Research Article

Fourier Series Approximations to J_2 -Bounded Equatorial Orbits

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The current paper offers a comprehensive dynamical analysis and Fourier series approximations of J_2 -bounded equatorial orbits. The initial conditions of heterogeneous families of J_2 -perturbed equatorial orbits are determined first. Then the characteristics of two types of J_2 -bounded orbits, namely, pseudo-elliptic orbit and critical circular orbit, are studied. Due to the ambiguity of the closed-form solutions which comprise the elliptic integrals and Jacobian elliptic functions, showing little physical insight into the problem, a new scheme, termed Fourier series expansion, is adopted for approximation herein. Based on least-squares fitting to the coefficients, the solutions are expressed with arbitrary high-order Fourier series, since the radius and the flight time vary periodically as a function of the polar angle. As a consequence, the solutions can be written in terms of elementary functions such as cosines, rather than complex mathematical functions. Simulations enhance the proposed approximation method, showing bounded and negligible deviations. The approximation results show a promising prospect in preliminary orbits design, determination, and transfers for low-altitude spacecrafts.

1. Introduction

The motion of a particle is known to be integrable by quadratures in a central force field, wherein the potential displays a spherical symmetry in the mathematical expression, written in the following form via a finite or infinite expansion:

$$V(r) = \sum_{n=-\infty}^{+\infty} a_n r^n, \quad n \in \mathbb{Z} \setminus \{0\}, \quad (1)$$

thus the motion of the particle depends on its distance r from the center of the planet only. Such potentials are determined by the density of the planets, usually taken to establish the dynamic models in stellar and galactic systems [1]. Several typical ones are the Hernquist-Newton potential, the Plummer potential, the spherical harmonic potential, the power-law potential, the logarithmic potential, and the Kepler potential [2]. Interestingly, whatever formats the potentials are, and despite the differences in mathematical expressions, analogous rosette-shaped bounded orbits always rise under the specific conditions. Moreover, potentials in

some special cases are also equivalent to be spherically symmetric, among which are the radial acceleration problems for orbital transfers in spacecrafts mission design [3–14], and the problems of a particle in equatorial orbits considering the J_2 term studied by some researchers recently [15–18].

In general, the total gravitational potential, including J_2 perturbative potential and Kepler potential, is axially symmetric, and it works significantly on a low-orbiting particle that moves around an ellipsoidal planet. While the particle's orbit is confined to the equatorial plane, the J_2 potential reduces to be spherically symmetric, and the problem becomes integrable. Due to its integrability in the sense of Liouville [17], the closed-form solutions of J_2 -bounded equatorial orbits were given in terms of elliptic integrals [15, 18], based on which the periodic and pseudo-elliptic bounded relative orbits were obtained. Lately, the solutions of J_2 -unbounded equatorial orbits were also studied with the help of elliptic functions, and two types of new unbounded orbits in equatorial plane, namely, pseudo-parabolic orbits and pseudo-hyperbolic orbits, were unveiled [16].

The dynamical analysis of the J_2 -bounded equatorial orbits, which may be of deep insight to the problem, is essential but scarce in the literature. Besides, although the elliptic integrals and elliptic functions may present compact closed-form solutions according to the previous studies, these mathematical functions complicate the practical use for preliminary mission design due to the lack of physical insight.

In view of this, the present paper first offers a comprehensive study on dynamical characteristics and initial conditions of J_2 -perturbed equatorial orbits. Then we establish a methodology to obtain arbitrary high-order approximations of the J_2 -bounded equatorial orbits by means of Fourier series expansions such as trigonometric functions, since the distance varies periodically as a function of time or polar angle. In this paper, we utilize the Fourier series of order 1, and order 2 via least-squares fitting to the coefficients. The deviations are quantifiable and negligible and are guaranteed to be bounded by imposing a constraint that the approximation solutions coincide with the exact solutions at both endpoints of the half period.

