A Novel Multiple Attribute Satisfaction Evaluation Approach with Hesitant Intuitionistic Linguistic Fuzzy Information

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This paper investigates the multiple attribute decision making (MADM) problems in which the attribute values take the form of hesitant intuitionistic linguistic fuzzy element (HILFE). Firstly, motivated by the idea of intuitionistic linguistic variables (ILVs) and hesitant fuzzy elements (HFEs), the concept, operational laws, and comparison laws of HILFE are defined. Then, some aggregation operators are developed for aggregating the hesitant intuitionistic linguistic fuzzy information, such as hesitant intuitionistic linguistic fuzzy weighted aggregation operators, hesitant intuitionistic linguistic fuzzy ordered weighted aggregation operators, and generalized hesitant intuitionistic linguistic fuzzy weighted aggregation operators. Moreover, some desirable properties of these operators and the relationships between them are discussed. Based on the hesitant intuitionistic linguistic fuzzy weighted average (HILFWA) operator and the hesitant intuitionistic linguistic fuzzy weighted geometric (HILFWG) operator, an approach for evaluating satisfaction degree is proposed under hesitant intuitionistic linguistic fuzzy environment. Finally, a practical example of satisfaction evaluation for milk products is given to illustrate the application of the proposed method and to demonstrate its practicality and effectiveness.

1. Introduction

Multiattribute decision making (MADM), which addresses the problem of making an optimal choice that has the highest degree of satisfaction from a set of feasible alternatives that are characterized in terms of their attributes, both quantitative and qualitative, is a usual task in human activities. Due to the inherent vagueness of human preferences as well as the objects being fuzzy and uncertain or data about the decision-making problems domain, the attributes involved in the decision problems are not always expressed as crisp numbers, and some of them are more suitable to be denoted by fuzzy numbers [1–6]. The fuzzy set theory originally proposed by Zadeh [7] is a very useful tool to describe uncertain information. However, in some real decision-making situations, the fuzzy set is imprecise resulting from characterizing the fuzziness just by a membership degree. On the basis of the fuzzy set theory, Atanassov [8, 9] proposed the intuitionistic fuzzy set characterized by a membership function and a nonmembership function. Obviously, the intuitionistic fuzzy set can describe and characterize the fuzzy essence of the objective world more exquisitely, and it has received more and more attention since its appearance [10–20].

However, in the real world, decision makers usually cannot completely express their opinions by quantitative numbers, and some of them are more appropriately described by qualitative linguistic terms. Since linguistic variables [21] have been proposed, so far, a number of linguistic approaches have been defined such as 2-tuple linguistic [22], interval-valued 2-tuple linguistic [23], uncertain linguistic [24], trapezoid fuzzy linguistic [25], and trapezoid fuzzy 2-tuple linguistic [26] approaches. In order to express the uncertainty and ambiguity as accurate as possible, Wang and Li [27] proposed the concept of intuitionistic linguistic
set based on linguistic variables and intuitionistic fuzzy set, which can overcome the defects for intuitionistic fuzzy set, which can only roughly represent criteria's membership and nonmembership to a particular concept, such as "good" and "bad," and for linguistic variables which usually implies that membership degree is 1, and the nonmembership degree and hesitation degree of decision makers cannot be expressed.

In real decision-making processes, we often encounter such a situation that the decision makers are hesitant among a set of possible values which makes the outcome of decision making inconsistent. To solve this problem, the hesitant fuzzy set (HFS), an extension of fuzzy sets [7], was proposed by Torra and Narukawa [28] and Torra [29], which permits the membership degree of an element to a given set to be represented by several possible numerical values. To accommodate more complex environments, several extensions of HFS have been presented, such as interval-valued hesitant fuzzy set (IVHFS) [30, 31], hesitant triangular fuzzy set (HTFS) [32], hesitant multiplicative set (HMS) [33], hesitant fuzzy linguistic term set (HFLTS) [34], and hesitant fuzzy uncertain linguistic set (HFULS) [35]. In particular, considering the human judgments including preference information may be stated by a linguistic variable or an uncertain linguistic variable which permits the membership having a set of possible crisp values, Lin et al. [36] proposed the concepts of hesitant fuzzy linguistic set (HFLS) and hesitant fuzzy uncertain linguistic set (HFULS). Furthermore, Liu et al. [37] developed the hesitant intuitionistic fuzzy linguistic set (HIFLS) and the hesitant intuitionistic fuzzy uncertain linguistic set (HIFULS) which permit the possible membership degree and nonmembership degree of an element to a linguistic term and an uncertain linguistic term having sets of intuitionistic fuzzy numbers.

To the best of our knowledge, the existing methods under hesitant fuzzy environment are not suitable for dealing with MADM problems under hesitant intuitionistic linguistic fuzzy environment. In fact, when decision makers give their assessments on attributes which are in the form of intuitionistic linguistic variables (ILVs), they may also be hesitant among several possible ILVs. Therefore, inspired by the idea of the HFS, based on the ILVs, we propose a new fuzzy variable called hesitant intuitionistic linguistic fuzzy element (HILFE) which is composed of a set of ILVs. The main advantage of the HILFE is that it can describe the uncertain information by several linguistic variables in qualitative and intuitionistic fuzzy numbers adopted to demonstrate how much degree that an attribute value belongs and does not belong to a linguistic term in quantitative. For example, for a predefined linguistic set $S = \{s_0, s_1, \ldots, s_6\}$ set, $s_0$ means extremely low, $s_1$ means very low, $s_2$ means low, $s_3$ means medium, $s_4$ means high, $s_5$ means extremely high, when we can evaluate the "growth" of a company, we can utilize a HILFE $\{\langle s_3, 0.6, 0.3 \rangle, \langle s_4, 0.6, 0.2 \rangle, \langle s_5, 0.5, 0.4 \rangle\}$. Obviously, $s_3$, $s_4$, and $s_5$ indicate that the "growth" of a company may be "medium", "high", and "very high", and the intuitionistic fuzzy numbers "(0.6, 0.3)", "(0.6, 0.2)", and "(0.5, 0.4)" explain the degree that the "growth" of a company belongs to and does not belong to $s_3$, $s_4$, and $s_5$, respectively.

The remainder of this paper is organized as follows: some basic definitions of intuitionistic linguistic set and hesitant fuzzy set are briefly reviewed in Section 2. In Section 3, the concept, operational laws, score function, and accuracy function of the hesitant intuitionistic linguistic fuzzy element are defined. In Section 4, some hesitant intuitionistic linguistic fuzzy aggregation operators are proposed, and then some desirable properties of the proposed operators are investigated. In Section 5, we develop an approach to evaluate satisfaction degree with hesitant intuitionistic linguistic fuzzy information based on the proposed operators. In Section 6, a numerical example is given to illustrate the application of the proposed method. The paper is concluded in Section 7.

2. Preliminaries

To facilitate the following discussion, some basic definitions related to intuitionistic linguistic set and hesitant fuzzy set are briefly reviewed in this section.

Let $S = \{s_0, s_1, \ldots, s_g\}$ be a finite linguistic term set with odd cardinality, where $s_i$ represents a possible value for a linguistic term and $g + 1$ is the cardinality of $S$. For example, when $g = 6$, a set of seven terms $S$ can be given as follows.

$S = \{s_0 = \text{extremely low}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{medium}, s_4 = \text{high}, s_5 = \text{very high}, s_6 = \text{extremely high}\}$.

In general, for any linguistic term set $S$, it is required that $s_i$ and $s_j$ satisfy the following properties [38, 39].

1. The set is ordered: $s_i > s_j$, if and only if $i > j$.
2. There is the negation operator: $\text{Neg}(s_i) = s_j$, such that $j = g - i$.
3. Maximum operator is $\max\{s_i, s_j\} = s_i$, if $i \geq j$.
4. Minimum operator is $\min\{s_i, s_j\} = s_j$, if $i \geq j$.

