Research Article

Existence of Solutions for Nonlinear Impulsive Fractional Differential Equations with p -Laplacian Operator

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This paper studies the existence of solutions for a nonlinear boundary value problem of impulsive fractional differential equations with p -Laplacian operator. Our results are based on some standard fixed point theorems. Examples are given to show the applicability of our results.

1. Introduction

Fractional calculus (differentiation and integration of arbitrary order) has proved to be an important tool in the modeling of dynamical systems associated with phenomena such as fractals and chaos. In fact, this branch of calculus has found its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, and fitting of experimental data. Fractional differential equations serve as an excellent tool for the description of hereditary properties of various materials and processes. The interest in the study of fractional-order differential equations lies in the fact that fractional-order models are found to be more accurate than the classical integer-order models; that is, there are more degrees of freedom in the fractional-order models. In consequence, the subject of fractional differential equations is gaining more and more attention. For some recent development on the topic, see [1–11] and the references therein.

Since the p -Laplacian operator and fractional calculus arise from many applied fields such as turbulent filtration in

porous media, blood flow problems, rheology, modeling of viscoplasticity, and material science, it is worth studying the fractional p -Laplacian differential equations. The research of boundary value problems for p -Laplacian equations of fractional order has just begun in recent years; see [12–16].

The theory of impulsive differential equations is adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jumps change of their state. Impulsive differential equations have become an important area of research in recent years of the needs of modern technology, engineering, economic and physics. Moreover, impulsive differential equations are richer in applications compared to the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations. For the introduction of the basic theory of impulsive equations, see [17–22] and the references therein. It is worthwhile mentioning that impulsive differential equations of fractional order have not been much studied and many aspects of these equations are yet to be explored. The recent results on impulsive fractional differential equations can be found in [23–30].

The investigation on fractional p -Laplacian impulsive differential equations has not been appreciated well enough.

As far as we know, there is only one paper that has considered nonlinear fractional impulsive differential equation with p -Laplacian operator; see [31].

Motivated by the above mentioned work on p -Laplacian and impulsive boundary value problems of fractional order, in this study, we consider the following problem:

$$\begin{aligned} (\phi_p({}^C D_{0+}^\alpha u(t)))' &= f(t, u(t)), \quad 1 < \alpha \leq 2, t \in J', \\ \Delta u(t_k) &= I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u'(t_k)), \\ k &= 1, 2, \dots, m, \\ au(0) - bu'(0) &= 0, \quad cu(1) + du'(1) = 0, \end{aligned} \quad (1)$$

where ${}^C D_{0+}^\alpha$ is the Caputo fractional derivative, ϕ_p is a p -Laplacian operator, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, ϕ_p is invertible, and $(\phi_p)^{-1} = \phi_q$ where $(1/p) + (1/q) = 1$, $f \in \mathcal{C}([0, 1] \times \mathbb{R}, \mathbb{R})$, $I_k, J_k \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $a, b, c, d \in [0, \infty)$ with $ac + ad + bc > 0$, $J = [0, 1]$, $0 = t_0 < t_1 < \dots < t_k < \dots < t_m < t_{m+1} = 1$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ ($k = 1, 2, \dots, m$), respectively. $\Delta u'(t_k)$ have a similar meaning for $u'(t)$.

This paper is organized as follows. In Section 2, we provide some necessary definitions and preliminary lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give some examples to demonstrate our main results.

2. Preliminaries

In this section, we present auxiliary lemmas which will be used later.

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\dots$, $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, 1]$, and we introduce the following spaces.

$PC(J, \mathbb{R}) = \{u : J \rightarrow \mathbb{R} \mid u \in \mathcal{C}(J_k), k = 0, 1, \dots, m \text{ and } u(t_k^+) \text{ exist, } k = 1, 2, \dots, m\}$ with the norm $\|u\| = \sup_{t \in J} |u(t)|$, and $PC^1(J, \mathbb{R}) = \{u : J \rightarrow \mathbb{R} \mid u \in \mathcal{C}^1(J_k), k = 0, 1, \dots, m, \text{ and } u(t_k^+), u'(t_k^+) \text{ exist, } k = 1, 2, \dots, m\}$ with the norm $\|u\|_{PC^1} = \max\{\|u\|, \|u'\|\}$. Obviously, $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ are Banach spaces.