As a result, the solutions are expressed in terms of elementary functions such as cosines, rather than complex mathematical functions, and the proposed methods are demonstrated to be effective in simulation.

2. Problem Formulation and Mathematic Model

The equatorial geopotential comprising J_2 perturbative potential and Kepler potential can be expressed as [19]

$$V(r) = -\frac{\mu}{r} \left[1 + \frac{J_2}{2} \left(\frac{R_e}{r} \right)^2 \right], \quad (2)$$

where R_e represents the mean equatorial radius of the planet, J_2 is the second zonal harmonics coefficient, μ is the gravitational parameter, and r denotes the modulus of position vector $\mathbf{r}$.

Since we are dealing only with gravitational forces in this paper, the orbit of a particle in a given field does not depend on its mass. Hence, we examine the dynamics of a particle of unit mass, so the quantities such as momentum, angular momentum, and energy, and functions such as the Lagrangian and Hamiltonian, are normally written per unit mass. The Lagrangian L of the system is formed by subtracting the potential energy $V(r)$ from the kinetic energy T , where polar coordinates (r, θ) are mostly conveniently used as follows:

$$L = T - V(r) = \frac{1}{2} \left[\dot{r}^2 + (r\dot{\theta})^2 \right] + \frac{\mu}{r} \left[1 + \frac{J_2}{2} \left(\frac{R_e}{r} \right)^2 \right]. \quad (3)$$

Obviously, θ is an ignorable coordinate from (3); the canonical momentum, also referred to as the first integral constant, yields from

$$\frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} = h = \text{const.} \quad (4)$$

The Euler-Lagrange equation for r , therefore, becomes the radial force equation

$$\ddot{r} - r\dot{\theta}^2 = F(r), \quad (5)$$

where $F(r) = -\mu(1 + 3J_2 R_e^2 / 2r^2) / r^2$, denoting the acceleration per mass that resulted from the center force or imposed by Kepler and J_2 perturbative forces along $\mathbf{r}$. Also the second integral constant is easily obtained from the Hamiltonian

$$\varepsilon = T + V = \frac{1}{2} \dot{r}^2 + \frac{1}{2} (r\dot{\theta})^2 - \frac{\mu}{r} - \frac{J_2 \mu R_e^2}{2r^3} = \text{const}, \quad (6)$$

as the energy is conserved.

Denote the effective potential energy of the radial motion as

$$V_{\text{eff}} = \frac{1}{2} (r\dot{\theta})^2 + V(r) = \frac{h^2}{2r^2} - \frac{\mu}{r} - \frac{J_2 \mu R_e^2}{2r^3}. \quad (7)$$

Substituting (6) into (7) yields

$$\dot{r} = \pm \sqrt{2(\varepsilon - V_{\text{eff}})}. \quad (8)$$

3. Initial Conditions of J_2 -Perturbed Equatorial Orbits

The particle motion in equatorial plane, subject to J_2 perturbation, is therefore established in Section 2. It may be checked that if we set $\dot{r} = 0$, a unified equation is obtained according to (8):

$$\varepsilon r^3 + \mu r^2 - \frac{h^2}{2} r + \frac{\mu J_2 R_e^2}{2} = 0, \quad (9)$$

comprising three main cases that need to be discussed. Note that (9) is a cubic equation if $\varepsilon \neq 0$. To have three real roots, the discriminant of (9) satisfies

$$\Delta = \left(-\frac{\mu h^2}{12\varepsilon^2} - \frac{\mu^3}{27\varepsilon^3} - \frac{J_2 \mu R_e^2}{4\varepsilon} \right)^2 + \left(-\frac{h^2}{6\varepsilon} - \frac{\mu^2}{9\varepsilon^2} \right)^3 \leq 0, \quad (10)$$

where the equality sign indicates the existence of a couple of identical roots.

