Research Article

An Analytical Solution for Lateral Buckling Critical Load Calculation of Leaning-Type Arch Bridge

Ai-rong Liu, Yong-hui Huang, Qi-cai Yu, and Rui Rao

Guangzhou University-Tamkang University Joint Research Center for Engineering Structure Disaster Prevention and Control, Guangzhou University, Guangzhou 510006, China

Correspondence should be addressed to Yong-hui Huang; kunull@foxmail.com

Received 14 March 2014; Revised 1 May 2014; Accepted 1 May 2014; Published 25 May 2014

Academic Editor: Sarp Adali

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An analytical solution for lateral buckling critical load of leaning-type arch bridge was presented in this paper. New tangential and radial buckling models of the transverse brace between the main and stable arch ribs are established. Based on the Ritz method, the analytical solution for lateral buckling critical load of the leaning-type arch bridge with different central angles of main arch ribs and leaning arch ribs under different boundary conditions is derived for the first time. Comparison between the analytical results and the FEM calculated results shows that the analytical solution presented in this paper is sufficiently accurate. The parametric analysis results show that the lateral buckling critical load of the arch bridge with fixed boundary conditions is about 1.14 to 1.16 times as large as that of the arch bridge with hinged boundary condition. The lateral buckling critical load increases by approximately 31.5% to 41.2% when stable arch ribs are added, and the critical load increases as the inclined angle of stable arch rib increases. The differences in the center angles of the main arch rib and the stable arch rib have little effect on the lateral buckling critical load.

1. Introduction

Leaning-type arch bridge is a relatively new type of spatial tied-arch bridge developed from x-type arch bridge. It is composed of two load-bearing ribs (called main arch ribs) which are perpendicular to the bridge deck and two leaning arch ribs (called stable arch ribs) on the sides of the main arch ribs. These two types of arch ribs constitute a space stable system when connected by transverse braces between them. This type of bridge is one of the most competitive urban bridges because it is stylish and unique, gives an open and clear view for drivers, and is economically efficient. As the vertical stiffness of the main arch rib is much higher than that of the stable arch rib, the major portion of the dead and live loads is supported by the main arch ribs. Stable arch rib supports only a small portion of the live load, and its main function is to assure the lateral stability of the main arch rib. Because there are no transverse braces placed between the main arch ribs, and the stable arch rib's contribution to the improvement of the lateral stability of bridge is very limited, the lateral stability often becomes a key factor deciding the bridge's safety [1]. The world's first leaning-type arch bridge named

Bacde Road Bridge was built in 1992 in Barcelona. It was designed by the Spanish architect, Santiago Calatrava. This bridge is 52 m long, 25.8 m wide, and the inclined angle of stable arch rib is 30° . The vertical main arch ribs are hinged and the inclined stable arch ribs are fixed at both ends [2]. Over the past decades, leaning-type arch bridges have developed rapidly. At present, more than 20 leaning-type arch bridges have been built around the world, most of them in China. Usually, fixed boundary conditions are adopted for both the main and stable arch ribs, such as the Kangfu Bridge in Yiyang city, China, Hanjiang North Bridge in Chaozhou city, China, and Shengli Bridge in Jiangmen city, China (Figure 1). However, under special circumstances, in order to reduce main arch rib's internal force and horizontal thrust or adapt to the local environmental conditions [3], hinged boundary condition is applied at the arch ends of the main arch, such as Yufeng Bridge in Kunshan city, China, and Danxi Bridge in Yiwu city, China.

At present, the derivation of the analytical solution of the arch structure's buckling problem is mainly focused on the individual arch rib or conventional arch bridges with symmetrical arch ribs. Closed-form solutions for out-of-plane

FIGURE 1: Shenli Bridge in Jiangmen City of China.

buckling of arches, subjected to uniform bending or uniform compression, were obtained by Lim and Kang [4]. Pi et al. [5, 6] investigated the flexural torsional buckling of shallow arches with an open thin-walled section, subjected to a radial load uniformly distributed around the arch axis. In-plane static and dynamic buckling of shallow pin-ended parabolic arches with a horizontal cable was investigated by Chen and Feng [7]. La et al. [8] presented an experimental investigation of the elastic-plastic out-of-plane buckling response of roller bent circular steel arches subjected to a single force applied to the crown. Bradford and Pi [9] derived a new unified analytical solution for the lateral-torsional buckling load of pin-ended arches by accounting for the combined bending and axial compressive action. Dou et al. [10] investigated the sectional rigidities of trusses and the out-of-plane buckling loads of pin-ended circular steel tubular truss arches in uniform axial compression and in uniform bending. In addition, Jin and Zhao [11] derived the lateral buckling critical load for X-type twin ribbed arch braced with transverse beams. Moreover, Liu and Lu [12] derived the analytical solution of the lateral buckling critical load for the conventional tied-arch bridge with transverse braces and discussed the effect of structural parameters on lateral stability. Additionally, Xiang [13] presented a formula of the lateral buckling critical load for a half-through bridge at service stage and the effect of beam location for vehicle lane on the stability of this type of bridge was further studied.

Compared with individual arch rib or conventional arch bridges with symmetrical arch ribs, the leaning-type arch rib system is different in several aspects including an inclined angle between the main and stable arch ribs, different stiffness of main and stable arch ribs, more significant spatial effect, and more complicated loading conditions. These factors result in difficulties in deriving the analytical solution of the lateral buckling critical load. Especially that the mechanical model of transverse brace is very different from those of the conventional arch bridge when a lateral buckling occurs. Presently, researchers have conducted preliminary studies on the lateral stability of leaning-type arch bridge based on finite element method (FEM) [14]. The derivation of the analytical solution formula of the lateral buckling critical load of leaning-type arch ribs system has been preliminarily studied by Liu et al. [15]. However, in their studies the

influence of the bridge deck system and hanger tensions on leaning-type arch bridge was not taken into consideration. The central angles of the main and stable arch ribs were assumed to be the same in their studies, but in fact they are different in some conditions. And in their studies the inclined angles of stable arch ribs should be less than 15 degrees, while the inclined angles larger than 15 degrees; the precision of the solution is bad. Compared with the FEM calculation results, the relationship among different design parameters of the bridge can be clearly revealed by the analytical solution and the optimization of structure design is made easier; the complicated process of constructing the FE model can also be simplified.

In order to consider the influence of the components of leaning-type arch bridge comprehensively, the global transverse brace deformation parameter β is considered and the central angles of the main arch ribs and stable arch ribs are assumed different, and the tangential and radial mechanical models of the transverse brace between the main and stable arch ribs are established, then the analytical solution of the lateral buckling critical load for leaning-type arch rib system is derived in this paper. Compared with the analytical solution derived by Liu et al. [15], the one derived in this paper has a wider scope of applications. The deformation energy of the main arch rib, stable arch rib, and transverse braces between them are constructed; and the potential energy caused by the hangers tensions are also established for both the main arch rib and stable arch rib under fixed and hinged boundary conditions. And the total potential energy of the bridge in a buckling process is obtained thereafter. Based on the stationary energy principle, the analytical solution for the lateral buckling critical load of leaning-type arch bridge is obtained. In the end, parametric analysis is carried out in order to investigate how changes in certain design parameters would affect the critical load of the leaning-type arch bridges, which could lead to an optimum design of this type of bridge structures.