Definition 1. For a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (2)$$

$$n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α .

Definition 2. The Riemann-Liouville fractional integral of order α is defined as

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \quad (3)$$

provided the integral exists.

Lemma 3 (see [7]). Let $\alpha > 0$; then the differential equation

$${}^C D_{0+}^\alpha f(t) = 0 \quad (4)$$

has solution $f(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 4 (see [7]). Let $\alpha > 0$, $f \in L(0, 1)$; then

$${}^C D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t). \quad (5)$$

Definition 5. A function $u \in PC^1(J, \mathbb{R})$ with its Caputo derivative of order α existing on J is a solution of (1) if it satisfies (1).

We need the following known results to prove the existence of solutions for (1).

Theorem 6 (see [32]). Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \rightarrow E$ be a completely continuous operator such that

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega. \quad (6)$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 7 (see [32]). Let E be a Banach space. Assume that $T : E \rightarrow E$ is a completely continuous operator and the set $V = \{u \in E \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E .

Lemma 8. For a given $y \in \mathcal{C}[0, 1]$, a function u is a solution of the impulsive boundary value problem of fractional order

$$\begin{aligned} (\phi_p({}^C D_{0+}^\alpha u(t)))' &= y(t), \quad 1 < \alpha \leq 2, t \in J', \\ \Delta u(t_k) &= I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u'(t_k)), \\ k &= 1, 2, \dots, m, \\ au(0) - bu'(0) &= 0, \quad cu(1) + du'(1) = 0, \end{aligned} \quad (7)$$

if and only if u is a solution of the impulsive fractional integral equation

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s y(r) dr \right) ds \\ - \frac{at+b}{ac+ad+bc} \\ \times \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \\ \times \phi_q \left(\int_0^s y(r) dr \right) ds - \mathcal{A}, \\ t \in J_0; \\ \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s y(r) dr \right) ds \\ - \frac{at+b}{ac+ad+bc} \\ \times \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \\ \times \phi_q \left(\int_0^s y(r) dr \right) ds \\ + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\ \times \phi_q \left(\int_0^s y(r) dr \right) ds \\ \left. + I_i(u(t_i)) \right] \\ + \sum_{i=1}^{k-1} (t_k - t_i) \\ \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ \left. + J_i(u(t_i)) \right] \\ + \sum_{i=1}^k (t - t_k) \\ \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ \left. + J_i(u(t_i)) \right] - \mathcal{A}, \\ t \in J_k, \quad k=1, 2, \dots, m, \end{cases} \quad (8)$$

where

$$\mathcal{A} = \frac{at+b}{ac+ad+bc} \\ \times \left\{ \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \right. \\ \times \phi_q \left(\int_0^s y(r) dr \right) ds \\ \left. \left. + cI_i(u(t_i)) + dJ_i(u(t_i)) \right] \right\}$$

$$+ c \sum_{i=1}^{m-1} (t_m - t_i) \\ \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ \left. + J_i(u(t_i)) \right] \\ + c \sum_{i=1}^m (1 - t_m) \\ \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ \left. + J_i(u(t_i)) \right] \Bigg\}. \quad (9)$$

Proof. Let u be a solution of (7). Then, for $t \in J_0$, there exist constants $c_0, c_1 \in \mathbb{R}$ such that

$$u(t) = I_0^\alpha \phi_q \left(\int_0^t y(s) ds \right) - c_0 - c_1 t \\ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s y(r) dr \right) ds - c_0 - c_1 t, \quad (10)$$

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \phi_q \left(\int_0^s y(r) dr \right) ds - c_1. \quad (11)$$

For $t \in J_1$, there exist constants $d_0, d_1 \in \mathbb{R}$ such that

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s y(r) dr \right) ds \\ - d_0 - d_1 (t - t_1), \\ u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^t (t-s)^{\alpha-2} \phi_q \left(\int_0^s y(r) dr \right) ds - d_1. \quad (12)$$

Then we have

$$u(t_1^-) = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} \phi_q \left(\int_0^s y(r) dr \right) ds - c_0 - c_1 t_1, \\ u(t_1^+) = -d_0, \\ u'(t_1^-) = \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} \phi_q \left(\int_0^s y(r) dr \right) ds - c_1, \\ u'(t_1^+) = -d_1. \quad (13)$$