The three roots are given by

$$\begin{aligned}
 x_1 &= -\frac{\mu}{3\epsilon} + \sqrt[3]{-\frac{\mu h^2}{12\epsilon^2} - \frac{\mu^3}{27\epsilon^3} - \frac{J_2 \mu R_e^2}{4\epsilon}} + \sqrt{\Delta} \\
 &\quad + \sqrt[3]{-\frac{\mu h^2}{12\epsilon^2} - \frac{\mu^3}{27\epsilon^3} - \frac{J_2 \mu R_e^2}{4\epsilon}} - \sqrt{\Delta} \\
 x_2 &= -\frac{\mu}{3\epsilon} + \frac{-1 + \sqrt{3}i}{2} \sqrt[3]{-\frac{\mu h^2}{12\epsilon^2} - \frac{\mu^3}{27\epsilon^3} - \frac{J_2 \mu R_e^2}{4\epsilon}} + \sqrt{\Delta} \\
 &\quad + \frac{-1 - \sqrt{3}i}{2} \sqrt[3]{-\frac{\mu h^2}{12\epsilon^2} - \frac{\mu^3}{27\epsilon^3} - \frac{J_2 \mu R_e^2}{4\epsilon}} - \sqrt{\Delta} \\
 x_3 &= -\frac{\mu}{3\epsilon} + \frac{-1 - \sqrt{3}i}{2} \sqrt[3]{-\frac{\mu h^2}{12\epsilon^2} - \frac{\mu^3}{27\epsilon^3} - \frac{J_2 \mu R_e^2}{4\epsilon}} + \sqrt{\Delta} \\
 &\quad + \frac{-1 + \sqrt{3}i}{2} \sqrt[3]{-\frac{\mu h^2}{12\epsilon^2} - \frac{\mu^3}{27\epsilon^3} - \frac{J_2 \mu R_e^2}{4\epsilon}} - \sqrt{\Delta}.
 \end{aligned} \tag{11}$$

The other special case is $\epsilon = 0$, and (9) reduces to be a quadratic equation:

$$r^2 - \frac{h^2}{2\mu}r + \frac{J_2 R_e^2}{2} = 0. \tag{12}$$

Accordingly, the discriminant of (9) satisfies

$$\Delta = \left(\frac{h^2}{2\mu}\right)^2 - 2J_2 R_e^2 > 0. \tag{13}$$

The two roots are

$$x_{1,2} = \frac{h^2/2\mu \pm \sqrt{h^4/4\mu^2 - 2J_2 R_e^2}}{2}. \tag{14}$$

Assume that the J_2 zonal harmonics is taken as $J_2 = 1.08263 \times 10^{-3}$ herein, r_0 denotes the initial radius, and $(\tilde{r})$ represents the normalized variables: $\tilde{r} = r/R_e$, $\tilde{t} = t/\sqrt{R_e^3/\mu}$, $\tilde{h} = h/\sqrt{R_e\mu}$, $\tilde{\epsilon} = \epsilon/\sqrt{R_e\mu}$. As the total energy ϵ and angular momentum h are the two integral constant values, (4) and (6) can be translated in a graphical form in $(\tilde{r}_0 \sim \tilde{\epsilon})$ plane for different values of the fixed $\tilde{h}$, as shown in Figure 1.

Observe that all the contour lines illustrated in Figure 1 pass through the horizontal line $\epsilon = 0$. In other words, it implies the existence of heterogeneous families of orbits: bounded orbits for $\epsilon < 0$ and unbounded orbits for $\epsilon = 0$ and $\epsilon > 0$; the orbital shape is totally determined by the initial conditions. This is confirmed by Figure 2, in which the phase space portrait is depicted, and the corresponding potential well is outlined.

Generally, the curve of the potential energy intersects the constant energy line twice for $\epsilon \geq 0$ and three times for $\epsilon < 0$. However, the points associated with the minimum roots of (8) are found to be located below the surface of the planet and are of less physical meaning, so they are

FIGURE 1: Representation of motion in the plane $(\tilde{r}_0 \sim \tilde{\epsilon})$ with different values of the fixed $\tilde{h}$.