2.1. Intuitionistic Linguistic Set

**Definition 1** (see [27]). Let $S = \{s_i \mid s_0 \leq s_i \leq s_g, i \in [0, g]\}$ be the continuous form of $S$ and let $X$ be in a given domain; an intuitionistic linguistic set $A$ in $X$ can be defined as

$$ A = \{\langle x, [a_A(x), v_A(x)] \rangle \mid x \in X\}, \quad (1) $$

where $a_A(x) \in S$, $a_A(x) \in [0, 1]$, $v_A(x) \in [0, 1]$, and 0 $\leq a_A(x) + v_A(x) \leq 1$. $a_A(x)$ and $v_A(x)$ represent the membership degree and the nonmembership degree of the element $x$ to the linguistic variable $s_{a_A(x)}$, respectively. Let $\pi(x) = 1 - a_A(x) - v_A(x)$, $\pi(x) \in [0, 1]$, $\forall x \in X$; then, $\pi(x)$ is called a hesitancy degree of $x$ to the linguistic variable $s_{\pi(x)}$.

In (1), $(a(s_{\pi(x)}), (u(a), v(a)))$ is an intuitionistic linguistic variable (ILV). For convenience, $a = (s_{\pi(D)}, (u(a), v(a)))$ is used to represent an ILV.

**Definition 2** (see [27]). Let $a_1 = (s_{\pi(D)}, (u(a_1), v(a_1)))$ and $a_2 = (s_{\pi(D)}, (u(a_2), v(a_2)))$ be two ILVs and $\lambda \geq 0$; then, the operational laws of ILVs are defined as follows:

1. $a_1 \oplus \lambda a_2 = (s_{\pi(D) + \lambda \pi(D)}, (u(a_1) + u(a_2) - u(a_1)u(a_2), v(a_1)v(a_2)))$;
2. $a_1 \otimes a_2 = (s_{\pi(D)\pi(D)}, (u(a_1)u(a_2), v(a_1) + v(a_2) - v(a_1)v(a_2)))$;
\( (3) \lambda_{a_1} = \langle s_{\lambda\theta(a_1)}, (1 - (1 - u(a_1))^\beta), (v(a_1))^\gamma \rangle; \)
\( (4) a_1^\gamma = \langle s_{\theta(a_1)}, ((u(a_1))^\beta, 1 - (1 - v(a_1))^\gamma) \rangle. \)

**Definition 3** (see [40]). Let \( a_1 = \langle s_{\theta(a_1)}, (u(a_1), v(a_1)) \rangle \) be an ILV; the score function \( S(a_1) \) and the accuracy function \( H(a_1) \) of \( a_1 \) are defined, respectively, as follows:

\[
S(a_1) = \frac{1}{2} \times (u(a_1) + v(a_1)) \times \frac{\theta(a_1)}{g}, \tag{2}
\]
\[H(a_1) = (u(a_1) + v(a_1)) \times \frac{\theta(a_1)}{g}.
\]

**Theorem 4** (see [40]). Let \( a_1 = \langle s_{\theta(a_1)}, (u(a_1), v(a_1)) \rangle \) and \( a_2 = \langle s_{\theta(a_2)}, (u(a_2), v(a_2)) \rangle \) be two ILVs; then, they can be compared by the following laws:

(1) if \( S(a_1) > S(a_2) \), then \( a_1 > a_2 \);
(2) if \( S(a_1) = S(a_2) \), then

\[
\begin{align*}
&\text{if } H(a_1) > H(a_2), \text{ then } a_1 > a_2; \\
&\text{if } H(a_1) = H(a_2), \text{ then } a_1 = a_2. 
\end{align*}
\]

2.2. Hesitant Fuzzy Set

**Definition 5** (see [29]). Let \( X \) be a fixed set; then, a hesitant fuzzy set (HFS) on \( X \) is in terms of a function that when applied to \( X \) returns a subset of \([0, 1]\), which can be represented by the following mathematical symbol:

\[
E = \left\{ (x, \tilde{h}(x)) \mid x \in X \right\},
\]

where \( \tilde{h}(x) = \bigcup_{r \in \tilde{h}(x)} \{f(x) \} \) is a set of some values in \([0, 1]\), denoting the possible membership degrees of the element \( x \in X \) to the set \( E \). For convenience, Liu et al. [37] called \( \tilde{h}(x) \) a hesitant fuzzy element (HFE) and \( E \) the set of all HFEs.

**Definition 6** (see [41]). Let \( \tilde{h} = \bigcup_{r \in \tilde{h}} \{F_r(x)\} \), \( \tilde{h}_1 = \bigcup_{r \in \tilde{h}_1} \{F_r(x)\} \), and \( \tilde{h}_2 = \bigcup_{r \in \tilde{h}_2} \{F_r(x)\} \) be any three HFSs, and \( \beta \geq 0 \); then, some operational laws of HFEs are defined as follows:

(1) \( \tilde{h}^\beta = \bigcup_{r \in \tilde{h}} \{r^\beta\} \);
(2) \( \beta \tilde{h} = \bigcup_{r \in \tilde{h}} \{1 - (1 - r)^\beta\} \);
(3) \( \tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{r \in \tilde{h}_1} \bigcup_{s \in \tilde{h}_2} \{F_{r+s}(x)\} \);
(4) \( \bar{\tilde{h}}_1 \oplus \tilde{h}_2 = \bigcup_{r \in \tilde{h}_1} \bigcup_{s \in \tilde{h}_2} \{F_{r+s}(x)\} \).

**Definition 7** (see [37]). For two HFEs \( \tilde{h}_1 = \bigcup_{r \in \tilde{h}_1} \{F_r(x)\} \) and \( \tilde{h}_2 = \bigcup_{r \in \tilde{h}_2} \{F_r(x)\} \), let \( S(\tilde{h}_i) = (1/\#\tilde{h}_i) \sum_{r \in \tilde{h}_i} t_i \) and \( V(\tilde{h}_i) = \sqrt{(1/\#\tilde{h}_i) \sum_{r \in \tilde{h}_i} (t_i - S(\tilde{h}_i))^2} \) be the score function and variance function of \( \tilde{h}_i \) \( (i = 1, 2) \), respectively, where \( \#\tilde{h}_i \) is the number of elements in \( \tilde{h}_i \) \( (i = 1, 2) \); then,

(1) if \( S(\tilde{h}_1) > S(\tilde{h}_2) \), then \( \tilde{h}_1 \) is superior to \( \tilde{h}_2 \), denoted by \( \tilde{h}_1 > \tilde{h}_2 \);

(2) if \( S(\tilde{h}_1) = S(\tilde{h}_2) \), then

\[
\begin{align*}
&\text{(a) if } V(\tilde{h}_1) > V(\tilde{h}_2), \text{ then } \tilde{h}_1 \text{ is inferior to } \tilde{h}_2; \\
&\text{(b) if } V(\tilde{h}_1) = V(\tilde{h}_2), \text{ then } \tilde{h}_1 \text{ is equivalent to } \tilde{h}_2, \\
&\text{denoted by } \tilde{h}_1 = \tilde{h}_2.
\end{align*}
\]

3. Hesitant Intuitionistic Linguistic Fuzzy Set

Based on the intuitionistic linguistic set and the hesitant fuzzy set, we propose the definition of the hesitant intuitionistic linguistic fuzzy set, as well as the operational laws, score function, and accuracy function in what follows.