2. Lateral Buckling Critical Load under the Fixed Boundary Condition of Main Arch Rib

2.1. Calculation Model of Leaning-Type Arch Bridge. The following assumptions are made in the derivation process:

FIGURE 2: Calculation sketch of leaning-type arch bridge.

(1) the main arch ribs and the stable arch ribs are fixed in their ends; (2) the axis of main and stable arch ribs are arc-shaped curves; (3) the stiffness of arch ribs and transverse braces are constants; (4) the axial deformation of arch rib in the buckling process is neglected; (5) the external load is evenly and vertically distributed along the bridge deck and is transmitted to the arch rib via the hangers; and (6) the force acting on arch ribs satisfies the film tension assumption.

The simplified calculation model of a leaning-type arch bridge is shown in Figure 2. In the figure, x - y - z is the global coordinate system and u - v - w is the local coordinate system. u , v , and w represent the lateral, radial, and tangential displacement of arch ribs, respectively, ϕ_0 represents the angle between the main arch rib and the stable arch rib, φ is the angle of a position in the arch ribs, α_1 is the central angle of the main arch ribs, α_2 is the central angle of the stable arch rib, and R is the radius of the main and stable arch ribs. $b(x)$ is the distance between the main arch rib and the stable arch rib. In the arch crown position the distance is b_0 and $b(x) = b_0 + 2R \sin \phi_0 (1 - \cos \varphi)$. The transverse braces between the main and stable arch ribs are equidistant with a distance of d ; the hangers are also arranged equidistantly with a distance of d_h .

According to the basic assumption and calculation scheme, the relationship between the lateral displacement u_1^I , u_2^{II} and the radial displacement v_1^I , v_2^{II} of the main arch ribs and the stable arch ribs under the local coordinate is given, respectively, as

$$\begin{aligned} u_1^I &= u_1, & u_2^{II} &= \frac{y'_{02}}{y'_{01} + y'_{02}} b(x) \beta \sin \phi_0 + u_1 \cos \phi_0 \\ v_1^I &= \frac{y'_{01}}{y'_{01} + y'_{02}} b(x) \beta, \\ v_2^{II} &= -\frac{y'_{02}}{y'_{01} + y'_{02}} b(x) \beta \cos \phi_0 + u_1 \sin \phi_0, \end{aligned} \quad (1)$$

where u_1 represents the lateral displacement of the arch axis of the main arch, β represents the global torsional angle of the transverse brace caused by deformation of arch rib in radial plane, $b(x)$ is the distance between the main arch rib, and y'_{01} and y'_{02} represent the distance from the contraflexure point of

FIGURE 3: Radial bending deformation of transverse brace.

transverse brace's radial deformation to the main and stable arch ribs, respectively (shown in Figure 3).

When there is a lateral deformation in the arch rib system, axial strain of the main arch is given as

$$\varepsilon = -\frac{v_1^I}{R} + \frac{dw_1^I}{ds} + \frac{1}{2} \left[\left(\frac{du_1^I}{ds} \right)^2 + \left(\frac{dv_1^I}{ds} \right)^2 \right], \quad (2)$$

where w_1^I represents the tangential displacement of the main arch ribs under the local coordinate.

According to the basic assumption $\varepsilon = 0$,

$$\frac{dw_1^I}{ds} = \frac{v_1^I}{R} - \frac{1}{2} \left[\left(\frac{du_1^I}{ds} \right)^2 + \left(\frac{dv_1^I}{ds} \right)^2 \right]. \quad (3)$$

Because the arches are fixed at both ends, $\int_s dw_1^I = 0$, so

$$\int_s \left(\frac{v_1^I}{R} \right) ds = \frac{1}{2} \int_s \left(\frac{du_1^I}{ds} \right)^2 ds + \frac{1}{2} \int_s \left(\frac{dv_1^I}{ds} \right)^2 ds. \quad (4)$$

Assuming the ends of the main arch rib and the stable arch rib are perfectly fixed, the torsional angle and the lateral displacements of arch axis of the main and stable arch ribs are given, respectively, as

$$\theta_1 = C_1 \left(1 - \cos \frac{2\pi\varphi}{\alpha_1} \right) \quad (5a)$$

$$\theta_2 = C_3 \left(1 - \cos \frac{2\pi\varphi}{\alpha_2} \right), \quad (5b)$$

$$u_1 = C_2 \left(1 - \cos \frac{2\pi\varphi}{\alpha_1} \right) \quad (6a)$$

$$u_2 = C_2 \left(1 - \cos \frac{2\pi\varphi}{\alpha_2} \right). \quad (6b)$$

The lateral displacement of bridge deck system is given as

$$u_d = C_4 \left(1 - \cos \frac{2\pi\varphi}{\alpha_1} \right). \quad (7)$$

The global torsional angle of the transverse brace caused by deformation of arch rib in radial plane is given as

$$\beta = C_5 \left(1 - \cos \frac{2\pi\varphi}{\alpha_1} \right). \quad (8)$$

The above equations satisfy the following displacement boundary conditions.

- (1) When $\varphi = 0$ and $\varphi = \alpha_1$, $\theta_1 = 0$, $\theta'_1 = 0$, $u_1 = u_d = \beta = 0$, and $u'_1 = u'_d = \beta' = 0$.
- (2) When $\varphi = 0$ and $\varphi = \alpha_2$, $\theta_2 = 0$, $\theta'_2 = 0$, $u_2 = 0$, and $u'_2 = 0$.

The lateral deflection curvatures along the v axis of the main and stable arch ribs are given as [16]

$$\begin{aligned} K_u^I &= \frac{\theta_1}{R} - \frac{d^2 u_1^I}{ds^2}, \\ K_u^{II} &= \frac{\theta_2}{R} - \frac{d^2 u_2^{II}}{ds^2}, \end{aligned} \quad (9)$$

where K_v^I and K_v^{II} represent the lateral deflection curvature along v axis of the main and stable arch ribs respectively.

The torsional deflection curvatures along w axes are also given as [16]

$$\begin{aligned} K_w^I &= \frac{d\theta_1}{ds} + \frac{1}{R} \frac{d^2 u_1^I}{ds}, \\ K_w^{II} &= \frac{d\theta_2}{ds} + \frac{1}{R} \frac{d^2 u_2^{II}}{ds}, \end{aligned} \quad (10)$$

where K_w^I and K_w^{II} represent the torsional deflection curvatures along w axes of the main and stable arch ribs, respectively.