In view of $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$ and $\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = J_1(u(t_1))$, we have

$$\begin{aligned} -d_0 &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad - c_0 - c_1 t_1 + I_1(u(t_1)), \\ -d_1 &= \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad - c_1 + J_1(u(t_1)). \end{aligned} \quad (14)$$

Consequently,

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad + \frac{(t-t_1)}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad + I_1(u(t_1)) + (t-t_1) J_1(u(t_1)) - c_0 - c_1 t, \quad t \in J_1. \end{aligned} \quad (15)$$

By a similar process, we can get

$$\begin{aligned} u(t) &= \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \right. \\ &\quad \quad \left. \times \left(\int_0^s y(r) dr \right) ds + I_i(u(t_i)) \right] \\ &\quad + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ &\quad \quad \left. \times \left(\int_0^s y(r) dr \right) ds + J_i(u(t_i)) \right] \\ &\quad + \sum_{i=1}^k (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ &\quad \quad \left. + J_i(u(t_i)) \right] - c_0 - c_1 t, \\ &\quad t \in J_k, \quad k = 1, 2, \dots, m. \end{aligned} \quad (16)$$

By conditions $au(0) - bu'(0) = 0$ and $cu(1) - du'(1) = 0$, we have

$$\begin{aligned} c_0 &= \frac{b}{ac + ad + bc} \\ &\quad \times \left\{ \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \\ &\quad \quad \left. \times \phi_q \left(\int_0^s y(r) dr \right) ds \right. \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \\ &\quad \times \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad \left. + cI_i(u(t_i)) + dJ_i(u(t_i)) \right] \\ &+ c \sum_{i=1}^{m-1} (t_m - t_i) \\ &\quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ &\quad \quad \left. + J_i(u(t_i)) \right] \\ &+ c \sum_{i=1}^m (1 - t_m) \\ &\quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s y(r) dr \right) ds \right. \\ &\quad \quad \left. + J_i(u(t_i)) \right] \Bigg\}, \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{a}{ac + ad + bc} \\ &\quad \times \left\{ \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \\ &\quad \times \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad + \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \\ &\quad \times \phi_q \left(\int_0^s y(r) dr \right) ds \\ &\quad \left. + cI_i(u(t_i)) + dJ_i(u(t_i)) \right] \\ &\quad + c \sum_{i=1}^{m-1} (t_m - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ &\quad \quad \times \left(\int_0^s y(r) dr \right) ds \\ &\quad \quad \left. + J_i(u(t_i)) \right] \\ &\quad \left. + c \sum_{i=1}^m (1 - t_m) \right. \\ &\quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ &\quad \quad \times \left(\int_0^s y(r) dr \right) ds + J_i(u(t_i)) \Bigg\}. \end{aligned} \quad (17)$$

Substituting the value of c_i ($i = 0, 1$) in (10) and (16), we can get (8). Conversely, assume that u is a solution of the impulsive fractional integral equation (8); then by a direct computation, it follows that the solution given by (8) satisfies (7). This completes the proof. $\square$

3. Main Results

For the sake of convenience, we set

$$\lambda_1(t) = \frac{at+b}{ac+ad+bc}, \quad \lambda'_1(t) = \lambda_2 = \frac{a}{ac+ad+bc}. \quad (18)$$

Define the operator $T : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$ as

$$\begin{aligned} Tu(t) = & \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \\ & - \lambda_1(t) \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \phi_q \\ & \quad \times \left(\int_0^s f(r, u(r)) dr \right) ds \\ & + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \right. \\ & \quad \times \left. \left(\int_0^s f(r, u(r)) dr \right) ds + I_i(u(t_i)) \right] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \right. \\ & \quad \left. + J_i(u(t_i)) \right] \\ & + \sum_{i=1}^k (t - t_k) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \right. \\ & \quad \left. + J_i(u(t_i)) \right] \\ & - \lambda_1(t) \left\{ \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \right. \\ & \quad \times \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \\ & \quad \left. \left. + cI_i(u(t_i)) + dJ_i(u(t_i)) \right] \right\} \end{aligned}$$

$$\begin{aligned} & + c \sum_{i=1}^{m-1} (t_m - t_i) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ & \quad \times \left. \left(\int_0^s f(r, u(r)) dr \right) ds \right. \\ & \quad \left. + J_i(u(t_i)) \right] \\ & + c \sum_{i=1}^m (1 - t_m) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ & \quad \times \left. \left(\int_0^s f(r, u(r)) dr \right) ds \right. \\ & \quad \left. + J_i(u(t_i)) \right] \Bigg\}. \quad (19) \end{aligned}$$

Then the problem (1) has a solution if and only if the operator T has a fixed point.