**Definition 8.** Let \( X \) be a nonempty set of the universe and \( \tilde{S} \) a continuous linguistic term set of \( S = \{s_0, s_1, \ldots, s_g\} \); then, a hesitant intuitionistic linguistic fuzzy set (HILFS) on \( X \) can be expressed by the mathematical symbol as follows:

\[
H = \{ (x, h(x)) \mid x \in X \},
\]

where \( h(x) = \bigcup_{r \in h(x)} \{r(x)\} \) is a set of ILVs; that is, \( r(x) = \langle s_{\theta(x)}, (\mu(x), v(x)) \rangle \), denoting the possible membership degrees of the element \( x \in X \) to the set \( H \). For convenience, one calls \( h = \bigcup_{r \in h} \) a hesitant intuitionistic linguistic fuzzy element (HILFE) and \( H \) the set of all HILFEs.

**Definition 9.** Let \( h, h_1, \) and \( h_2 \) be any three HILFEs, and \( \beta \geq 0 \); then, the operational laws of HILFEs are defined as follows:

(1) \( h_1 \oplus h_2 = \bigcup_{r \in h_1} \bigcup_{s \in h_2} \{s_{\theta(r+s)}, (\mu(r_1) + \mu(r_2), v(r_1) + v(r_2)) \}; \)
(2) \( h_1 \otimes h_2 = \bigcup_{r \in h_1} \bigcup_{s \in h_2} \{s_{\theta(r+s)}, (\mu(r_1) \mu(r_2), v(r_1) + v(r_2) + v(r_1)v(r_2)) \}; \)
(3) \( \beta h = \bigcup_{r \in h} \{s_{\theta(r)}, (1 - (1 - \mu(r))^\beta, v(r)^\beta) \}; \)
(4) \( h^\beta = \bigcup_{r \in h} \{s_{\theta(r)}, (\mu(r)^\beta, 1 - (1 - v(r))^\beta) \}. \)

Obviously, the above operational results are still HILFEs.

**Theorem 10.** Let \( h_1 \) and \( h_2 \) be two HILFEs, and \( \lambda \geq 0 \); the calculation rules are shown as follows:

(1) \( h_1 \oplus h_2 = h_2 \oplus h_1; \)
(2) \( h_1 \otimes h_2 = h_2 \otimes h_1; \)
(3) \( \lambda h_1 \oplus \lambda h_2 = \lambda (h_1 \oplus h_2); \)
(4) \( h_1^\beta \oplus h_2^\beta = (h_1 \oplus h_2)^\beta. \)

**Definition 11.** Let \( h \) be a HILFE; then, the score function \( S(h) \) of \( h \) can be represented as follows:

\[
S(h) = \frac{1}{\#h} \sum_{r \in h} \frac{\theta(h) (1 + \mu(h) - v(h))}{2g},
\]

where \( \#h \) is the number of ILVs in \( h \) and \( g+1 \) is the cardinality of linguistic term set \( S \).
Definition 12. Let \( h \) be a HILFE; then, the accuracy function \( P(h) \) of \( h \) can be represented as follows:

\[
P(h) = \frac{1}{\# h} \sum_{r \in h} \theta(h)(\mu(h) + \nu(h)),
\]

where \( \# h \) is the number of ILVs in \( h \) and \( g+1 \) is the cardinality of linguistic term set \( S \).

Theorem 13. Let \( h_1 \) and \( h_2 \) be two HILFEs and let \( S(h_1) \) and \( P(h_1) \) be the score value and accuracy degree of \( h_i \) (\( i = 1, 2 \), respectively; then, one has the following:

1. If \( S(h_1) < S(h_2) \), then \( h_1 \) is smaller than \( h_2 \), denoted by \( h_1 < h_2 \).
2. If \( S(h_1) = S(h_2) \), then one has the following:
   a. if \( P(h_1) = P(h_2) \), then \( h_1 \) is equal to \( h_2 \), denoted by \( h_1 = h_2 \);
   b. if \( P(h_1) < P(h_2) \), then \( h_1 \) is smaller than \( h_2 \), denoted by \( h_1 < h_2 \).

4. Hesitant Intuitionistic Linguistic Fuzzy Aggregation Operators

Motivated by the operational laws of HILFEs, in the following, some aggregation operators are developed for aggregating the hesitant intuitionistic linguistic fuzzy information.

4.1. Hesitant Intuitionistic Linguistic Fuzzy Weighted Aggregation Operators

Definition 14. Let \( h_i \) (\( i = 1, 2, \ldots, n \)) be a collection of HILFEs and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \); then, the hesitant intuitionistic linguistic fuzzy weighted average (HILFWA) operator is a mapping HILFWA: \( H^n \rightarrow H \), and

\[
\text{HILFWA} (h_1, h_2, \ldots, h_n) = \frac{1}{n} \sum_{i=1}^{n} w_i h_i,
\]

where \( H \) is a hesitant intuitionistic linguistic fuzzy set.

Theorem 15. Let \( h_i \) (\( i = 1, 2, \ldots, n \)) be a collection of HILFEs, and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \); then, their aggregated value by the HILFWA operator is still a HILFE, and

\[
\text{HILFWA} (h_1, h_2, \ldots, h_n) = \bigcup_{r \in h_1 \cap h_2 \cap \ldots \cap h_n} \left\{ \left( \sum_{i=1}^{n} w_i \theta(r_i), (1 - \prod_{i=1}^{n} (1 - \mu(r_i))^w_i, \prod_{i=1}^{n} \nu(r_i)^w_i) \right) \right\} = \{ (s_\theta, (\mu, \nu)) \}.
\]

In particular, if \( w = (1/n, 1/n, \ldots, 1/n)^T \), then the HILFWA operator reduces to the hesitant intuitionistic linguistic fuzzy average (HILFA) operator:

\[
\text{HILFA} (h_1, h_2, \ldots, h_n) = \frac{1}{n} \sum_{i=1}^{n} h_i.
\]

Some desirable properties of the HILFWA operator are given as follows.

Theorem 16 (idempotency). If all HILFEs \( h_i \) (\( i = 1, 2, \ldots, n \)) are equal and \( h_i = \{ (s_\theta, (\mu, \nu)) \} \) for all \( i = 1, 2, \ldots, n \), then

\[
\text{HILFWA} (h_1, h_2, \ldots, h_n) = \{ (s_\theta, (\mu, \nu)) \}.
\]

Proof. Since \( h_i = \{ (s_\theta, (\mu, \nu)) \} \) for all \( i = 1, 2, \ldots, n \), then

\[
\text{HILFWA} (h_1, h_2, \ldots, h_n) = \bigcup_{r \in h_1 \cap h_2 \cap \ldots \cap h_n} \left\{ \left( \sum_{i=1}^{n} w_i \theta(r_i), (1 - \prod_{i=1}^{n} (1 - \mu(r_i))^w_i, \prod_{i=1}^{n} \nu(r_i)^w_i) \right) \right\}
\]

where \( \theta \) is the hesitant intuitionistic linguistic fuzzy average (HILFA) operator:

\[
H^n \rightarrow H = \frac{1}{n} \sum_{i=1}^{n} h_i.
\]

Theorem 17 (boundedness). Let \( h_i \) (\( i = 1, 2, \ldots, n \)) be a collection of HILFEs; if \( s_\theta = \min_{1 \leq i \leq n} \theta(r_i) \) \( r_i \in h_i \), \( s_\theta = \max_{1 \leq i \leq n} \theta(r_i) \) \( r_i \in h_i \), \( \mu = \min_{1 \leq i \leq n} \mu(r_i) \) \( r_i \in h_i \), and \( \nu = \max_{1 \leq i \leq n} \nu(r_i) \) \( r_i \in h_i \), then
$$\mu^+ = \max_{1 \leq i \leq n} \{\mu(r_i) \mid r_i \in h_i\}, \quad v^- = \min_{1 \leq i \leq n} \{v(r_i) \mid r_i \in h_i\},$$
and
$$v^+ = \max_{1 \leq i \leq n} \{v(r_i) \mid r_i \in h_i\},$$
then
$$\langle s_{\theta^*}, (\mu^+, v^-) \rangle \leq \text{HILFWA} \left(h_1, h_2, \ldots, h_n\right) \leq \langle s_{\theta^*}, (\mu^+, v^+) \rangle. \quad (13)$$