FIGURE 2: Orbital motions in the phase space $(r, \dot{r})$ and potential energy well.

ignored in Figure 2. If $\epsilon < 0$ and $\partial V_{\text{eff}}/\partial r \neq 0$, the particle will be trapped in the well, oscillating radially, and the radius range is $r_{\min} \leq r \leq r_{\max}$. Here $r_{\min}$ and $r_{\max}$ correspond to $\dot{r} = 0$ in the phase space $(r, \dot{r})$, referred to as the lower limit and upper limit. As the energy decreases until the effective energy arrives at its local minimum, namely, $\partial V_{\text{eff}}/\partial r = 0$, the orbital shape degenerates from a pseudo-ellipse to a circle with a radius r_0 , corresponding to the fixed point in Figure 2. Otherwise if the energy increases till $\epsilon \geq 0$, pseudo-parabolic or pseudo-hyperbolic unbounded orbits will occur, which lead to bifurcations in phase space. In this paper, we only focus on the behaviors of the bounded orbits, whereas the unbounded orbits are left for the further research.

4. Fundamental Characteristics of J_2 -Bounded Equatorial Orbits

A comprehensive analysis on initial conditions of heterogeneous families of J_2 -perturbed equatorial orbits is discussed. In this section, the focus is J_2 -bounded equatorial orbits'

fundamental characteristics. To begin, radial and azimuthal kinetic energy are defined for convenience:

$$\varepsilon_r = \frac{1}{2} \dot{r}^2, \quad \varepsilon_\theta = \frac{1}{2} (r\dot{\theta})^2. \quad (15)$$

4.1. Pseudo-Elliptic Orbit. For pseudo-elliptic bounded orbit, $\varepsilon_r = 0$ is satisfied for a certain r and the equation $\dot{r} = 0$ normally has two roots $r_{\min}$ and $r_{\max}$ provided that $\partial V_{\text{eff}}/\partial r \neq 0$, known as pericenter and apocenter, between which the particle oscillates radially as it revolves. The radial period, and the azimuthal angle increases by an amount, together with the azimuthal period and the distance r can be computed as [2]

$$\begin{aligned} T_r &= \sqrt{\frac{2}{-\varepsilon}} \left[\lambda_1 F\left(\frac{\pi}{2}, k\right) + \lambda_2 E\left(\frac{\pi}{2}, k\right) + \lambda_3 \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \right], \\ \Delta\theta &= \frac{2h}{\lambda_2} \sqrt{\frac{2}{-\varepsilon}} F\left(\frac{\pi}{2}, k\right), \\ T_\theta &= \frac{2\pi}{\Delta\theta} T_r = \left(\pi \lambda_2 \left[\lambda_1 F\left(\frac{\pi}{2}, k\right) + \lambda_2 E\left(\frac{\pi}{2}, k\right) \right. \right. \\ &\quad \left. \left. + \lambda_3 \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \right] \right) \\ &\quad \times \left(h F\left(\frac{\pi}{2}, k\right) \right)^{-1}, \\ r(\theta) &= (r_{\min} r_{\max}) \\ &\quad \times \left(r_{\min} + (r_{\max} - r_{\min}) \operatorname{sn}^2 \right. \\ &\quad \left. \times \left[F\left(\frac{\pi}{2}, k\right) - \frac{\lambda_2 \sqrt{-2\varepsilon}}{2h} \theta, k \right] \right)^{-1}, \end{aligned} \quad (16)$$

where $F(\pi/2, k)$, $E(\pi/2, k)$, $\Pi(\pi/2, \alpha^2, k)$ are the complete elliptic integrals of the first, second, and third kind, respectively. The coefficients are as follows:

$$\begin{aligned} \lambda_1 &= -r_{\max} \sqrt{\frac{r_{\min}}{r_{\max} - r_*}}, \quad \lambda_2 = \sqrt{r_{\min} (r_{\max} - r_*)}, \\ \lambda_3 &= \frac{r_{\max} (r_{\max} + r_{\min} + r_*)}{\sqrt{r_{\min} (r_{\max} - r_*)}}. \end{aligned} \quad (17)$$

