lambda}) - A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda})}\right]^2, \tag{31}$$

where a_0 is an arbitrary constant and $\xi = (1/\ell)x + (2(3+4\lambda)/(Aa_0 - B))(t/\tau)$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (31), we have the periodic solution (Figure 2)

$$u(\xi) = a_0 + \frac{6\lambda(Aa_0 - B)}{A(3+4\lambda)}\tan^2(\xi\sqrt{\lambda}), \tag{32}$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solution

$$u(\xi) = a_0 + \frac{6\lambda(Aa_0 - B)}{A(3+4\lambda)}\cot^2(\xi\sqrt{\lambda}). \tag{33}$$

Note that our results (32) and (33) are in agreement with the results u_{13} and u_{14} obtained in page 1251 of [47] with interchanges $\lambda \leftrightarrow qr$ and with the results (33) and (34) in page 1544 of [48] with the interchanges $\lambda \leftrightarrow \mu$.

We close this case with the remark that our results (27)–(29) and (31) are new and are not reported elsewhere.

Case 3 (rational function solutions ($\lambda = 0$)). If $\lambda = 0$, substituting (15) into (14) and using (3) and (9), the left-hand side of (14) becomes a polynomial in ϕ and ψ. Setting the

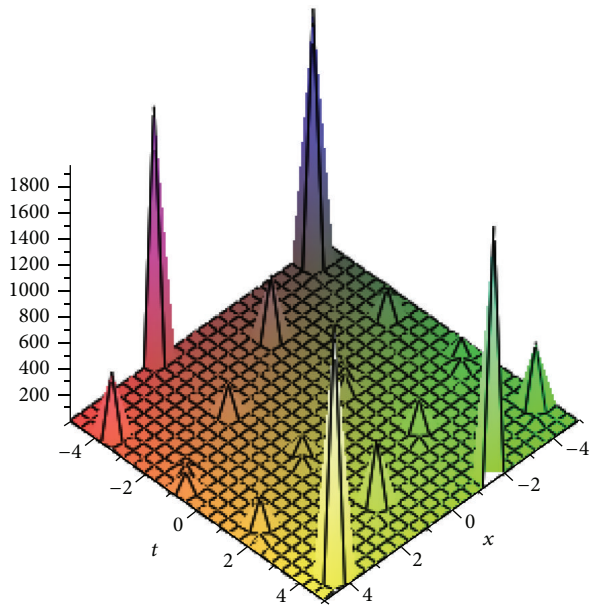

FIGURE 2: The plot of solution (32) when $\lambda = 1$, $A = 8$, $B = 7$, $a_0 = 1$, $\tau = \ell = 1/2$.

coefficients of this polynomial to be zero yields a system of algebraic equations in $a_0, a_1, a_2, b_1, b_2, \mu$, and c as follows:

$$\begin{aligned}
\phi^5 &: -24a_2 - \frac{2Acb_2^2}{A_1^2 - 2\mu A_2} - 2Aca_2^2 = 0,\\
\phi^4 &: \frac{36b_2\mu}{A_1^2 - 2\mu A_2} - 6a_1 - \frac{3Acb_1b_2}{A_1^2 - 2\mu A_2}\\
&\quad - 3Aca_1a_2 + \frac{3Aca_2b_2\mu}{A_1^2 - 2\mu A_2} = 0,\\
\phi^4\psi &: -24b_2 - 4Aca_2b_2 = 0,\\
\phi^3 &: -\frac{24a_2\mu^2}{A_1^2 - 2\mu A_2} + \frac{6b_1\mu}{A_1^2 - 2\mu A_2} - \frac{2Acb_2^2\mu^2}{(A_1^2 - 2\mu A_2)^2}\\
&\quad - 2Aca_0a_2 - \frac{Acb_1^2}{A_1^2 - 2\mu A_2} - Aca_1^2 + \frac{2Aca_1b_2\mu}{A_1^2 - 2\mu A_2}\\
&\quad + \frac{2Acb_1a_2\mu}{A_1^2 - 2\mu A_2} - 12a_2 + 2Bca_2 = 0,\\
\phi^3\psi &: -6b_1 + 54a_2\mu - 3Aca_1b_2 + 2Aca_2^2\mu\\
&\quad - 3Acb_1a_2 + \frac{5Acb_2^2\mu}{A_1^2 - 2\mu A_2} = 0,\\
\phi^2 &: -\frac{3a_1\mu^2}{A_1^2 - 2\mu A_2} + \frac{12b_2\mu^3}{(A_1^2 - 2\mu A_2)^2} - Aca_0a_1\\
&\quad + \frac{Aca_0b_2\mu}{A_1^2 - 2\mu A_2} + \frac{Acb_1a_1\mu}{A_1^2 - 2\mu A_2} - \frac{2Acb_1b_2\mu^2}{(A_1^2 - 2\mu A_2)^2}\\
&\quad - 6a_1 + \frac{6b_2\mu}{A_1^2 - 2\mu A_2} - \frac{Bcb_2\mu}{A_1^2 - 2\mu A_2}\\
&\quad + Bca_1 + Ca_2 = 0,\\
\phi^2\psi &: -\frac{78b_2\mu^2}{A_1^2 - 2\mu A_2} + 12a_1\mu - 2Aca_0b_2 - 2Aca_1b_1\\
&\quad + \frac{7Acb_1b_2\mu}{A_1^2 - 2\mu A_2} - \frac{6Aca_2b_2\mu^2}{A_1^2 - 2\mu A_2} + 3Aca_1a_2\mu\\
&\quad - 12b_2 + 2Bcb_2 = 0,\\
\phi &: Ca_1 = 0,\\
\phi\psi &: \frac{48a_2\mu^3}{A_1^2 - 2\mu A_2} - \frac{12b_1\mu^2}{A_1^2 - 2\mu A_2} - Aca_0b_1 + Aca_1^2\mu\\
&\quad - \frac{4Aca_1b_2\mu^2}{A_1^2 - 2\mu A_2} + 2Aca_0a_2\mu - \frac{4Acb_1a_2\mu^2}{A_1^2 - 2\mu A_2}\\
&\quad + \frac{4Acb_2^2\mu^3}{(A_1^2 - 2\mu A_2)^2} + \frac{2Acb_1^2\mu}{A_1^2 - 2\mu A_2}\\
&\quad + 12a_2\mu - 6b_1 - 2Bca_2\mu + Bcb_1 + Cb_2 = 0,\\
\psi &: \frac{6a_1\mu^3}{A_1^2 - 2\mu A_2} - \frac{24b_2\mu^4}{(A_1^2 - 2\mu A_2)^2} + Aca_0a_1\mu\\
&\quad - \frac{2Aca_0b_2\mu^2}{A_1^2 - 2\mu A_2} - \frac{2Acb_1a_1\mu^2}{A_1^2 - 2\mu A_2} + \frac{4Acb_1b_2\mu^3}{(A_1^2 - 2\mu A_2)^2}\\
&\quad + 6a_1\mu - \frac{12b_2\mu^2}{A_1^2 - 2\mu A_2} - Bca_1\mu\\
&\quad + \frac{2Bcb_2\mu^2}{A_1^2 - 2\mu A_2} + Cb_1 = 0,\\
\phi^0 &: Ca_0 = 0.
\end{aligned} \tag{34}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Result 1. Consider

$$\begin{gathered}
a_0 = \frac{Bc - 6}{Ac}, \qquad a_1 = 0, \qquad a_2 = -\frac{12}{Ac},\\
b_1 = 0, \qquad b_2 = 0, \qquad c = c, \qquad \mu = 0, \qquad C = 0.
\end{gathered} \tag{35}$$

From (8), (15), and (35), we deduce the traveling wave solution of (1) as follows:

$$u(\xi) = \frac{Bc - 6}{Ac} - \frac{12}{Ac}\left[\frac{A_1}{A_1\xi + A_2}\right]^2. \tag{36}$$