**Proof.** Since $$s_{\theta^*} = \min_{1 \leq i \leq n} \{s_{\theta}(r_i) \mid r_i \in h_i\}, s_{\theta^*} = \max_{1 \leq i \leq n} \{s_{\theta}(r_i) \mid r_i \in h_i\}, \mu^* = \min_{1 \leq i \leq n} \{\mu(r_i) \mid r_i \in h_i\}, \mu^* = \max_{1 \leq i \leq n} \{\mu(r_i) \mid r_i \in h_i\},$$
$$v^- = \min_{1 \leq i \leq n} \{v(r_i) \mid r_i \in h_i\},$$
and
$$v^+ = \max_{1 \leq i \leq n} \{v(r_i) \mid r_i \in h_i\},$$
we have
$$\theta^- \leq \theta(r_i) \leq \theta^+, \quad \mu^- \leq \mu(r_i) \leq \mu^+, \quad v^- \leq v(r_i) \leq v^+,$$
\forall i = 1, 2, \ldots, n. \quad (14)

Then,
$$\sum_{i=1}^{n} w_i \theta(r_i) \geq \sum_{i=1}^{n} w_i \theta^- = \theta^-,$$

$$1 - \prod_{i=1}^{n} (1 - \mu(r_i))^{w_i} \geq 1 - \prod_{i=1}^{n} (1 - \mu^-)^{w_i} = \mu^-,$$ \quad (15)

$$\prod_{i=1}^{n} v(r_i)^{w_i} \leq \prod_{i=1}^{n} (v^+)^{w_i} = v^+.$$ That is,
$$\frac{1}{\#h} \sum_{r \in h} \left( \left( \sum_{i=1}^{n} w_i \cdot \theta(r_i) \right) \left( 1 + 1 - \prod_{i=1}^{n} (1 - \mu(r_i))^{w_i} \right) - \prod_{i=1}^{n} v(r_i)^{w_i} \right) \right) \times (2g)^{-1} \geq \frac{(1 + \mu^- - v^+) \times \theta^-}{2g},$$

where \#h is the numbers of ILVs in HILFWA(h_1, h_2, \ldots, h_n) and g + 1 is the cardinality of linguistic term set S. Therefore, according to Theorem 13, we obtain
$$\text{HILFWA} \left(h_1, h_2, \ldots, h_n\right) \succeq \langle s_{\theta^*}, (\mu^+, v^-) \rangle. \quad (17)$$

Similarly,
$$\text{HILFWA} \left(h_1, h_2, \ldots, h_n\right) \leq \langle s_{\theta^*}, (\mu^+, v^+) \rangle. \quad (18)$$

Therefore,
$$\langle s_{\theta^*}, (\mu^+, v^-) \rangle \leq \text{HILFWA} \left(h_1, h_2, \ldots, h_n\right) \leq \langle s_{\theta^*}, (\mu^+, v^+) \rangle. \quad (19)$$
be the balancing coefficient which plays a role of balance; then, one has
\[
\text{HILFWG} (h_1, h_2, \ldots, h_n) \leq \text{HILFWA} (h_1, h_2, \ldots, h_n). \tag{24}
\]

Proof. According to Lemma 20, for any \( r_i = (s_{\theta(r_i)}, (u(r_i), v(r_i))) \in h_i, i = 1, 2, \ldots, n \), we have
\[
\prod_{i=1}^{n} \theta(r_i)^w \leq \sum_{i=1}^{n} w_i \theta(r_i),
\]
\[
\prod_{i=1}^{n} \mu(r_i)^w \leq \sum_{i=1}^{n} w_i \mu(r_i)
\]
\[
eq 1 - \sum_{i=1}^{n} w_i (1 - \mu(r_i)) \leq 1 - \prod_{i=1}^{n} (1 - \mu(r_i))^w,
\]
\[
1 - \prod_{i=1}^{n} (1 - v(r_i))^w \geq 1 - \sum_{i=1}^{n} w_i (1 - v(r_i)) = \sum_{i=1}^{n} w_i v(r_i) \geq \prod_{i=1}^{n} v(r_i)^w.
\]

That is,
\[
\frac{1}{#h_1} \sum_{r_i \in h_i} \left( \prod_{i=1}^{n} \theta(r_i)^w \times \left( 1 + \prod_{i=1}^{n} \mu(r_i)^w \right) - 1 - \prod_{i=1}^{n} (1 - v(r_i))^w \right) \times (2g)^{-1}
\]
\[
\leq \frac{1}{#h_2} \sum_{r_i \in h_i} \left( \sum_{i=1}^{n} w_i \theta(r_i) \times \left( 1 + 1 - \prod_{i=1}^{n} (1 - \mu(r_i))^w \right) - \prod_{i=1}^{n} v(r_i)^w \right) \times (2g)^{-1},
\]
where \( #h_1 \) and \( #h_2 \) are the numbers of ILVs in HILFWG\((h_1, h_2, \ldots, h_n)\) and HILFWA\((h_1, h_2, \ldots, h_n)\), respectively, and \( g + 1 \) is the cardinality of linguistic term set \( S \). Therefore, according to Theorem 13, we obtain
\[
\text{HILFWG} (h_1, h_2, \ldots, h_n) \leq \text{HILFWA} (h_1, h_2, \ldots, h_n). \tag{27}
\]

4.2. Hesitant Intuitionistic Linguistic Fuzzy Ordered Weighted Aggregation Operators

Definition 22. Let \( h_i \ (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the aggregation-associated vector such that \( \omega_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} \omega_i = 1 \); then, the hesitant intuitionistic linguistic fuzzy ordered weighted average (HILFOWA) operator is a mapping HILFOWA: \( H^n \rightarrow H \), and
\[
\text{HILFOWA} (h_1, h_2, \ldots, h_n) = \oplus_{i=1}^{n} (h_{\sigma(i)})^{\omega_i}, \tag{28}
\]
where \( H \) is a hesitant intuitionistic linguistic fuzzy set. \( h_{\sigma(i)} \) is the \( i \)th largest element in \( h_i \ (i = 1, 2, \ldots, n) \).

Theorem 23. Let \( h_i \ (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the aggregation-associated vector such that \( \omega_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} \omega_i = 1 \); then, their aggregated value by the HILFOWA operator is still a HILFE, and
\[
\text{HILFOWA} (h_1, h_2, \ldots, h_n) = \bigcup_{r_1 \in h_1, r_2 \in h_2, r_3 \in h_3, \ldots, r_n \in h_n} \left\{ \left( \prod_{i=1}^{n} (1 - \mu(r_{\sigma(i)})^{\omega_i}), \prod_{i=1}^{n} v(r_{\sigma(i)})^{\omega_i} \right) \right\}. \tag{29}
\]

Similar to the HILFA operator, the HILFOWA operator also has the properties of idempotency and boundedness under some conditions, which can be proved similar to Theorems 16 and 17. Furthermore, the HILFOWA operator also has the property of commutativity.