2.2. Energy Equations. The lateral deformation energy of a leaning-type arch bridge can be written as

$$\begin{aligned} W &= U_u^I + U_u^{II} + U_w^I + U_w^{II} + U_{cu}^I + U_{cu}^{II} + U_{bH} \\ &\quad + U_{bv} + V_H + U_d + V, \end{aligned} \quad (11)$$

where U_u^I , U_u^{II} , U_w^I , U_w^{II} , U_{cu}^I , and U_{cu}^{II} represent the total lateral bending deformation energy, torsional deformation energy, and local bending deformation energy of the main and stable arch ribs, respectively; U_{bv} and U_{bH} are the bending deformation energy of the transverse braces in radial and tangential directions along the main arch rib's axis; V_H is the elastic potential energy caused by the horizontal part of the tension of the hangers; U_d represents the elastic potential energy of bridge deck system; and V represents the potential energy of external loading applied to the arch bridge.

The total lateral bending deformation energy of the main and stable arch ribs are given in

$$\begin{aligned} U_u^I + U_u^{II} &= \frac{1}{2} E^I I_u^I \int_{s_1} (K_u^I)^2 ds + \frac{1}{2} E^{II} I_u^{II} \int_{s_2} (K_u^{II})^2 ds \\ &= \frac{1}{2} E^I I_u^I \left[C_1^2 A_{11} + \frac{C_2^2}{R^3} A_{12} + \frac{C_1 C_2}{R^2} A_{13} \right] \\ &\quad + \frac{1}{2} E^{II} I_u^{II} \left[\frac{C_2^2}{R^3} A_{21} + \frac{C_3^2}{R} A_{22} + \frac{C_5^2}{R} A_{23} \right. \\ &\quad \left. + \frac{C_2 C_3}{R^2} A_{24} + \frac{C_2 C_5}{R^2} A_{25} + \frac{C_3 C_5}{R} A_{26} \right], \end{aligned} \quad (12)$$

where $E^I I_u^I$ and $E^{II} I_u^{II}$ are the lateral bending stiffness of the main and stable arch ribs. Consider the following:

$$\begin{aligned} A_{11} &= \int_0^{\alpha_1} \left(1 - \cos \frac{2\pi\varphi}{\alpha_1} \right)^2 d\varphi \\ A_{12} &= \int_0^{\alpha_1} \left(\frac{2\pi}{\alpha_1} \right)^4 \cos^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\ A_{13} &= \int_0^{\alpha_1} -2 \left(\frac{2\pi}{\alpha_1} \right)^2 \left(1 - \cos \frac{2\pi\varphi}{\alpha_1} \right) \cos \frac{2\pi\varphi}{\alpha_1} d\varphi \\ A_{21} &= \int_0^{\alpha_2} \cos \phi_0 \left(\frac{2\pi}{\alpha_2} \right)^4 \cos^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\ A_{22} &= \int_0^{\alpha_2} \left(1 - \cos \frac{2\pi\varphi}{\alpha_2} \right)^2 d\varphi \end{aligned}$$

$$\begin{aligned}
A_{23} &= \int_0^{\alpha_2} \frac{1}{R^2} \left(\frac{2\pi}{\alpha_1} \right)^4 \frac{b(x)^2 \sin^2 \phi_0}{(1+e_2)^2} \cos^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\
A_{24} &= \int_0^{\alpha_2} -2 \cos \phi_0 \left(\frac{2\pi}{\alpha_2} \right)^2 \cos \frac{2\pi\varphi}{\alpha_2} \left(1 - \cos \frac{2\pi\varphi}{\alpha_2} \right) d\varphi \\
A_{25} &= \int_0^{\alpha_2} \frac{2b(x) \sin \phi_0 \cos \phi_0}{R(1+e_2)} \\
&\quad \times \left(\frac{2\pi}{\alpha_1} \right)^2 \left(\frac{2\pi}{\alpha_2} \right)^2 \cos \frac{2\pi\varphi}{\alpha_1} \cos \frac{2\pi\varphi}{\alpha_2} d\varphi \\
A_{26} &= \int_0^{\alpha_2} -\frac{2b(x) \sin \phi_0}{R(1+e_2)} \left(\frac{2\pi}{\alpha_1} \right)^2 \cos \frac{2\pi\varphi}{\alpha_1} \left(1 - \cos \frac{2\pi\varphi}{\alpha_2} \right) d\varphi.
\end{aligned} \tag{13}$$

The torsional deformation energy of the main and stable arch ribs is given in

$$\begin{aligned}
U_w^I + U_w^{II} &= \frac{1}{2} G^I T^I \int_{s_1} (K_w^I)^2 ds + \frac{1}{2} G^{II} T^{II} \int_{s_2} (K_w^{II})^2 ds \\
&= \frac{1}{2} G^I T^I \left[\frac{C_1^2}{R} B_{11} + \frac{C_2^2}{R^3} B_{12} + \frac{C_1 C_2}{R^2} B_{13} \right] \\
&\quad + \frac{1}{2} G^{II} T^{II} \left[\frac{C_2^2}{R^3} B_{21} + \frac{C_3^2}{R} B_{22} + \frac{C_5^2}{R} B_{23} \right. \\
&\quad \left. + \frac{C_2 C_3}{R^2} B_{24} + \frac{C_2 C_5}{R^2} B_{25} + \frac{C_3 C_5}{R} B_{26} \right],
\end{aligned} \tag{14}$$

where $G^I T^I$ and $G^{II} T^{II}$ are the torsional stiffness of the main and stable arch ribs. Consider the following:

$$\begin{aligned}
B_{11} &= \int_0^{\alpha_1} \left(\frac{2\pi\varphi}{\alpha_1} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\
B_{12} &= \int_0^{\alpha_1} \left(\frac{2\pi}{\alpha_1} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\
B_{13} &= \int_0^{\alpha_1} 2 \left(\frac{2\pi}{\alpha_1} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\
B_{21} &= \int_0^{\alpha_2} \cos^2 \phi_0 \left(\frac{2\pi}{\alpha_2} \right)^4 \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\
B_{22} &= \int_0^{\alpha_2} \left(\frac{2\pi}{\alpha_2} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_2} d\varphi \\
B_{23} &= \int_0^{\alpha_2} \frac{1}{R^2} \left(\frac{2\pi}{\alpha_1} \right)^2 \frac{b(x)^2 \sin^2 \phi_0}{(1+e_2)^2} \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \\
B_{24} &= \int_0^{\alpha_2} 2 \cos \phi_0 \left(\frac{2\pi}{\alpha_2} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_2} d\varphi
\end{aligned}$$

FIGURE 4: Bending deformation of arch ribs and transverse braces.