Result 2. Consider

$$a_0 = \frac{Bc-6}{Ac}, \qquad a_1 = 0, \qquad a_2 = -\frac{6}{Ac},$$
$$b_1 = \frac{36A_1^2 - A^2c^2b_2^2}{12cA_2A}, \qquad c = c, \qquad b_2 = b_2, \tag{37}$$
$$\mu = \frac{36A_1^2 - A^2c^2b_2^2}{72A_2}, \qquad C = 0.$$

In this result, we deduce the traveling wave solution of (1) as follows:

$$u(\xi) = \frac{Bc-6}{Ac} - \frac{(\mu\xi + A_1)[6(\mu\xi + A_1) - Acb_2]}{Ac((\mu/2)\xi^2 + A_1\xi + A_2)^2} + \frac{36A_1^2 - A^2c^2b_2^2}{12cA_2A((\mu/2)\xi^2 + A_1\xi + A_2)}, \tag{38}$$

where ξ in (36) and (38) is given by $\xi = (1/\ell)x - c(t/\tau)$. Our result (38) is new and is not reported elsewhere.

Remark 5. In all the above results, we have noted the term

$$C = 3\left(RZ^{-1} - G_0Z\right) = 0, \tag{39}$$

which is equivalent to the condition (21) of [45]. The condition (39) enables that the ohmic loss could be balanced by fresh ions injected from the nanopores in microtubules since these act ionic pumps in suitable voltage regime. Also, the authors [46] have obtained the condition (39) when they have discussed (1) using the extended tanh-function method. The same remark has been happened in [47, 48] using the improved Riccati equation mapping method and the improved (G'/G)-expansion method, respectively.

Remark 6. Since $C = 0$, then (14) becomes

$$u''' + Acuu' + (6 - Bc)u' = 0. \tag{40}$$

Integrating (40) once with respect to ξ with zero constant of integration, we get

$$u'' + \frac{1}{2}Acu^2 + (6 - Bc)u = 0. \tag{41}$$

Multiplying (41) by u' and integrating once with respect to ξ with zero constant of integration, we have

$$(u')^2 + \frac{1}{3}Acu^3 + (6 - Bc)u^2 = 0, \tag{42}$$

and hence we have

$$u' = \pm\frac{u}{\sqrt{3}}\sqrt{3(Bc-6) - Acu}. \tag{43}$$

Consequently, we have

$$\int \frac{du}{u\sqrt{3(Bc-6) - Acu}} = \pm\frac{(\xi + \xi_0)}{\sqrt{3}}, \tag{44}$$

where ξ_0 is a constant of integration.

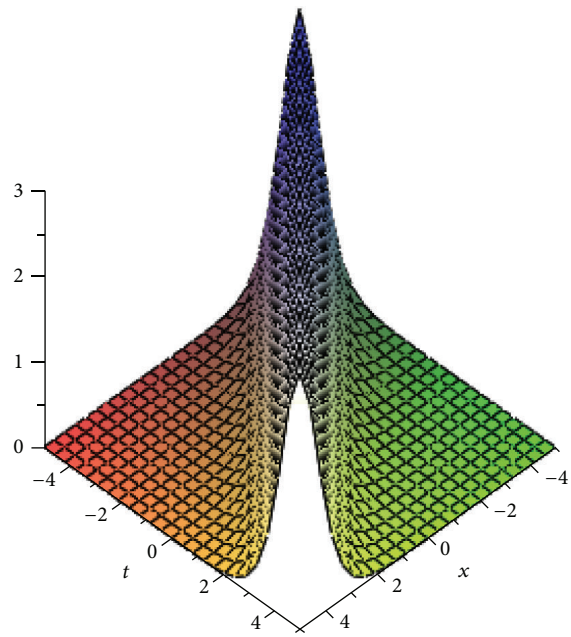

Figure 3: The plot of solution (45) when $c = 1$, $A = 1$, $B = 7$, $\tau = \ell = 1/2$, $\xi_0 = 0$.