Theorem 24 (commutativity). Let \( h_i \ (i = 1, 2, \ldots, n) \) be a collection of HILFEs. If \( \{h_1', h_2', \ldots, h_n'\} \) is any permutation of \( \{h_1, h_2, \ldots, h_n\} \), then
\[
\text{HILFOWA} (h_1', h_2', \ldots, h_n') = \text{HILFOWA} (h_1, h_2, \ldots, h_n). \tag{30}
\]

Proof. Since \( \{h_1', h_2', \ldots, h_n'\} \) is a permutation of \( \{h_1, h_2, \ldots, h_n\} \), we have \( h_{\sigma(i)}' = h_{\sigma(i)} \) for all \( i = 1, 2, \ldots, n \). Then, based on Definition 22, we obtain
\[
\text{HILFOWA} (h_1', h_2', \ldots, h_n') = \text{HILFOWA} (h_1, h_2, \ldots, h_n). \tag{31}
\]

Definition 25. Let \( h_i \ (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the aggregation-associated vector such that \( \omega_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} \omega_i = 1 \); then, the hesitant intuitionistic linguistic fuzzy ordered weighted geometric (HILFWG) operator is a mapping HILFWG: \( H^n \rightarrow H \), and
\[
\text{HILFWG} (h_1, h_2, \ldots, h_n) = \oplus_{i=1}^{n} (h_{\sigma(i)})^{\omega_i}, \tag{32}
\]
where \( H \) is a hesitant intuitionistic linguistic fuzzy set. \( h_{\sigma(i)} \) is the \( i \)th largest element in \( h_i \ (i = 1, 2, \ldots, n) \).
Theorem 26. Let $h_i$ ($i = 1, 2, \ldots, n$) be a collection of HILFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$; then, their aggregated value by the HILFOWG operator is still a HILFE, and
\[
\text{HILFOWG}(h_1, h_2, \ldots, h_n) = \bigcup_{r_{\sigma(1)} \in h_1, r_{\sigma(2)} \in h_2, \ldots, r_{\sigma(n)} \in h_n} \left\{ \left( \frac{\sum_{i=1}^{n} \omega_i \mu(r_{\sigma(i)})}{\sum_{i=1}^{n} \omega_i}, \prod_{i=1}^{n} \left(1 - \nu(r_{\sigma(i)})\right)^{\omega_i} \right) \right\}.
\]
(36)

In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the HILFOWG operator reduces to the HILFWA operator in (8); if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the HILFOWG operator reduces to the HILFOWA operator in (28).

Similar to the HILFWA operator, the HILFOWG operator also has the properties of idempotency and boundedness under some conditions, which can be proved similar to Theorems 16, 17, and 24.

Theorem 27. Let $h_i$ ($i = 1, 2, \ldots, n$) be a collection of HILFEs, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$, and let $\nu$ be the balancing coefficient which plays a role of balance; then, based on the aggregation-associated vector

\[
\text{HILFOWG}(h_1, h_2, \ldots, h_n) \leq \text{HILFOWA}(h_1, h_2, \ldots, h_n)
\]
(34)

which can be proved similar to Theorem 21.

4.3. Hesitant Intuitionistic Linguistic Fuzzy Hybrid Aggregation Operators

Definition 28. Let $h_i$ ($i = 1, 2, \ldots, n$) be a collection of HILFEs, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of them, with $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$, and let $\nu$ be the balancing coefficient which plays a role of balance; then, based on the aggregation-associated vector

\[
\text{HILFOWG}(h_1, h_2, \ldots, h_n) = \bigcup_{r_{\sigma(1)} \in h_1, r_{\sigma(2)} \in h_2, \ldots, r_{\sigma(n)} \in h_n} \left\{ \left( \frac{\sum_{i=1}^{n} \omega_i \mu(r_{\sigma(i)})}{\sum_{i=1}^{n} \omega_i}, \prod_{i=1}^{n} \left(1 - \nu(r_{\sigma(i)})\right)^{\omega_i} \right) \right\}.
\]
(35)

where $H$ is a hesitant intuitionistic linguistic fuzzy set, $h_{\sigma(j)}$ is the jth largest element of hesitant intuitionistic linguistic fuzzy weighted arguments $h_j$ ($h_j = h_{j\omega}$), $j = 1, 2, \ldots, n$.

Theorem 29. Let $h_i$ ($i = 1, 2, \ldots, n$) be a collection of HILFEs, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of them, with $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$; then, their aggregated value by the HILFHA operator is still a HILFE, and

\[
\text{HILFHA}(h_1, h_2, \ldots, h_n)
\]

\[
= \bigcup_{r_{\sigma(1)} \in h_1, r_{\sigma(2)} \in h_2, \ldots, r_{\sigma(n)} \in h_n} \left\{ \left( \frac{\sum_{i=1}^{n} \omega_i \theta(r_{\sigma(i)})}{\sum_{i=1}^{n} \omega_i}, \prod_{i=1}^{n} \left(1 - \nu(r_{\sigma(i)})\right)^{\omega_i} \right) \right\}.
\]
(37)

where $H$ is a hesitant intuitionistic linguistic fuzzy set, $h_{\sigma(j)}$ is the jth largest element of hesitant intuitionistic linguistic fuzzy weighted arguments $h_j$ ($h_j = h_{j\omega}$), $j = 1, 2, \ldots, n$.

Theorem 30. Let $h_i$ ($i = 1, 2, \ldots, n$) be a collection of HILFEs, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of them, with $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \omega_i = 1$; then, their aggregated value by the HILFHA operator is still a HILFE, and

\[
\text{HILFHA}(h_1, h_2, \ldots, h_n)
\]

\[
= \bigcup_{r_{\sigma(1)} \in h_1, r_{\sigma(2)} \in h_2, \ldots, r_{\sigma(n)} \in h_n} \left\{ \left( \frac{\sum_{i=1}^{n} \omega_i \theta(r_{\sigma(i)})}{\sum_{i=1}^{n} \omega_i}, \prod_{i=1}^{n} \left(1 - \nu(r_{\sigma(i)})\right)^{\omega_i} \right) \right\}.
\]
(38)
In particular, if \( \omega = (1/n, 1/n, \ldots, 1/n)^T \), then the HILFHG operator reduces to the HILFWG operator in (20). If \( w = (1/n, 1/n, \ldots, 1/n)^T \), then the HILFHG operator reduces to the HILFWG operator in (32).

Similar to the HILFWG operator, the HILFHG operator also has the properties of idempotency and boundedness under some conditions, which can be proved similar to Theorems 16 and 17.

**Theorem 32.** Let \( h_i (i = 1, 2, \ldots, n) \) be a collection of HILFEs, let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \), let \( \omega = (w_1, w_2, \ldots, w_n)^T \) be the aggregation-associated vector such that \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \), and let \( n \) be the balancing coefficient which plays a role of balance; then, one has

\[
\text{HILFHG}(h_1, h_2, \ldots, h_n) \leq \text{HILFA}(h_1, h_2, \ldots, h_n)
\]

which can be proved similar to Theorem 21.

**4.4. Generalized Hesitant Intuitionistic Linguistic Fuzzy Weighted Aggregation Operators**

**Definition 33.** Let \( h_i (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \); \( \lambda \) is a parameter such that \( \lambda \in (0, +\infty) \); then, the generalized hesitant intuitionistic linguistic fuzzy weighted average (GHILFWA) operator is a mapping GHILFW: \( H^n \rightarrow H \), and

\[
\text{GHILFW}_\lambda(h_1, h_2, \ldots, h_n) = \left( \frac{1}{\lambda} \sum_{i=1}^{n} w_i (h_i)^\lambda \right)^{1/\lambda}, \tag{40}
\]

where \( H \) is a hesitant intuitionistic linguistic fuzzy set.

**Theorem 34.** Let \( h_i (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \); then, their aggregated value by the GHILFWA operator is still a HILFE, and

\[
\text{GHILFW}_\lambda(h_1, h_2, \ldots, h_n)
\]

In particular, if \( \lambda = 1 \), then the GHILFWA operator reduces to the HILFWA operator in (8).