Lemma 9. The operator $T : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$ defined by (19) is completely continuous.

Proof. It is obvious that T is continuous in view of the continuity of f , I_k , and J_k . Let $\Omega \subset PC(J, \mathbb{R})$ be bounded. Then, there exist positive constants $M_i > 0$ ($i = 1, 2, 3$) such that $|f(t, u)| \leq \phi_p(M_1)$, $|I_k(u)| \leq M_2$ and $|J_k(u)| \leq M_3$, $\forall u \in \Omega$. Thus, $\forall u \in \Omega$, and we have

$$\begin{aligned} |Tu(t)| \leq & \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\ & + \lambda_1(t) \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \\ & \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\ & + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\ & \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\ & \quad \left. + |I_i(u(t_i))| \right] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \right. \\ & \quad \left. + |J_i(u(t_i))| \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^k (t - t_k) \\
& \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \right. \\
& \quad \left. + |J_i(u(t_i))| \right] \\
& + \lambda_1(t) \left\{ \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \right. \\
& \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. + c |I_i(u(t_i))| + d |J_i(u(t_i))| \right] \\
& + c \sum_{i=1}^{m-1} (t_m - t_i) \\
& \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\
& \quad \times \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. + |J_i(u(t_i))| \right] \\
& + c \sum_{i=1}^m (1 - t_m) \\
& \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\
& \quad \times \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. + |J_i(u(t_i))| \right] \Bigg\} \\
& \leq M_1 \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + M_1 \lambda_1(t) \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) ds \\
& + \sum_{i=1}^m \left[M_1 \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} ds + M_2 \right] \\
& + \sum_{i=1}^{m-1} (t_m - t_i) \left[M_1 \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \\
& + \sum_{i=1}^k (t - t_k) \left[M_1 \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right]
\end{aligned}$$

$$\begin{aligned}
& + \lambda_1(t) \left\{ \sum_{i=1}^m \left[M_1 \int_{t_{i-1}}^{t_i} \left(\frac{c(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} \right. \right. \right. \\
& \quad \left. \left. + \frac{d(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) ds \right. \\
& \quad \left. + c M_2 + d M_3 \right] \\
& + c \sum_{i=1}^{m-1} (t_m - t_i) \\
& \quad \times \left[M_1 \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \\
& + c \sum_{i=1}^m (1 - t_m) \\
& \quad \times \left[M_1 \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \Bigg\} \\
& \leq \frac{(1+m)(1+c\lambda_1(1))M_1}{\Gamma(\alpha+1)} \\
& + \frac{[(2m-1)(1+c\lambda_1(1)) + d\lambda_1(1)(1+m)]M_1}{\Gamma(\alpha)} \\
& + (1+c\lambda_1(1))mM_2 \\
& + [(2m-1)(1+c\lambda_1(1)) + md\lambda_1(1)]M_3. \tag{20}
\end{aligned}$$

Since $t \in [0, 1]$, therefore there exists a positive constant M , such that $\|Tu\| \leq M$, which implies that the operator T is uniformly bounded.

On the other hand, for any $t \in J_k$, $0 \leq k \leq m$, we have

$$\begin{aligned}
|(Tu)'(t)| & \leq \int_{t_k}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& + \lambda_2 \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \\
& \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \right. \\
& \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. + |J_i(u(t_i))| \right]
\end{aligned}$$

Hence, for $t_1, t_2 \in J_k, t_1 < t_2, 0 \leq k \leq m$, we have

Thus, it follows that $\|Tu\| \leq \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 6, the operator T has at least one fixed point, which in turn implies that the problem (1) has at least one solution $u \in \overline{\Omega}$. $\square$

Theorem 11. Assume that there exist positive constants M_i ($i = 1, 2, 3$) such that

$$|f(t, u)| \leq \phi_p(M_1), \quad |I_k(u)| \leq M_2, \quad |J_k(u)| \leq M_3, \\ \text{for } t \in J, u \in \mathbb{R} \text{ and } k = 1, 2, \dots, m. \quad (25)$$

Then the problem (1) has at least one solution.