Here, r_* is another root of (9), and

$$k = \sqrt{\frac{(r_{\max} - r_{\min}) r_*}{(r_{\max} - r_*) r_{\min}}}, \quad \alpha = \sqrt{1 - \frac{r_{\max}}{r_{\min}}}, \quad (18)$$

where $\operatorname{sn}(\cdot)$ is the Jacobian elliptic function.

Such pseudo-elliptic bounded orbit displays a rosette shape in polar frame that bifurcates from the circular orbit and wraps around it, enclosed in between and tangential to two concentric circles of radii $r_{\min}$ and $r_{\max}$.

The flight time less than one-half radial period, as a function of the polar angle, is given by

$$\begin{aligned} t &= \int_{r_{\min}}^r \frac{dt}{dr} dr = \int_{r_{\min}}^r \frac{1}{\sqrt{2(\varepsilon - V_{\text{eff}})}} dr = \frac{1}{\sqrt{-2\varepsilon}} \\ &\quad \times \left[\sqrt{\frac{\varepsilon - V_{\text{eff}}}{r}} + \lambda_1 F(\theta, k) \right. \\ &\quad \left. + \lambda_2 E(\theta, k) + \lambda_3 \Pi(\theta, \alpha^2, k) \right]. \end{aligned} \quad (19)$$

One should take modulo if $t > T/2$, namely,

$$\begin{aligned} t &= \begin{cases} nT_r + \int_{r_{\min}}^r \frac{dt}{dr} dr, & t \in \left[nT_r, \left(n + \frac{1}{2}\right)T \right] \\ (n+1)T_r - \int_r^{r_{\max}} \frac{dt}{dr} dr, & t \in \left[\left(n + \frac{1}{2}\right)T_r, (n+1)T \right] \end{cases} \\ \theta &= \begin{cases} n\Delta\theta + \int_{r_{\min}}^r \frac{d\theta}{dr} dr, & t \in \left[nT_r, \left(n + \frac{1}{2}\right)T \right] \\ (n+1)\Delta\theta - \int_r^{r_{\max}} \frac{d\theta}{dr} dr, & t \in \left[\left(n + \frac{1}{2}\right)T_r, (n+1)T \right], \end{cases} \end{aligned} \quad (20)$$

where $n \in \mathbb{N}$.

4.2. Critical Circular Orbit. Consider a special case in which the curve of the potential energy is tangent to the constant energy line, indicating two equivalent roots that appear in couple with their radii $(r)_{1,2} = r_0$. Equivalently, the potential energy arrives at its minimum. Applying $\partial V_{\text{eff}}/\partial r = 0$ yields

$$\begin{aligned} (r_0)_1 &= \frac{h^2 + \sqrt{h^4 - 6J_2\mu^2 R_e^2}}{2\mu}, \\ (r_0)_2 &= \frac{h^2 - \sqrt{h^4 - 6J_2\mu^2 R_e^2}}{2\mu}, \end{aligned} \quad (21)$$

which is in agreement with Humi's conclusion [20], and $(r_0)_2$ is an extraneous root that should be discarded. The angular velocity and the orbital period of the spacecraft are given by

$$\omega = \frac{h}{r_0^2}, \quad T = \frac{2\pi r_0^2}{h}. \quad (22)$$

Substituting r_0 into $\partial^2 V_{\text{eff}}/\partial r^2$ yields

$$\left. \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \right|_{r=r_0} = -\frac{2\mu}{r_0^3} - \frac{6J_2\mu R_e^2}{r_0^5} + \frac{3h^2}{r_0^4} > 0, \quad (23)$$

indicating that the J_2 -circular orbit is stable according to the theorems of the classical mechanics [21].