Similar to the HILFWA operator, the GHILFWA operator also has the properties of idempotency and boundedness under some conditions, which can be proved similar to Theorems 16 and 17.

**Definition 35.** Let \( h_i (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \); \( \lambda \) is a parameter such that \( \lambda \in (0, +\infty) \); then, the generalized hesitant intuitionistic linguistic fuzzy weighted geometric (GHILFWG) operator is a mapping GHILFWG: \( H^n \rightarrow H \), and

\[
\text{GHILFWG}_\lambda(h_1, h_2, \ldots, h_n) = \frac{1}{\lambda} \left( \sum_{i=1}^{n} w_i (\lambda h_i)^{\lambda w_i} \right), \tag{42}
\]

where \( H \) is a hesitant intuitionistic linguistic fuzzy set.

**Theorem 36.** Let \( h_i (i = 1, 2, \ldots, n) \) be a collection of HILFEs and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them, with \( w_i \in [0, 1], i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \); then, their aggregated value by the GHILFWG operator is still a HILFE, and

\[
\text{GHILFWG}_\lambda(h_1, h_2, \ldots, h_n)
\]

In particular, if \( \lambda = 1 \), then the GHILFWG operator reduces to the HILFWG operator in (20).

Similar to the HILFWG operator, the GHILFWG operator also has the properties of idempotency and boundedness under some conditions, which can be proved similar to Theorems 16 and 17.
Theorem 37. Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HILFEs, let $w = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of them, with $\omega_i \in [0, 1], i = 1, 2, \ldots, n$, and \( \sum_{i=1}^{n} \omega_i = 1 \), and let $n$ be the balancing coefficient which plays a role of balance; then, one has

$$
\text{GHILFWG}(h_1, h_2, \ldots, h_n) \leq \text{GHILFWA}(h_1, h_2, \ldots, h_n)
$$

which can be proved similar to Theorem 21.

4.5. Generalized Hesitant Intuitionistic Linguistic Fuzzy Ordered Weighted Aggregation Operators

Definition 38. Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HILFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1], i = 1, 2, \ldots, n$, and \( \sum_{i=1}^{n} \omega_i = 1 \); $\lambda$ is a parameter such that $\lambda \in (0, +\infty)$; then, the generalized hesitant intuitionistic linguistic fuzzy ordered weighted geometric (GHILFWG) operator is a mapping GHILFWG: $H^n \to H$, and

$$
\text{GHILFWG}_\lambda(h_1, h_2, \ldots, h_n) = \left( \frac{n}{\sum_{i=1}^{n} \omega_i(h_{\sigma(i)})^{1/\lambda}} \right)^{1/\lambda}
$$

where $H$ is a hesitant intuitionistic linguistic fuzzy set. $h_{\sigma(i)}$ is the $i$th largest element in $h_i \ (i = 1, 2, \ldots, n)$.

Theorem 39. Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HILFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1], i = 1, 2, \ldots, n$, and \( \sum_{i=1}^{n} \omega_i = 1 \); then, their aggregated value by the GHILFWA operator is still a HILFE, and

$$
\text{GHILFWA}_\lambda(h_1, h_2, \ldots, h_n) = \left( \frac{n}{\sum_{i=1}^{n} \omega_i(h_{\sigma(i)})^{1/\lambda}} \right)^{1/\lambda},
$$

where $H$ is a hesitant intuitionistic linguistic fuzzy set. $h_{\sigma(i)}$ is the $i$th largest element in $h_i \ (i = 1, 2, \ldots, n)$.

Theorem 40. Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HILFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1], i = 1, 2, \ldots, n$, and \( \sum_{i=1}^{n} \omega_i = 1 \); then, their aggregated value by the GHILFWA operator is still a HILFE, and

$$
\text{GHILFWA}_\lambda(h_1, h_2, \ldots, h_n) = \left( \frac{n}{\sum_{i=1}^{n} \omega_i(h_{\sigma(i)})^{1/\lambda}} \right)^{1/\lambda}
$$

where $H$ is a hesitant intuitionistic linguistic fuzzy set. $h_{\sigma(i)}$ is the $i$th largest element in $h_i \ (i = 1, 2, \ldots, n)$.

Theorem 41. Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HILFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1], i = 1, 2, \ldots, n$, and \( \sum_{i=1}^{n} \omega_i = 1 \); then, their aggregated value by the GHILFWA operator is still a HILFE, and

$$
\text{GHILFWA}_\lambda(h_1, h_2, \ldots, h_n) = \left( \frac{n}{\sum_{i=1}^{n} \omega_i(h_{\sigma(i)})^{1/\lambda}} \right)^{1/\lambda}
$$

where $H$ is a hesitant intuitionistic linguistic fuzzy set. $h_{\sigma(i)}$ is the $i$th largest element in $h_i \ (i = 1, 2, \ldots, n)$.

Theorem 42. Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HILFEs, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1], i = 1, 2, \ldots, n$, and \( \sum_{i=1}^{n} \omega_i = 1 \), and let $n$ be the balancing coefficient which plays a role of balance; then, one has

$$
\text{GHILFWG}(h_1, h_2, \ldots, h_n) \leq \text{GHILFWA}(h_1, h_2, \ldots, h_n)
$$

which can be proved similar to Theorem 21.
5. An Approach for Satisfaction Evaluation with Hesitant Intuitionistic Linguistic Fuzzy Information

For a MADM problem, let \( A = \{A_1, A_2, \ldots, A_m\} \) be a finite set of \( m \) alternatives and let \( C = \{C_1, C_2, \ldots, C_n\} \) be the set of \( n \) attributes. Suppose that all values assigned to alternatives with respect to attributes are expressed by a hesitant intuitionistic linguistic fuzzy decision matrix denoted by \( H = (h_{ij})_{m \times n} \), where elements \( h_{ij} = \bigcup_{\{s_1, s_2, \ldots, s_g\}} \{s_1\Theta(h_{ij}), (u(h_{ij}), v(h_{ij}))\} \) are HILFEs provided for the rating of the alternative \( A_i \) (\( i = 1, 2, \ldots, m \)) with respect to the attribute \( C_j \) (\( j = 1, 2, \ldots, n \)), with \( s(h_{ij}) \in S = \{s_0, s_1, \ldots, s_g\} \). If the information about attribute weights is completely known as \( w = (w_1, w_2, \ldots, w_n)^T \), with \( w_j \in [0, 1], j = 1, 2, \ldots, n \), and \( \sum_{j=1}^{n} w_j = 1 \), then to determine the most desirable alternative(s), the HILFWA operator or the HILFWG operator is utilized to propose an approach to MADM under hesitant intuitionistic linguistic fuzzy environment, which involves the following steps.

Step 1. Aggregate the hesitant intuitionistic linguistic fuzzy assessment values \( h_{ij} \) of the alternative \( A_i \) (\( i = 1, 2, \ldots, m \)) on all attributes \( C_j \) (\( j = 1, 2, \ldots, n \)) into the overall assessment value \( h_i \) of the alternative \( A_i \) (\( i = 1, 2, \ldots, m \)) based on the HILFWA operator or the HILFWG operator in (50) and (51), respectively. Consider

\[
h_i = \text{HILFWA}(h_{i1}, h_{i2}, \ldots, h_{in}) = \bigoplus_{j=1}^{n} w_j h_{ij}
\]

or

\[
h_i = \text{HILFWG}(h_{i1}, h_{i2}, \ldots, h_{in}) = \bigoplus_{j=1}^{n} w_j h_{ij}^{w_j}
\]

where \( h_i \) is in the form of HILFEs and it can be denoted by

\[
h_i = \{s_1\Theta(h_{ij}), (u(h_{ij}), v(h_{ij}))\}.
\]

Step 2. Calculate the score values \( S(h_i) \) of overall assessment values \( h_i \) (\( i = 1, 2, \ldots, m \)) by

\[
S(h_i) = \frac{1}{\#h_i} \sum_{(s(h_{ij}), (u(h_{ij}), v(h_{ij})) \in h_i} \theta(h_i) (\mu(h_i) + v(h_i))
\]

where \( \#h_i \) is the number of ILVs in \( h_i \) and \( g + 1 \) is the cardinality of linguistic term set \( S \). If there is no difference between two score values \( S(h_i) \) and \( S(h_k) \), then we need to calculate the accuracy degrees \( P(h_i) \) and \( P(h_k) \) of the overall assessment values \( h_i \) and \( h_k \) (\( i, k = 1, 2, \ldots, m \) and \( i \neq k \)), respectively, according to

\[
P(h_i) = \frac{1}{\#h_i} \sum_{(s(h_{ij}), (u(h_{ij}), v(h_{ij})) \in h_i} \theta(h_i) (\mu(h_i) + v(h_i))
\]

Step 3. Rank all feasible alternatives \( A_i \) (\( i = 1, 2, \ldots, m \)) according to Theorem 4 and select the most desirable alternative(s).