From (44) we have the following solution (Figure 3):

$$u(\xi) = \frac{3(Bc-6)}{Ac}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{Bc-6}(\xi + \xi_0)\right], \tag{45}$$

which is in agreement with the solution (17) obtained in [48] when $a_0 = 0$, $\mu = 0$ using the improved (G'/G)-expansion method. After some simple calculation, the solution (45) is equivalent to the solution (34) obtained in [45].

Remark 7. Integrating (40) twice with respect to ξ, we get

$$(u')^2 = -\frac{Ac}{3}u^3 - (6 - Bc)u^2 + K_1u + K_2, \tag{46}$$

where K_1 and K_2 are nonzero constants of integration.Setting

$$u(\xi) = -\frac{(6 - Bc) + 12P(\xi)}{Ac}, \tag{47}$$

we obtain the Weierstrass equation [49, 50]

$$P'^2 = 4P^3 - g_2P - g_3, \tag{48}$$

where $g_2 = (AcK_1 + (6 - Bc)^2)/12$, $g_3 = (6 - Bc)^3/216 + (AcK_1(6 - Bc) - A^2c^2K_2)/144$, while P is the Weierstrass elliptic P-function.

If $g_2 = g_3 = 0$, (48) reduces to the equation $P'^2 = 4P^3$, which admits the elementary solution

$$P(\xi) = \frac{1}{(\xi + \xi_0)^2}, \tag{49}$$

where ξ_0 is a constant of integration. From (47) and (49), we get

$$u(\xi) = \frac{(Bc-6)}{Ac} - \frac{12}{Ac(\xi + \xi_0)^2}, \tag{50}$$

which is equivalent to the solution (36) for $A_1 = 1$, $A_2 = \xi_0$.

From the well-known solutions of (48), see, for example, [50], we rewrite the solution (47) in terms of the Jacobi elliptic functions as follows:

$$u_1(\xi) = -\frac{(6-Bc)}{Ac} - \frac{12}{Ac}\left[e_2 - (e_2 - e_3)\,\mathrm{cn}^2(R\xi; m)\right],$$
$$u_2(\xi) = -\frac{(6-Bc)}{Ac} - \frac{12}{Ac}\left[e_3 + (e_1 - e_3)\,\mathrm{ns}^2(R\xi; m)\right], \tag{51}$$

where $R = \sqrt{e_1 - e_3}$, $m^2 = (e_2 - e_3)/(e_1 - e_3)$ is the modulus of the Jacobi elliptic functions and e_i, $i = 1, 2, 3$, $e_1 > e_2 > e_3$ are three roots of the cubic equation $4z^3 - g_2 z - g_3 = 0$. When $m \to 1$, that is, $e_1 \to e_2$, we get $\mathrm{ns}(R\xi; m) \to \coth(R\xi)$ and $\mathrm{cn}(R\xi; m) \to \mathrm{sech}(R\xi)$. Finally, we note that the solutions (51) are new and are not reported elsewhere.

4. Conclusions

The two-variable $(G'/G, 1/G)$-expansion method is used in this paper to obtain some new solutions of a selected nonlinear partial differential equation of nanoionic currents along MTs (1). As the two constants A_1 and A_2 take special values, we obtain the solitary wave solutions. When $\mu = 0$ and $b_i = 0$ in (2) and (13), the two-variable $(G'/G, 1/G)$-expansion method reduces to the (G'/G)-expansion method. So the two-variable $(G'/G, 1/G)$-expansion method is an extension of the (G'/G)-expansion method. The proposed method in this paper is more effective and more general than the (G'/G)-expansion method because it gives exact solutions in more general forms. In summary, the advantage of the two-variable $(G'/G, 1/G)$-expansion method over the (G'/G)-expansion method is that the solutions using the first method recover the solutions using the second one. On comparing our results obtained in this paper with the well-known results obtained in [45, 47, 48] we deduce that some of these results are the same while the others are new which are not reported elsewhere. Finally, all solutions obtained in this paper have been checked with the Maple by putting them back into the original equation (14).

Glossary

ER: elementary ring
MT: microtubule
NP: nanopore
ODE: ordinary differential equation
PDE: partial differential equation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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