Proof. Let us show that the set $V = \{u \in PC(J, \mathbb{R}) \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Let $u \in V$, and then $u = \mu Tu$, $0 < \mu < 1$. For any $t \in J$, we have

$$\begin{aligned} u(t) = & \int_{t_k}^t \frac{\mu(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \\ & - \lambda_1(t) \int_{t_m}^1 \left(\frac{\mu c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \\ & \times \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \\ & + \sum_{i=1}^k \mu \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\ & \quad \times \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \\ & \quad \left. + I_i(u(t_i)) \right] \\ & + \sum_{i=1}^{k-1} \mu(t_k - t_i) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \right. \\ & \quad \left. + J_i(u(t_i)) \right] \\ & + \sum_{i=1}^k \mu(t - t_k) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \right. \\ & \quad \left. + J_i(u(t_i)) \right] \\ & - \lambda_1(t) \left\{ \sum_{i=1}^m \mu \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \right. \\ & \quad \times \phi_q \left(\int_0^s f(r, u(r)) dr \right) ds \\ & \quad \left. \left. + cI_i(u(t_i)) + dJ_i(u(t_i)) \right] \right\} \end{aligned}$$

$$\begin{aligned} & + c \sum_{i=1}^{m-1} \mu(t_m - t_i) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ & \quad \times \left(\int_0^s f(r, u(r)) dr \right) ds \\ & \quad \left. + J_i(u(t_i)) \right] \\ & + c \sum_{i=1}^m \mu(1 - t_m) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\ & \quad \times \left(\int_0^s f(r, u(r)) dr \right) ds \\ & \quad \left. + J_i(u(t_i)) \right] \Bigg\}. \quad (26) \end{aligned}$$

Combining (25) and (26) and employing the procedure used to obtain (20), we obtain

$$\begin{aligned} |u(t)| = & \mu |Tu(t)| \\ \leq & \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\ & + \lambda_1(t) \int_{t_m}^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \\ & \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\ & + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\ & \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\ & \quad \left. + |I_i(u(t_i))| \right] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \\ & \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \right. \\ & \quad \left. + |J_i(u(t_i))| \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^k (t - t_k) \\
& \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \right. \\
& \quad \left. + |J_i(u(t_i))| \right] \\
& + \lambda_1(t) \left\{ \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} \left(\frac{c(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \right. \right. \\
& \quad \times \phi_q \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. \left. + c |I_i(u(t_i))| + d |J_i(u(t_i))| \right] \right\} \\
& + c \sum_{i=1}^{m-1} (t_m - t_i) \\
& \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\
& \quad \times \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. + |J_i(u(t_i))| \right] \\
& + c \sum_{i=1}^m (1 - t_m) \\
& \quad \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \right. \\
& \quad \times \left(\int_0^s |f(r, u(r))| dr \right) ds \\
& \quad \left. + |J_i(u(t_i))| \right] \Bigg\} \\
& \leq \left\{ \frac{(1+m)(1+c\lambda_1(1))M_1}{\Gamma(\alpha+1)} \right. \\
& \quad + \frac{[(2m-1)(1+c\lambda_1(1)) + d\lambda_1(1)(1+m)]M_1}{\Gamma(\alpha)} \\
& \quad + (1+c\lambda_1(1))mM_2 \\
& \quad \left. + [(2m-1)(1+c\lambda_1(1)) + md\lambda_1(1)]M_3 \right\} := M, \tag{27}
\end{aligned}$$

which implies that $\|u\| \leq M$ for any $t \in J$. So, the set V is bounded. Thus, by Theorem 7, the operator T has at least one fixed point. Hence the problem (1) has at least one solution. $\square$