To detect the orbital evolution when small radial perturbations act on the particle, the substitution $u = 1/r$ is employed. Thus, the orbital dynamic equation is expressed as

$$\frac{d^2 u}{d\theta^2} + u = J(u), \quad (24)$$

where

$$J(u) = -\frac{1}{h^2} \frac{d}{du} V\left(\frac{1}{u}\right) = \frac{1}{h^2} \left(\frac{3}{2} J_2 \mu R_e^2 u^2 - h^2 u + \mu \right). \quad (25)$$

Since motion is stable, u is surely bounded and varies within a small region around u_0 . To proceed, we define $\Delta u \triangleq u - u_0$, denoting the deviation between perturbed and unperturbed motion. Expanding $J(u)$ in a Taylor series at u_0 and neglecting all terms in $J(u)$ of second and higher orders yield

$$J(u) = u_0 + (u - u_0) \frac{dJ}{du_0} + O[(u - u_0)^2], \quad (26)$$

then the equation of orbital motion around the J_2 -circular orbit reduces to

$$\frac{d^2 \Delta u}{d\theta^2} + \Delta u = \Delta u \frac{dJ}{du_0}. \quad (27)$$

For the sake of compactness, denote

$$\chi^2 = 1 - \frac{dJ}{du_0} = 2 - \frac{3J_2 \mu R_e^2 u_0}{h^2}, \quad (28)$$

then a simplified solution to (27) is obtained

$$\Delta u = u - u_0 = A \cos(\chi\theta), \quad (29)$$

where A is related to the external perturbation imposed on the particle. Through transformation, we have

$$r = \frac{r_0}{1 + A r_0 \cos(\chi\theta)}. \quad (30)$$

Usually, χ is an irrational number. Hence, the particle will come to a pseudo-elliptic orbit and oscillate radially within a small region.

5. Fourier Series Approximations

In the previous section, the closed-form analytical solutions for J_2 -bounded equatorial orbits were derived in terms of elliptic integrals and elliptic functions. However, due to the lack of physical insight regarding these mathematical functions, instead of the elementary functions, it is difficult to put them into practical use for mission design. Here, a Fourier series expansion is resorted for analytical approximations.

Equation (5) shows that the radius is an even function of θ , with half radial period $\Delta\theta$. Hence, the generalized Fourier series evaluated on the interval of $[0, \Delta\theta]$ can be written by means of cosines alone [3]

$$r_{\text{app}}^{(\infty)}(\theta) = \frac{r_{\min}}{1 + \sum_{i=0}^{\infty} C_i \cos(i\pi(\theta/\Delta\theta))}, \quad (31)$$

where C_i are the Fourier coefficients, expressed as

$$C_0 = \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[\frac{r_{\min}}{r(\theta)} - 1 \right] d\theta, \quad (32)$$

$$C_i = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{r_{\min}}{r(\theta)} - 1 \right] \cos\left(i\pi \frac{\theta}{\Delta\theta}\right) d\theta.$$

Consider a limiting case in which the first-order approximation is merely retained in (31) under the assumption that $e_{J_2} \ll 1$. As such, the truncated expression of (31) is given by

$$r_{\text{app}}^{(1)}(\theta) = \frac{r_{\min}}{1 + B_0 + B_1 \cos(\pi(\theta/\Delta\theta))}, \quad (33)$$

where B_0 and B_1 are constants and different from the Fourier coefficients C_i in (31). Assume that the approximation solution coincides with the exact solution at both endpoints of the half period $[0, \Delta\theta]$; thus, the maximum deviation will remain bounded.