$$\begin{aligned}
B_{25} &= \int_0^{\alpha_2} \frac{2b(x) \sin \phi_0 \cos \phi_0}{R(1+e_2)} \frac{2\pi}{\alpha_1} \frac{2\pi}{\alpha_2} \sin \frac{2\pi\varphi}{\alpha_1} \sin \frac{2\pi\varphi}{\alpha_2} d\varphi \\
B_{26} &= \int_0^{\alpha_2} \frac{2b(x) \sin \phi_0}{R(1+e_2)} \frac{2\pi}{\alpha_1} \frac{2\pi}{\alpha_2} \sin \frac{2\pi\varphi}{\alpha_1} \sin \frac{2\pi\varphi}{\alpha_2} d\varphi.
\end{aligned} \tag{15}$$

Transverse brace bending deformation occurs in the tangential direction along arch axis when there is a lateral buckling in the arch rib system as shown in Figure 4. In this case, the tangential bending deformation energy of the single transverse brace is given as

$$\bar{U}_{bH} = \frac{1}{2E_b I_{bh}} \int_0^{y_{01}} M_{bh1}^2 dy_0 + \frac{1}{2E_b I_{bh}} \int_0^{y_{02}} M_{bh2}^2 dy_0, \tag{16}$$

where $E_b I_{bh}$ is transverse brace's bending stiffness along tangential direction of arch rib and y_{01} and y_{02} represent the distances from the contraflexure point of transverse brace tangential deformation to the main and stable arch ribs, respectively. Therefore, the length of transverse brace is $b(x) = y_{01} + y_{02}$.

From Figure 4, the tangential bending moment of transverse brace near the main and stable arch ribs is given by the superposition principle.

$$\begin{aligned}
M_{bh1} &= \frac{4E_b I_{bh}}{b} \gamma_1 + \frac{2E_b I_{bh}}{b} \gamma_3, \\
M_{bh2} &= \frac{2E_b I_{bh}}{b} \gamma_1 + \frac{4E_b I_{bh}}{b} \gamma_3,
\end{aligned} \tag{17}$$

where γ_1 represents the tangential angular rotation of the transverse brace at the intersection points between the main arch rib and the transverse brace and γ_3 represents the tangential angular rotation of the transverse brace at the intersection points between the stable arch rib and the transverse brace.

It can be further derived as

$$\begin{aligned}
y_{01} &= \frac{2\gamma_1 + \gamma_3}{3(\gamma_1 + \gamma_3)} b(x), \\
y_{02} &= \frac{\gamma_1 + 2\gamma_3}{3(\gamma_1 + \gamma_3)} b(x),
\end{aligned} \tag{18}$$

$b(x)$ can be replaced by a constant h , which is the distance between the main and stable arch ribs at quarter span for calculation simplification. Although the length of transverse brace $b(x)$ is not a constant, the length of the transverse brace h at quarter span is very close to the average length of the transverse braces of the entire bridge. Such simplification can make the derivation process become simple.

The values of γ_{01} and γ_{02} are closely related to the bending stiffness of arch ribs, panel length, bending stiffness, and length of transverse braces. Due to the inclined angle and the different stiffness of leaning-type arch bridge's main and stable arch ribs, the tangential and radial deformation of the transverse brace between the main and stable arch ribs differ from that of the conventional arch bridges. It is not a simple "S" shape. For leaning-type arch bridges, bending stiffness of the main arch rib is larger than that of the stable arch rib, so $\gamma_{01} > \gamma_{02}$, as shown in Figure 4, as an exceptional case, for conventional arch bridges, $\gamma_{01} = \gamma_{02}$. In order to obtain the relationship between γ_{01} , γ_{02} , a tangential mechanical model of transverse brace along the arch axis is established when a lateral buckling occurs, as shown in Figure 5(a); its corresponding bending moment is shown in Figure 5(b), from which (19) can be given as

$$\frac{M_{bh1}}{M_{bh2}} = \frac{E^I E_b I_u^I I_{bh} d + 6E^I E^{II} I_u^{II} I_{bh} b(x) \cos \phi_0}{6E^I E^{II} I_u^{II} I_{bh} b(x) \cos \phi_0 + E^{II} E_b I_u^{II} I_{bh} d \cos \phi_0} = e_1, \quad (19)$$

where e_1 is a constant.

From (17) and (19), it can be obtained that

$$\frac{M_{bh1}}{M_{bh2}} = \frac{2\gamma_1 + \gamma_3}{\gamma_1 + 2\gamma_3} = e_1. \quad (20)$$

Assuming $\gamma_3 = a_1 \gamma_1$, then

$$a_1 = \frac{\gamma_3}{\gamma_1} = \frac{2 - e_1}{2e_1 - 1}; \quad (21)$$

thus, from (18) and (21), the relationship of γ_{01} and γ_{02} can be obtained as

$$\gamma_{01} = \frac{2 + a_1}{1 + 2a_1} \gamma_{02}. \quad (22)$$

Substituting (17) and (18) into (16), the equation of tangential bending energy of the transverse braces in the full arch rib range is given as

$$\begin{aligned} U_{bH} &= \int_s \frac{\bar{U}_{bH}}{d} ds \\ &= \frac{2E_b I_{bh}}{9dh} \left[\int_{s_1} \frac{2\gamma_1 + \gamma_3}{\gamma_1 + \gamma_3} (4\gamma_1^2 + 4\gamma_1 \gamma_3 + \gamma_3^2) ds \right. \\ &\quad \left. + \int_{s_2} \frac{\gamma_1 + 2\gamma_3}{\gamma_1 + \gamma_3} (\gamma_1^2 + 4\gamma_1 \gamma_3 + 4\gamma_3^2) ds \right]. \end{aligned} \quad (23)$$

When the tangential local deformation occurs in the main and stable arch ribs, assuming the bending moments of the main and stable arch ribs along the radial direction are M_{cu1} and M_{cu2} , respectively, the single-panel section arch rib's local bending energy of the main and stable arch ribs can be expressed as

$$\begin{aligned} \bar{U}_{cu}^I + \bar{U}_{cu}^{II} &= \frac{1}{2E^I I_u^I} \int_d M_{cu1}^2 dx_0 + \frac{1}{2E^{II} I_u^{II}} \int_d M_{cu2}^2 dx_0 \\ &= \frac{6E^I I_u^I}{d} \gamma_2^2 + \frac{6E^{II} I_u^{II}}{d} \gamma_4^2. \end{aligned} \quad (24)$$

If the local bending energy of each single-panel section arch rib is the same, the full-arch-rib local bending deformation energy can be written as

$$\begin{aligned} U_{cu}^I + U_{cu}^{II} &= \int_s \frac{\bar{U}_{cu}^I}{d} ds + \int_s \frac{\bar{U}_{cu}^{II}}{d} ds \\ &= \int_s \frac{3E_b I_{bh}}{dh} \gamma_1 \gamma_2 ds + \int_s \frac{3E_b I_{bh}}{dh} \gamma_3 \gamma_4 ds. \end{aligned} \quad (25)$$