4. Examples

Example 1. For $1 < \alpha \leq 2$, consider the following fractional order impulsive boundary value problem:

$$\begin{aligned}
& (\phi_3({}^C D_{0+}^\alpha u(t)))' = t \arctan^3 u(t) + e^{2t} u^5(t), \\
& 0 < t < 1, \quad t \neq t_1, \quad 0 < t_1 < 1, \\
& \Delta u(t_1) = \sin(1 - \cos u(t_1)), \\
& \Delta u'(t_1) = e^{u^2(t_1)} - \cos^2 u(t_1), \\
& u(0) - 2u'(0) = 0, \quad 3u(1) + u'(1) = 0.
\end{aligned} \tag{28}$$

Here $m = 1$, $p = 3$, $a = d = 1$, $b = 2$, $c = 3$, $f(t, u) = t \arctan^3 u(t) + e^{2t} u^5(t)$, $I_1(u(t_1)) = \sin(1 - \cos u)$, and $J_1(u(t_1)) = e^{u^2} - \cos^2 u$. Clearly

$$\begin{aligned}
\lim_{u \rightarrow 0} \frac{f(t, u)}{\phi_p(u)} &= \lim_{u \rightarrow 0} \frac{t \arctan^3 u(t) + e^{2t} u^5(t)}{u |u|} \\
&= \begin{cases} \lim_{u \rightarrow 0^+} \frac{t \arctan^3 u(t) + e^{2t} u^5(t)}{u^2}, \\ \lim_{u \rightarrow 0^-} \frac{t \arctan^3 u(t) + e^{2t} u^5(t)}{-u^2}, \end{cases} \\
&= \begin{cases} 0, \\ 0, \end{cases} \\
&= 0,
\end{aligned} \tag{29}$$

$$\lim_{u \rightarrow 0} \frac{I_1(u)}{u} = \lim_{u \rightarrow 0} \frac{\sin(1 - \cos u)}{u} = 0,$$

$$\begin{aligned}
\lim_{u \rightarrow 0} \frac{J_1(u)}{u} &= \lim_{u \rightarrow 0} \frac{e^{u^2} - \cos^2 u}{u} \\
&= \lim_{u \rightarrow 0} 2ue^{u^2} + 2 \cos u \sin u = 0.
\end{aligned}$$

Furthermore, in this case, δ_i ($i = 1, 2$) given by (23) satisfy the inequality

$$\left(\frac{19}{5\Gamma(\alpha+1)} + \frac{5}{2\Gamma(\alpha)} \right) \delta_1 + \frac{19}{10} \delta_2 + \frac{11}{5} \delta_3 \leq 1. \tag{30}$$

Thus all the assumptions of Theorem 10 hold. Hence, the conclusion of Theorem 10 applies and the impulsive fractional boundary value problem (28) has at least one solution.

Example 2. Consider the following fractional order impulsive boundary value problem:

$$\begin{aligned}
& (\phi_4({}^C D_{0+}^\alpha u(t)))' = \frac{e^{3t} \cos^3 u(t)}{1 + u^2(t)}, \quad 0 < t < 1, \quad t \neq \frac{1}{3}, \\
& \Delta u\left(\frac{1}{3}\right) = \frac{2\cos^2 u(1/3)}{u^2(1/3) + 5}, \\
& \Delta u^{(1/3)} = e^{-2u^2(1/3)} \left(1 + \sin\left(1 + e^{2u(1/3)}\right) \right), \\
& 2u(0) - u'(0) = 0, \quad u(1) + 5u'(1) = 0.
\end{aligned} \tag{31}$$

Here $1 < \alpha \leq 2$, $m = 1$, $p = 4$, $a = 2$, $b = c = 1$, $d = 5$, $|f(t, u)| = |e^{3t} \cos^3 u / (1 + u^2)| \leq e^3 = \phi_4(e)$, $|I_1(u)| = |2 \cos^2 u / (u^2 + 5)| \leq 2/5$, and $|J_1(u)| = |e^{-2u^2} (1 + \sin(1 + e^{2u}))| \leq 2$.

In this case, $M_1 = e$, $M_2 = 2/5$, $M_3 = 2$, and the conditions of Theorem 11 can readily be verified. Thus, by the conclusion of Theorem 11, the problem (28) has at least one solution.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally in this paper. They read and approved the paper.

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