Suppose the particle starts at pericenter and ends at apocenter on a short timescale $[0, \Delta\theta]$; then B_0 and B_1 are calculated as follows:

$$B_0 = -\frac{r_{\max} - r_{\min}}{2r_{\max}}, \quad B_1 = \frac{r_{\max} - r_{\min}}{2r_{\max}}. \quad (34)$$

To seek for a brief format of approximation, denote

$$a_{J_2} = \frac{r_{\max} + r_{\min}}{2}, \quad e_{J_2} = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}, \quad (35)$$

$$p_{J_2} = a_{J_2} (1 - e_{J_2}^2).$$

Substituting B_0 , B_1 , and (35) into (33) yields

$$r_{\text{app}}^{(1)}(\theta) = \frac{p_{J_2}}{1 + e_{J_2} \cos(\pi(\theta/\Delta\theta))}. \quad (36)$$

Accordingly, the first-order approximation of flight time as a function of polar angle can be computed by substituting equation (36) into equation $d\theta/dt = h/r^2$:

$$t_{\text{app}}^{(1)}(\theta) \approx \int_0^\theta \frac{[r_{\text{app}}^{(1)}(\theta)]^2}{h} d\theta = \frac{\Delta\theta p_{J_2}^2}{\pi h (1 - e_{J_2}^2)^{3/2}} \times \left\{ 2 \arctan \left[\sqrt{\frac{1 - e_{J_2}}{1 + e_{J_2}}} \tan\left(\frac{\pi\theta}{2\Delta\theta}\right) \right] \right. \\ \left. - \frac{e_{J_2} \sqrt{1 - e_{J_2}^2} \sin(\pi\theta/\Delta\theta)}{1 + e_{J_2} \cos(\pi\theta/\Delta\theta)} \right\}. \quad (37)$$

To evaluate the accuracy of approximation, it is necessary to introduce the distance deviation

$$\varepsilon_{\Delta}^{(1)} = \frac{|r_{\text{app}}^{(1)}(\theta) - r|}{r}. \quad (38)$$

TABLE 1: Least-squares fitting coefficients B_0, B_1, B_2 vary with $e_{J_2}^{(2)}$.

$e_{J_2}^{(2)}$	B_0	B_1	B_2	$\min J$
0.1111	$-9.99953e-2$	$9.99966e-2$	$-1.25461e-6$	$3.47383e-8$
0.1428	$-1.24991e-2$	$1.24994e-2$	$-2.99121e-6$	$1.11865e-7$
0.1765	$-1.49988e-1$	$1.49991e-1$	$-3.18501e-6$	$2.36149e-7$
0.2121	$-1.74984e-1$	$1.74988e-1$	$-3.91574e-6$	$5.03017e-7$
0.2500	$-1.99789e-1$	$1.99984e-1$	$-4.71406e-6$	$9.65574e-7$

FIGURE 3: Variations of the deviation $\varepsilon_{\Delta}^{(1)}$ for $e_{J_2} = 4 \times 10^{-6} \ll 1$, and $\tilde{\tau} \in [0, 5]$.

And the flight time deviation is also constructed as

$$\varepsilon_{\Delta t}^{(1)} = \frac{|t_{\text{app}}^{(1)}(\theta) - t|}{t}. \quad (39)$$

Figure 3 shows that for $\tilde{h} = 1.1205$, $J_2 = 1.08263 \times 10^{-3}$, and $e_{J_2} = 4 \times 10^{-6} \ll 1$, $\varepsilon_{\Delta}^{(1)}$ varies periodically, and the magnitude of $\varepsilon_{\Delta}^{(1)}$ is less than 8×10^{-6} , implying the effectiveness of the first-order Fourier series approximation for small e_{J_2} .

As e_{J_2} increases gradually even if $e_{J_2} > 0.1$, the first-order Fourier series approximation of flight time seems to be still effective; the magnitude of $\varepsilon_{\Delta t}^{(1)}$ is less than 3×10^{-5} , as shown in Figure 4. Nevertheless, the radius approximation becomes to be inadequate, especially for $e_{J_2} > 0.1$, as displayed in Figure 5. Naturally it necessitates us to retain high-order terms in the proceedings of approximating.