It could be obtained from Figure 4 that

$$2M_{cu1} = M_{bh1}, \quad 2M_{cu2} = M_{bh2}. \quad (26)$$

Therefore, from (23), (25), and (26), the local bending energy of arch rib and the tangential bending energy of transverse brace are obtained as

$$\begin{aligned} U_{bH} + U_{cu}^I + U_{cu}^{II} &= \frac{3E_b I_{bh}}{dh} \int_{s_1} (N_1 \eta_1 + \eta_2) \eta_1 \gamma'^2 ds \\ &\quad + \frac{3E_b I_{bh}}{dh} \int_{s_2} (N_2 \eta_3 + \eta_4) \eta_3 \gamma''^2 ds \\ &= E_b I_{bh} \frac{C_2^2}{R^3} D_1 + E_b I_{bh} \frac{C_2^2}{R^3} D_2, \end{aligned} \quad (27)$$

where

$$\begin{aligned} D_1 &= \frac{3n}{bs} (N_1 \eta_1 + \eta_2) \eta_1 \int_0^{\alpha_1} R^2 \left(\frac{2\pi}{\alpha_2} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi, \\ D_2 &= \frac{3n}{bs} (N_2 \eta_3 + \eta_4) \eta_3 \int_0^{\alpha_2} R^2 \left(\frac{2\pi}{\alpha_2} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_2} d\varphi, \\ \gamma' &= \gamma_1 + \gamma_2, \\ \gamma'' &= \gamma_3 + \gamma_4, \\ \eta_1 &= \frac{\gamma_1}{\gamma'} = \frac{1}{1 + (dE_b I_{bh}) / (2hE^I I_u^I)}, \end{aligned}$$

FIGURE 5: Mechanical model and bending moment diagram in tangential direction.

$$\eta_2 = \frac{\gamma_2}{\gamma'} = \frac{1}{1 + (2hE^I I_u^I) / (dE_b I_{bh})},$$

$$\eta_3 = \frac{\gamma_3}{\gamma''} = \frac{1}{1 + (dE_b I_{bh}) / (2hE^{II} I_u^{II})},$$

$$\eta_4 = \frac{\gamma_4}{\gamma'''} = \frac{1}{1 + (2hE^{II} I_u^{II}) / (dE_b I_{bh})},$$

$$N_1 = \frac{2}{27} \cdot \frac{(a_1 + 2)}{(a_1 + 1)} (4 + 4a_1 + a_1^2),$$

$$N_2 = \frac{2}{27} \cdot \frac{(1/a_1 + 2)}{(1/a_1 + 1)} \left(\frac{1}{a_1^2} + \frac{4}{a_1} + 4 \right).$$

(28)

The mechanical model and moment diagram of arch ribs and transverse brace in radial direction are shown in Figure 3. And the radial deformation energy of a single transverse brace can be written as

$$\begin{aligned} \bar{U}_{bv} = & \frac{1}{2E_b I_b} \int_0^{\gamma'_{01}} \left(\frac{M_{bv1}}{\gamma'_{01}} \gamma'_0 \right)^2 d\gamma'_0 \\ & + \frac{1}{2E_b I_{bv}} \int_0^{\gamma'_{02}} \left(\frac{M_{bv2}}{\gamma'_{02}} \gamma'_0 \right)^2 d\gamma'_0, \end{aligned} \quad (29)$$

where $E_b I_{bv}$ is the transverse brace's bending stiffness along radial direction of arch rib and γ'_{01} and γ'_{02} represent the distance from the contraflexure point of transverse brace's radial deformation to the main and stable arch ribs, respectively.

The values of γ'_{01} and γ'_{02} are closely related to the bending stiffness of arch ribs, the length of the arch rib section, the bending stiffness, and length of the transverse braces. For conventional arch bridges the bending stiffness and inclined angle of two main arch ribs are the same, so $\gamma'_{01} = \gamma'_{02}$. However, for leaning-type arch bridges, as the

bending stiffness of main arch rib is far larger than that of stable arch rib, so $\gamma'_{01} > \gamma'_{02}$, as shown in Figure 3. In order to obtain the values of γ'_{01} and γ'_{02} , radial mechanical model of transverse brace along the arch central axis is established when a lateral buckling occurs, as shown in Figure 6(a). The bending moment caused by the transverse braces' radial deflection can be obtained by Castigliano's theorem of material mechanics, as shown in Figure 6(b), and (30) can be given as

$$e_2 = \frac{M_{bv1}}{M_{bv2}} = \frac{(E_b I_{bh} d + 2b \cos \phi_0 G^{II} T^{II}) G^I T^I}{(E_b I_{bh} d + 2b \cos \phi_0 G^I T^I) G^{II} T^{II}}. \quad (30)$$

According to the principle of similar triangles, we arrived at

$$\gamma'_{01} = \frac{e_2}{1 + e_2} b, \quad \gamma'_{02} = \frac{1}{1 + e_2} b. \quad (31)$$

The radial deformation energy of transverse brace can be expressed as

$$\begin{aligned} U_{bv} = & \frac{6E_b I_{bv}}{db} \frac{e_2}{1 + e_2} \int_s (\beta - \theta_1)^2 ds \\ & + \frac{6E_b I_{bv}}{db} \frac{1}{1 + e_2} \int_s (\beta - \theta_2)^2 ds \\ = & E_b I_{bv} \frac{(C_5 - C_1)^2}{R} D_3 + E_b I_{bv} \frac{C_3^2}{R} D_4 \\ & + E_b I_{bv} \frac{C_5^2}{R} D_4 + E_b I_{bv} \frac{C_3 C_5}{R} D_6, \end{aligned} \quad (32)$$

FIGURE 6: Mechanical model and moment diagram in radial direction.

where

$$\begin{aligned}
 D_3 &= \frac{6n}{bs} \frac{e_2}{1+e_2} \int_0^{\alpha_1} \left(1 - \cos \frac{2\pi\varphi}{\alpha_1}\right)^2 R^2 d\varphi \\
 D_4 &= \frac{6n}{bs} \frac{1}{1+e_2} \int_0^{\alpha_2} \left(1 - \cos \frac{2\pi\varphi}{\alpha_2}\right)^2 R^2 d\varphi \\
 D_5 &= \frac{6n}{bs} \frac{1}{1+e_2} \int_0^{\alpha_2} \left(1 - \cos \frac{2\pi\varphi}{\alpha_1}\right)^2 R^2 d\varphi, \\
 D_6 &= \frac{6n}{bs} \frac{1}{1+e_2} \int_0^{\alpha_2} -2 \left(1 - \cos \frac{2\pi\varphi}{\alpha_1}\right) \left(1 - \cos \frac{2\pi\varphi}{\alpha_2}\right) R^2 d\varphi.
 \end{aligned} \quad (33)$$

Assuming the distance between hangers of the main arch rib is d_h and its corresponding arc length of arch rib is d'_h , d_h is approximately equal to d'_h as it is previously assumed that the distance between hangers is small, and the distance between bridge deck and main arch rib is

$$y(\varphi) = R \left[\cos \left(\varphi - \frac{\alpha_1}{2} \right) - \cos \frac{\alpha_1}{2} \right]. \quad (34)$$

As shown in Figure 7 the tension of the hanger is

$$T = qd_h. \quad (35)$$

The horizontal component of hanger tensions is

$$H = qd_h \sin \phi_1. \quad (36)$$

As the lateral displacement is sufficiently small, one arrives at

$$\sin \phi_1 = \phi_1 = \frac{u - u_d}{y(\varphi)}, \quad (37)$$

where $y(\varphi)$ is the length of the hanger.