Step 4. End.

6. Numerical Example

In this section, a practical example of satisfaction evaluation for milk products is adapted to illustrate the application of the MADM method proposed in Section 5 and to demonstrate its feasibility and effectiveness in a realistic scenario.

To strengthen the competitiveness and enlarge the product lines, a milk and dairy company needs to know the consumer satisfaction of its products at first, so the market department organizes investigations in several supermarkets. There is a panel with four milk products: (1) \( A_1 \) is the milk beverage; (2) \( A_2 \) is the yoghurt; (3) \( A_3 \) is the cheese; (4) \( A_4 \) is the pasteurized milk. The milk and dairy company must make a decision according to the following four attributes: (1) \( C_1 \) is the price; (2) \( C_2 \) is the taste; (3) \( C_3 \) is the packaging; (4) \( C_4 \) is the storability, whose weight vector is given as \( w = (0.30, 0.35, 0.10, 0.25)^T \). The four possible alternatives \( \{A_1, A_2, A_3, A_4\} \) are evaluated by using the linguistic term set \( S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\} \) with \( s_0 \) is extremely poor, \( s_1 \) is very poor, \( s_2 \) is poor, \( s_3 \) is fair, \( s_4 \) is good, \( s_5 \) is very good, \( s_6 \) is extremely good. The hesitant intuitionistic linguistic fuzzy decision matrix \( H = (h_{ij})_{4 \times 4} \) is constructed as shown in Table 1.
<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>${\langle s_2, (0.2, 0.4)\rangle, \langle s_5, (0.6, 0.3)\rangle}$</td>
<td>${\langle s_1, (0.4, 0.5)\rangle, \langle s_5, (0.7, 0.3)\rangle, \langle s_7, (0.8, 0.2)\rangle}$</td>
<td>${\langle s_3, (0.5, 0.3)\rangle, \langle s_9, (0.7, 0.2)\rangle}$</td>
<td>${\langle s_2, (0.6, 0.4)\rangle, \langle s_4, (0.8, 0.2)\rangle}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>${\langle s_1, (0.6, 0.4)\rangle, \langle s_4, (0.7, 0.2)\rangle}$</td>
<td>${\langle s_1, (0.5, 0.3)\rangle, \langle s_5, (0.6, 0.3)\rangle}$</td>
<td>${\langle s_9, (0.9, 0.1)\rangle}$</td>
<td>${\langle s_6, (0.4, 0.3)\rangle, \langle s_9, (0.6, 0.2)\rangle}$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>${\langle s_5, (0.5, 0.3)\rangle, \langle s_4, (0.6, 0.3)\rangle}$</td>
<td>${\langle s_1, (0.6, 0.2)\rangle, \langle s_5, (0.8, 0.1)\rangle}$</td>
<td>${\langle s_2, (0.4, 0.2)\rangle, \langle s_7, (0.6, 0.3)\rangle}$</td>
<td>${\langle s_2, (0.3, 0.6)\rangle, \langle s_7, (0.5, 0.4)\rangle, \langle s_8, (0.7, 0.3)\rangle}$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>${\langle s_3, (0.4, 0.5)\rangle, \langle s_4, (0.6, 0.2)\rangle, \langle s_5, (0.8, 0.2)\rangle}$</td>
<td>${\langle s_6, (0.5, 0.4)\rangle}$</td>
<td>${\langle s_1, (0.7, 0.3)\rangle, \langle s_7, (0.8, 0.2)\rangle}$</td>
<td>${\langle s_5, (2.0, 0.4)\rangle, \langle s_6, (0.6, 0.3)\rangle}$</td>
</tr>
</tbody>
</table>
In the following, we utilize the proposed MADM method to rank the milk products according to the customer satisfaction evaluation with hesitant intuitionistic linguistic fuzzy information.

**Step 1.** Aggregate the hesitant intuitionistic linguistic fuzzy assessment values $h_{ij}$ of the milk products $A_i$ ($i = 1, 2, 3, 4$) on all attributes $C_j$ ($j = 1, 2, 3, 4$) into the overall assessment value $h_i$ of the milk products $A_i$ ($i = 1, 2, 3, 4$) based on the HILFWA operator in (50). Take the milk product $A_1$ for example; we have

$$h_1 = \text{HILFWA}(h_{11}, h_{12}, h_{13}, h_{14})$$

$$= \left\{ \left( s_{\tau_{i_0}}^{w_i}, \theta(h_{i_j}) \right) \right\} \bigcup_{(s_{\theta(h_{1_0})}, (u(h_{1_j}), v(h_{1_j}))) \in h_{1_j}} \left\{ \left( 1 - \prod_{j=1}^{4} \left( 1 - \mu(h_{i_j}) \right)^{w_j}, \right) \left( \prod_{j=1}^{4} v(h_{i_j})^{w_j} \right) \right\}$$

$$= \{ (s_{2.45}, (0.4196, 0.4202)), (s_{2.95}, (0.5120, 0.2971)), (s_{2.55}, (0.4485, 0.4035)), (s_{3.05}, (0.5363, 0.2853)), (s_{2.80}, (0.5447, 0.3514)), (s_{3.30}, (0.6171, 0.2485)), (s_{2.90}, (0.5673, 0.3375)), (s_{3.40}, (0.6362, 0.2386)), (s_{3.15}, (0.6049, 0.3049)), (s_{3.65}, (0.6678, 0.2156)), (s_{3.25}, (0.6246, 0.2928)), (s_{3.75}, (0.6843, 0.2071)), (s_{2.75}, (0.5286, 0.3855)), (s_{3.25}, (0.6036, 0.2726)), (s_{2.85}, (0.5521, 0.3702)), (s_{3.35}, (0.6233, 0.2617)), (s_{3.10}, (0.6302, 0.3224)), (s_{3.60}, (0.6890, 0.2280)), (s_{3.20}, (0.6486, 0.3096)), (s_{3.70}, (0.7045, 0.2189)), (s_{3.45}, (0.6791, 0.2797)), (s_{3.95}, (0.7301, 0.1978)), (s_{3.55}, (0.6951, 0.2686)), (s_{4.05}, (0.7436, 0.1899)) \right\} \quad (54)$$

Similarly, the overall assessment values $h_i$ of other milk products $A_i$ ($i = 2, 3, 4$) can be obtained. Consider

$$h_2 = \{ (s_{3.30}, (0.5833, 0.2930)), (s_{3.80}, (0.6235, 0.2648)), (s_{3.65}, (0.6146, 0.2930)), (s_{3.15}, (0.6518, 0.2648)), (s_{3.60}, (0.6178, 0.2380)), (s_{4.10}, (0.6546, 0.2151)) \} \quad (55)$$

**Step 2.** Calculate the score values $S(h_i)$ of overall assessment values $h_i$ ($i = 1, 2, 3, 4$) by (52), which are shown in Figure I.