To that end, we retain $m - 1$ orders ($m > 1$) of the series, written by

$$r_{\text{app}}^{(m)}(\theta) = \frac{r_{\min}}{1 + \sum_{i=0}^m B_i \cos(i\pi(\theta/\Delta\theta))}. \quad (40)$$

Once again, assume that the approximation solution coincides with the exact solution at both endpoints of the half-period $[0, \Delta\theta]$. Sequentially, the coefficients in (40) satisfy the following constraints:

$$\sum_{i=0}^m B_i = 0, \quad \sum_{i=0}^m B_i (-1)^i = \frac{r_{\min}}{r_{\max}} - 1. \quad (41)$$

It is noteworthy that these constraints enable $m-1$ degree-of-freedom. One rational approach is to apply least-squares fitting. The objective function is constructed as

$$\min J(B_0, B_1, \dots, B_m) = \sum_{j=1}^n \sum_{i=0}^m [r_{\text{app}}^{(m)}(B_i, \theta_j) - r(\theta_j)]^2, \quad (42)$$

where n denotes the quantity of total discrete points in the process of integration.

Table 1 shows the coefficients, with large e_{J_2} , obtained by Fourier series approximation of order 2, and the corresponding objective functions are also given. The fixed angular momentum $\tilde{h} = 1.1205$. One should not neglect the fact that due to the constraint $r_{\text{app}}^{(2)}(\theta) > R_e$, $e_{J_2} < 0.254$ must be satisfied.

Figure 6 displays the Fourier series approximation of order 2, via least-squares fitting. In contrast to the result by first-order Fourier series approximations, the maximum deviation is less than 0.025% of the orbital radius $r(\theta)$. In view of this, it is suitable for the high-accuracy required missions.

Figure 7 illustrates the particle's actual orbit (solid line) and orbit by Fourier series approximations (dashed line) in polar reference frame $(\tilde{\theta}, \tilde{r})$ with $e_{J_2} = 0.25$. In Figure 8, the left plane shows the evolution of the actual radius and radius of Fourier series approximation of order 2, via least-squares fitting. The right plane presents a magnified view of a local segment.

6. Conclusions

The main contribution of this paper is that a framework for approximating J_2 -bounded equatorial orbits is established with arbitrary high-order Fourier series expansions.

FIGURE 4: Variations of the deviation $\varepsilon_{\Delta t}^{(1)}$ for $e_{J_2} > 0.1$, and $\theta \in [0, 3]$. FIGURE 6: Variations of the deviation $\varepsilon_{\Delta}^{(2)}$ for $e_{J_2} > 0.1$, and $\tilde{t} \in [0, 5]$. FIGURE 5: Variations of the deviation $\varepsilon_{\Delta}^{(1)}$, and $\tilde{t} \in [0, 5]$. FIGURE 7: Actual orbit and orbit by Fourier series approximation for $e_{J_2} > 0.1$ in polar reference frame $(\tilde{\theta}, \tilde{r})$.

Since the distance and time vary periodically as a function of polar angle, the solutions are expressed in terms of elementary trigonometric functions, rather than Jacobian elliptic function and elliptic integrals that lack physical insight into the problem. For Fourier series expansion of second order or higher, the coefficients can be selected via least-squares fitting, and the deviations are guaranteed to be still bounded by imposing a constraint that the approximation solutions coincide with the exact solutions at both endpoints of the half period. Also, the approximation of closed-form relationship for the flight time as a function of the polar angle is given using an analytical approach.

The presented approximation method has a potential for space missions, such as novel orbits design, and computational efficiency improvement for long-term low-altitude orbits.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

FIGURE 8: Evolution of actual radius and radius by Fourier series approximation for $e_{J_2} = 0.25$.

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