FIGURE 7: Schematic diagram of the horizontal component of hanger tensions with lateral buckling.

The elastic potential energy of arch ribs and bridge deck system caused by the horizontal part of the tension of the hanger is given in

$$V_H = \frac{1}{2} \int_{-L}^L H (u_1 - u_d) \frac{d_x}{b_1} = q(C_2 - C_4)^2 E_1, \quad (38)$$

where

$$E_1 = \frac{1}{2} \int_0^{\alpha_1} \frac{R}{2f} \cos \varphi \left(1 - \cos \frac{2\pi\varphi}{\alpha_1}\right)^2 d\varphi, \quad (39)$$

f is the rise of main arch rib, and in order to simplify its integral, a conservative assumption of $y(\varphi) = f$ is adopted.

The lateral bending deformation energy of the bridge deck system is given in

$$U_d = \frac{1}{2} \int_L E_d I_d (u_d'')^2 dl = \frac{1}{2} \int_{s_1} E_d I_d \cos \varphi (u_d'')^2 ds \quad (40)$$

$$= E_d I_d \frac{C_4^2}{R^3} E_2,$$

where $E_d I_d$ is the lateral bending stiffness of the bridge deck system

$$E_2 = \frac{1}{2} \int_0^{\alpha_1} \left(\frac{2\pi}{\alpha_1} \right)^4 \cos \varphi \cos^2 \frac{2\pi\varphi}{\alpha_1} \cdot R^4 d\varphi. \quad (41)$$

Combined with (4), the potential energy of the external loading is

$$V = -\frac{1}{2} \int v q ds = -q (C_2^2 F_1 + C_5^2 F_2 R^2), \quad (42)$$

where

$$F_1 = \int_0^{\alpha_1} \left(\frac{2\pi}{\alpha_1} \right)^2 \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi \quad (43)$$

$$F_2 = \int_0^{\alpha_1} \left(\frac{2\pi}{\alpha_1} \right)^2 \frac{e_2^2}{(1+e_2)^2} \frac{b^2}{R^2} \sin^2 \frac{2\pi\varphi}{\alpha_1} d\varphi.$$

2.3. The Analytical Solution of Lateral Critical Buckling Loading. The total potential energy of the leaning-type arch bridge can be obtained based on (12), (14), (27), (32), and (38)–(42). According to the principle of stationary potential energy, the values of C_i minimizing the function $W(C_i)$ should therefore satisfy the algebraic equations:

$$\frac{\partial}{\partial C_i} (W) = 0 \quad (i = 1, 2, \dots, 5). \quad (44)$$

The existence of nontrivial solutions of (44) for C_i requires that the determinant of its coefficient matrix be equal to zero, then we obtained

$$H_1 \lambda_{cr}^3 + H_2 \lambda_{cr}^2 + H_3 \lambda_{cr} + H_4 = 0, \quad (45)$$

where

$$H_1 = 64F_1 F_2 S_1 S_3 E_1$$

$$H_2 = 8S_5^2 S_3 E_1 F_2 - 32S_1 S_3 S_4 F_1 E_1 - 32S_1 S_2 S_3 E_1 F_2$$

$$+ 64S_1 S_3 E_2 F_1 F_2 k_6 + 8S_1 S_7^2 E_1 F_2 + 8S_1 S_9^2 E_1 F_1$$

$$+ 8S_3 S_6^2 E_1 F_1 - 64S_1 S_3 E_1 E_2 F_2 k_6$$

$$H_4 = 4S_1 S_2 S_9^2 E_2 k_6 + S_3^2 S_9^2 E_2 k_6 - 4S_1 S_4 S_7^2 E_2 k_6$$

$$- 2S_5 S_6 S_7 S_9 E_2 k_6 - 4S_1 S_3 S_8^2 E_2 k_6 + 4S_3 S_5 S_6 S_8 E_2 k_6$$

$$+ 4S_1 S_7 S_8 S_9 E_2 k_6 - 4S_2 S_3 S_6^2 E_2 k_6 + S_6^2 S_7^2 E_2 k_6$$

$$+ 16S_1 S_2 S_3 S_3 E_2 k_6 - 4S_3 S_4 S_5^2 E_2 k_6. \quad (46)$$

The lateral buckling critical load coefficient was obtained by solving (45), and then the lateral critical buckling load of leaning-type arch bridge is

$$q_{cr} = \lambda_{cr} \frac{E^I I_u^I}{R^3}. \quad (47)$$

3. Lateral Buckling Critical Load under the Hinged Boundary Condition of Main Arch Rib

The following assumptions are made in the derivation process: the main arch ribs are hinged, the stable arch ribs are fixed, and the other assumptions are the same as those stated in Section 2.1. The variables without special explanation are the same as aforementioned.

The torsional angle of arch axis of the main and stable arch ribs is shown as

$$\theta_1 = C_1 \sin \frac{\pi\varphi}{\alpha_1}, \quad (48a)$$

$$\theta_2 = C_3 \left(1 - \cos \frac{2\pi\varphi}{\alpha_2} \right). \quad (48b)$$

The lateral displacements of arch axis of the main and stable arch ribs are given as

$$u_1 = C_2 \sin \frac{\pi\varphi}{\alpha_1}, \quad (49)$$

$$u_2 = C_2 \sin \frac{\pi\varphi}{\alpha_2}. \quad (50)$$

The lateral displacements of bridge deck system is given as

$$u_d = C_4 \sin \frac{\pi\varphi}{\alpha_1}. \quad (51)$$

The global torsional angle of the transverse brace in radial plane caused by arch rib's deflection is

$$\beta = C_5 \sin \frac{\pi\varphi}{\alpha_1}, \quad (52)$$

where C_1, C_2, C_3, C_4 , and C_5 are all constants.