**Step 3.** Rank all feasible milk products $A_i$ ($i = 1, 2, 3, 4$) in accordance with the descending order of corresponding score values. Since $S(h_4) > S(h_2) > S(h_1) > S(h_3)$, thus the ranking...
of all milk products is obtained as $A_4 > A_2 > A_1 > A_3$, where the symbol “$>$” means “superior to.” Therefore, the most desirable milk product is $A_4$ (pasteurized milk).

If we utilize the HILFWG operator in (51) to aggregate the hesitant intuitionistic linguistic fuzzy assessment values $h_{ij}$ of the milk products $A_i$ ($i = 1, 2, 3, 4$) on all attributes $C_j$ ($j = 1, 2, 3, 4$) into the overall assessment value $\tilde{h}_i$ of the milk products $A_i$ ($i = 1, 2, 3, 4$) in Step 1, we can obtain

$$\tilde{h}_1 = \{(s_{2.40}, (0.3677, 0.4283)), (s_{2.85}, (0.3951, 0.3674)), \}
\{s_{2.47}, (0.3803, 0.4207)), (s_{2.94}, (0.4086, 0.3589)), \}
\{s_{2.65}, (0.4472, 0.3569)), (s_{3.160}, (0.4806, 0.2883)), \}
\{s_{2.73}, (0.4625, 0.3483)), (s_{3.257}, (0.4970, 0.2787)), \}
\{s_{2.87}, (0.4686, 0.3261)), (s_{3.411}, (0.5036, 0.2542)), \}
\{s_{2.95}, (0.4847, 0.3171)), (s_{3.511}, (0.5208, 0.2442)), \}
\{s_{2.71}, (0.5112, 0.4013)), (s_{3.222}, (0.5493, 0.3374)), \}
\{s_{2.79}, (0.5287, 0.3932)), (s_{3.322}, (0.5681, 0.3285)), \}
\{s_{5.00}, (0.6218, 0.3265)), (s_{5.57}, (0.6682, 0.2546)), \}
\{s_{5.09}, (0.6431, 0.3174)), (s_{5.67}, (0.6911, 0.2446)), \}
\{s_{3.24}, (0.6516, 0.2942)), (s_{3.85}, (0.7002, 0.2189)), \}
\{s_{3.34}, (0.6739, 0.2848)), (s_{3.97}, (0.7241, 0.2084)), \};
\tilde{h}_2 = \{(s_{3.27}, (0.5297, 0.3146)), (s_{3.72}, (0.5862, 0.2914)), \}
\{s_{3.62}, (0.5646, 0.3146)), (s_{4.111}, (0.6248, 0.2914)), \}
\{s_{5.50}, (0.5548, 0.2528)), (s_{3.97}, (0.6139, 0.2275)), \}
\{s_{3.87}, (0.5913, 0.2528)), (s_{4.39}, (0.6544, 0.2275)), \};
\tilde{h}_3 = \{(s_{1.90}, (0.4587, 0.3537)), (s_{2.10}, (0.5212, 0.2848)), \}
\{s_{2.39}, (0.5669, 0.2566)), (s_{1.94}, (0.4777, 0.3623)), \}
\{s_{2.15}, (0.5428, 0.2942)), (s_{2.44}, (0.5904, 0.2665)), \}
\{s_{2.79}, (0.5073, 0.3265)), (s_{3.09}, (0.5764, 0.2546)), \}
\{s_{3.51}, (0.6270, 0.2254)), (s_{2.85}, (0.5283, 0.3354)), \}
\{s_{3.16}, (0.6002, 0.2645)), (s_{3.59}, (0.6529, 0.2356)), \}
\{s_{2.07}, (0.4845, 0.3031)), (s_{2.29}, (0.5505, 0.2287)), \}
\{s_{2.60}, (0.5988, 0.1984)), (s_{2.12}, (0.5045, 0.3123)), \}
\{s_{2.34}, (0.5733, 0.2390)), (s_{2.66}, (0.6236, 0.2091)), \};
weighted averaging (HFLWA) operator proposed by Zhang and Wu [35]. Consider

\[ h_1 = \{ s_{2.45}, s_{2.95}, s_{2.55}, s_{3.05}, s_{2.80}, s_{3.30}, s_{2.90}, \]  
\[ s_{3.40}, s_{3.15}, s_{3.65}, s_{3.25}, s_{3.75}, s_{2.75}, \]  
\[ s_{3.25}, s_{2.85}, s_{3.35}, s_{3.10}, s_{3.60}, s_{2.20}, \]  
\[ s_{3.70}, s_{3.45}, s_{3.95}, s_{3.55}, s_{4.05} \}; \]

\[ h_2 = \{ s_{3.30}, s_{3.80}, s_{3.45}, s_{4.15}, s_{3.60}, s_{4.10}, s_{3.95}, s_{4.45} \}; \]

\[ h_3 = \{ s_{2.15}, s_{2.40}, s_{2.90}, s_{2.25}, s_{2.50}, s_{3.00}, s_{2.85}, s_{3.10}, \]  
\[ s_{3.60}, s_{2.95}, s_{3.10}, s_{3.70}, s_{2.45}, s_{2.70}, s_{3.20}, s_{2.55}, \]  
\[ s_{2.80}, s_{3.30}, s_{3.15}, s_{3.40}, s_{4.30}, s_{3.25}, s_{3.50}, s_{4.00} \}; \]

\[ h_4 = \{ s_{4.35}, s_{4.60}, s_{4.55}, s_{4.80}, s_{4.65}, s_{4.90}, s_{4.85}, s_{5.10}, \]  
\[ s_{4.95}, s_{5.20}, s_{5.15}, s_{5.40} \}. \]

Thus, we can obtain the score values of the milk products \( A_i \) (i = 1, 2, 3, 4) as shown in Figure 3.

Since \( S(h_4) > S(h_2) > S(h_3) > S(h_1) \), the ranking of all milk products is obtained as \( A_4 > A_2 > A_1 > A_3 \). Therefore, the most desirable milk product is \( A_4 \) (pasteurized milk) as well, which demonstrates the feasibility and validity of the method proposed in this paper. Moreover, we can know the membership degree and the nonmembership degree of the possible assessment value from the results obtained by the new methods proposed in this paper.

7. Conclusions

With respect to MADM problems in which the attribute values take the form of HILFE, this paper studies the MADM approach under hesitant intuitionistic linguistic fuzzy environment. Firstly, the concept, operational laws, and comparison laws of HILFE are proposed. Then, some aggregation operators are developed for aggregating the hesitant intuitionistic linguistic fuzzy weighted aggregation operators, hesitant intuitionistic linguistic fuzzy ordered weighted aggregation operators, hesitant intuitionistic linguistic fuzzy hybrid aggregation operators, generalized hesitant intuitionistic linguistic fuzzy weighted aggregation operators, and generalized hesitant intuitionistic linguistic fuzzy ordered weighted aggregation operators. Based on the proposed HILFWA operator and HILFWG operator, an approach is proposed to solve MADM problems under hesitant intuitionistic linguistic fuzzy environment. Finally, a practical example is given to illustrate the application of the proposed method. The main advantage of our approach is that it can describe the uncertain information by several intuitionistic linguistic variables in which linguistic variables demonstrate whether an attribute is good or bad in qualitative and intuitionistic fuzzy numbers are adopted to demonstrate how much degree that an attribute value belongs and does not belong to a linguistic variable in quantitative. In future research, we will focus on expanding the hesitant intuitionistic linguistic decision-making approach to other domains such as supplier selection, location choice, project selection, and green supply chain evaluation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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