The above displacement functions should satisfy the following boundary conditions:

- (1) when $\varphi = 0$ and $\varphi = \alpha_1$, $\theta_1 = \beta = 0$, $u_1 = u_d = 0$, $\theta_1'' = \beta'' = 0$, and $u_1'' = u_d'' = 0$
- (2) when $\varphi = 0$ and $\varphi = \alpha_2$, $\theta_2 = 0$, $u_2 = 0$, $\theta_2' = 0$, and $u_2'' = 0$.

The derivation method of the lateral buckling critical load under the hinged boundary condition of main arch ribs is the same as stated above. Due to the limitation of the paper length, the derivation process is omitted; only the calculated results are discussed in the following section.

FIGURE 8: The FE model under the fixed boundary condition.

4. Verification Example

The leaning-type arch bridge, Shengli Bridge, with a span of 75 m in Jiangmen city of China is used to verify the accuracy of the derived analytical solution presented in this paper. A three-dimensional finite element model is established by using the Midas/Civil FEM software to calculate arch bridge's lateral buckling critical load for comparison. The main and stable arch ribs of this bridge are both fixed at the arch ends. The FE model is shown in Figure 8.

There are 284 elements and 217 nodes in this FE model. Spatial beam element with 6 degrees of freedom at each node is used to simulate the arch rib, transverse brace, girder, and transverse girder. Spatial truss element with 3 degrees of freedom at each node is used to simulate the hanger. The calculation parameters of the leaning-type bridge are listed in Table 1. Numerical analysis is carried out as the following steps. (1) 1N/m uniform load is applied to the middle of the transverse girder of the bridge deck system. (2) By using Midas/Civil's buckling eigenvalue solver, the eigenvalue λ of the bridge is obtained which indicates the lateral buckling critical load of the bridge.

Comparison of the FEM results and the analytical results of the leaning-type arch bridge under fixed and hinged boundary conditions when the stable rib inclined angle is 5° , 7° , 9° , 11° , 13° , 15° , 17° , 19° , 21° , 24° , 27° , and 30° , are shown in Tables 2 and 3. The contrastive results show that the analytical results agree well with the FEM results with the relative error no more than 3.47%, 3.20% under fixed boundary condition and hinged boundary condition, respectively, which indicate the accuracy of the analytical solution for the stable critical load of the leaning-type arch bridge presented in this paper.

5. Parametric Analysis

5.1. Effect of Main Arch Rib's Boundary Condition on the Critical Buckling Load. Figure 9 shows the lateral buckling critical load of a leaning-type arch bridge for both cases of fixed-end main arch ribs and hinged-end main arch ribs. It can be seen from this figure that the lateral buckling critical load of the leaning-type arch bridge with fixed main arch ribs is approximately 1.14 to 1.16 times that of the leaning-type arch bridge with hinged main arch ribs. As the inclined angle increases from 5° to 30° , the critical load q_{cr} increases by approximately 11.18% for cases of fixed-end main arch ribs

FIGURE 9: Comparison of critical buckling load under different boundary conditions.

and the critical load q_{cr} increases by approximately 19.57% for cases of hinged-end main arch ribs.

5.2. Effect of the Central Angle on the Critical Buckling Load. Figure 10 shows the critical load value q_{cr} when the central angle of the main arch rib and the stable arch rib are the same (the central angle of the main arch rib and the stable arch rib is 87.21° in this case), and different (the central angle of the main arch rib is 87.21° and the central angle of the stable arch rib is 104° in this case) as the inclined angle increases from 5° to 30° . It can be seen from Figure 10 that the difference of the critical load q_{cr} between same central angle model and the different central angle model is small enough to be neglected, the former's lateral buckling critical load is only 1.02 to 1.03 times of that of the latter. It indicates that the central angle of the stable arch has relatively less effect on the lateral buckling critical load of the arch bridge, and the central angles of the main arch rib and the stable arch rib can be considered to be the same.

5.3. Effect of Hanger Tensions and Bridge Deck on Critical Buckling Load. Figure 11 shows the critical load value with or without considering hanger tensions and bridge deck as the inclined angle increases from 5° to 30° . From this figure it can be seen that if the hanger tensions and bridge deck are considered, the critical load is 6.664 MN/m and 7.198 MN/m when inclined angle is $\phi_0 = 5^\circ$ and 30° , respectively. However, if the hanger tensions and the bridge deck are neglected, the critical load is 2.171 MN/m and 2.451 MN/m when inclined angle is $\phi_0 = 5^\circ$ and 30° , respectively. The critical load increases by 2.94 and 3.07 times, respectively, as compared with that of neglecting the hanger tensions and bridge deck. The results indicate that the hanger tensions and bridge deck can greatly improve the lateral stability of the leaning-type arch bridge.

TABLE 1: Calculation parameters used in the FE model.

Span (m)	75
Rise-span ratio	1/4
The central angle of main arch (°)	87.21
The central angle of stable arch (°)	104
Inclined angle of stable arch rib (°)	5, 7, 9, 11, 13, 15, 17, 19, 21, 24, 27 and 30
Transverse brace length on arch crown (m)	1.55
Number of transverse brace	6
Lateral bending stiffness of main arch rib ($\text{MN}\cdot\text{m}^2$)	1.414×10^4
Torsional stiffness of main arch rib ($\text{MN}\cdot\text{m}^2$)	1.111×10^3
Lateral bending stiffness of stable arch rib ($\text{MN}\cdot\text{m}^2$)	1.179×10^4
Torsional stiffness of main arch rib ($\text{MN}\cdot\text{m}^2$)	0.926×10^3
Transverse brace's bending stiffness along radial direction of arch rib ($\text{MN}\cdot\text{m}^2$)	36.84
Transverse brace's bending stiffness along tangential direction of arch rib ($\text{MN}\cdot\text{m}^2$)	1.058×10^2
Lateral bending stiffness of girder ($\text{MN}\cdot\text{m}^2$)	2.3×10^7
Vertical bending stiffness of girder ($\text{MN}\cdot\text{m}^2$)	5.52×10^4
Lateral bending stiffness of transverse beam ($\text{MN}\cdot\text{m}^2$)	1.15×10^4
Vertical bending stiffness of transverse beam ($\text{MN}\cdot\text{m}^2$)	7.187×10^2
The cross section area of hangers (m^2)	1.25664×10^{-3}
Poisson's ratio	0.2

TABLE 2: Comparison between the analytical solution and FEM results under fixed boundary condition.

Inclined angle	5°	7°	9°	11°	13°	15°	17°	19°	21°	24°	27°	30°
Analytical (MN/m)	6.35	6.42	6.51	6.59	6.66	6.72	6.79	6.85	6.90	6.97	7.02	7.06
FEM (MN/m)	6.42	6.45	6.48	6.51	6.53	6.55	6.59	6.62	6.71	6.74	6.87	6.90
Error (%)	1.09	0.47	0.46	1.23	1.99	2.60	3.03	3.47	2.83	3.41	2.18	2.32

FIGURE 10: Comparison of critical buckling load under the same and different central angle.

FIGURE 11: Comparison of critical buckling load with and without hanger tensions effect.

5.4. Effect of the Stable Arch Rib on Critical Buckling Load.

Figure 12 shows the critical load q_{cr} of the models with or without stable arch ribs as the inclined angle increases from 5° to 30°. From this figure it can be seen that the critical load of the models with stable arch ribs is always larger than that

of the models without stable arch ribs and the percentage of increase is from 31.5% and 42.1% as the inclined angle increased. The results indicate that the effect of stable arch rib on critical load is significant.

TABLE 3: Comparison between analytical solution and FEM results under hinged boundary condition.

Inclined angle	5°	7°	9°	11°	13°	15°	17°	19°	21°	24°	27°	30°
Analytical (MN/m)	5.57	5.63	5.73	5.86	6.03	6.17	6.27	6.34	6.45	6.52	6.60	6.66
FEM (MN/m)	5.70	5.75	5.87	5.90	5.96	6.00	6.09	6.19	6.25	6.32	6.40	6.47
Error (%)	2.28	2.09	2.38	0.68	1.17	2.83	2.96	2.42	3.20	3.16	3.13	2.93

FIGURE 12: Comparison of critical buckling load with and without stable arch rib.

6. Conclusions

This paper has derived an analytical solution for lateral buckling critical load of leaning-type arch bridge based on the Ritz method, and the accuracy of this solution has been verified through a numerical example. Moreover, parametric analysis is carried out in order to investigate how changes in certain design parameters would affect the critical load of the leaning-type arch bridges by using the analytical solution presented in this paper. The main conclusions are as follows.

- (1) The analytical solution present in this paper can be used to calculate the lateral buckling critical load of the leaning-type arch bridges in different cases, including the central angles of the main arch rib and stable arch rib which are different; both the main arch ribs and the stable arch ribs are fixed in their ends, the main arch ribs with hinged boundary condition while the stable arch ribs with fixed boundary conditions. From the comparison of the analytical results and the FEM results, the analytical solution presented in this paper is verified to be sufficiently accurate.
- (2) The lateral buckling critical load under fixed boundary condition is approximately 1.14 to 1.16 times as large as that under hinged boundary conditions, which indicate that the lateral stability of the former is better than that of the latter.
- (3) The critical load with the same central angles is slightly bigger than the one with different central

angles, and the former's lateral buckling critical load is 1.02 to 1.03 times as big as that of the latter. It indicates that the central angle of the stable arch has relatively less influence on the lateral buckling critical load of the leaning-type arch bridge, and therefore the central angles of the main arch and the stable arch can be considered to be the same for convenience.

- (4) Stable arch rib can significantly increase the lateral buckling critical load q_{cr} of leaning-type arch bridge by 31.5% to 42.1% when stable arch rib is considered under the fixed boundary condition, where the value of q_{cr} increases as the inclined angle of stable arch rib increases.
- (5) The hanger tensions and bridge deck have significant effect on the critical load, and when considering the effect of hanger tensions and bridge deck, the critical load can improve by 2.94 to 3.07 times.

Notations

- θ_1, θ_2 : The torsional angle of the main and stable arch ribs
- u_1, u_2 : The lateral displacement of the main and stable arch ribs under the global coordinate
- u_1^I, u_1^{II} : The lateral displacement of the main and stable arch ribs under the local coordinate
- u_d : The lateral displacement of bridge deck system
- ϕ_0 : The angle between the main arch rib and the stable arch rib
- φ : The angle of a position in the arch ribs
- α_1, α_2 : The central angle of the main and stable arch ribs
- β : The global torsional angle of the transverse brace caused by deformation of arch rib in radial plane
- R : The radius of the main and the stable arch ribs
- $b(x)$: The distance between the main arch rib and the stable arch rib
- b_0 : The length of the transverse brace at the arch crown
- h : The distance between the main and stable arch ribs at quarter span
- d : The distance between the transverse braces
- d_h : The distance between the hangers
- K_u^I, K_u^{II} : The lateral deflection curvature of the main and stable arch ribs respectively
- K_w^I, K_w^{II} : The torsional deflection curvatures of the main and stable arch ribs, respectively

U_u^I, U_u^{II} :	The lateral bending deformation energy of the main and stable arch ribs, respectively
U_w^I, U_w^{II} :	The torsional deformation energy of the main and stable arch ribs, respectively
U_{cu}^I, U_{cu}^{II} :	The local bending deformation energy of the main and stable arch ribs, respectively
U_{bv} :	The bending deformation energy of the transverse braces in radial directions
U_{bH} :	The bending deformation energy of the transverse braces in tangential directions
V_H :	The elastic potential energy of the arch ribs and the bridge deck system under the horizontal component of the hanger tensions
V_d :	The elastic potential energy of bridge deck system
V :	The potential energy of external loading
$\bar{U}_{cu}^I, \bar{U}_{cu}^{II}$:	The single-panel arch rib local bending energy of main and stable arch ribs
$\bar{U}_{bH}$:	The tangential bending deformation energy of the single transverse brace
$\bar{U}_{bv}$:	The radial deformation energy of a single transverse brace
$E^I I_u^I, E^{II} I_u^{II}$:	The lateral bending stiffness of the main and stable arch ribs
$G^I T^I, G^{II} T^{II}$:	The torsional stiffness of the main and stable arch ribs
$E_b I_{bh}$:	The bending stiffness of transverse brace along tangential direction of arch rib
$E_b I_{bv}$:	The bending stiffness of transverse brace along radial direction of arch rib
M_{cu1}, M_{cu2} :	The bending moments of main and stable arch ribs along radial direction
M_{bh1}, M_{bh2} :	The tangential bending moment of transverse brace near the main and stable arch ribs
M_{bv1}, M_{bv2} :	The vertical bending moment on both ends of transverse brace near the main and stable arch ribs
γ_{01}, γ_{02} :	The distances from the contraflexure point of transverse brace tangential deformation to main and stable arch ribs, respectively
$\gamma'_{01}, \gamma'_{02}$:	The distance from the vertical contraflexure point of transverse brace's radial deformation to main and stable arch ribs, respectively
γ_1, γ_3 :	The tangential angular rotation of the transverse brace at the intersection points between the main and stable arch rib and the transverse brace, respectively
γ_2, γ_4 :	The tangential angular rotation of the main and stable arch rib at the intersection points between the main and stable arch rib and the transverse brace, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This study was sponsored by the National Natural Science Foundation of China (nos. 11272095, 51378133, and 51208123), the Science and Technology Planning Major Project of Guangzhou City (no. 2011Y2-00006), the Key Technological Innovation Program of Guangdong Ministry Education (no. 2012CXZD0028), the Key Project supported by the Natural Science Foundation of Guangdong Province (no. S2011030002800), and the Talent Introduction Project supported by the Higher Education Department of Guangdong Province in 2012.

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