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Research Article

Robust Switched Control Design for Nonlinear Systems Using Fuzzy Models

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The paper proposes a new switched control design method for some classes of uncertain nonlinear plants described by Takagi-Sugeno fuzzy models. This method uses a quadratic Lyapunov function to design the feedback controller gains based on linear matrix inequalities (LMIs). The controller gain is chosen by a switching law that returns the smallest value of the time derivative of the Lyapunov function. The proposed methodology eliminates the need to find the membership function expressions to implement the control laws. The control designs of a ball-and-beam system and of a magnetic levitator illustrate the procedure.

1. Introduction

There has been much interest in recent years to study switched systems, mainly linear systems, as can be seen in [1–8]. This interest has also increased for nonlinear systems and several papers have been published on switched Takagi-Sugeno fuzzy systems. In general, these studies use switching rules based on regions that depend on the premise variables and/or membership functions and/or state variables [9–18].

Results on switching laws based on the premise variable can be seen in [9, 10, 17]. In [9, 10], a switched fuzzy system was used to represent the nonlinear dynamical model of a hovercraft vehicle and to design a switching fuzzy controller. Then, in [10] smoothness conditions were established, which avoid the phenomenon of discontinuity in the control signal. The problem of dynamic output feedback $\mathscr{H}_\infty$ control was addressed in [17]. Switching laws based on the values of the membership functions are considered in [11, 12, 14–16], where the switched control scheme presented in [14] is an extension of the parallel distributed compensation (PDC). A dynamic output feedback controller, which is based on switched dynamic parallel distributed compensation, was proposed in [15].

Switching laws based on the plant state vector were proposed, for instance, in [13, 18]. The control design presented in [13] uses local state feedback gains obtained from the solution of an optimization problem that assures a guaranteed cost performance. LMIs conditions for robust switched fuzzy parallel distributed compensation controller design and a $\mathscr{H}_\infty$ criterion were obtained in [18]. The procedure to design switching controllers described in [18] was based on the switched quadratic Lyapunov function proposed in [19].

This paper proposes a new method of switched control for some classes of uncertain nonlinear systems described by Takagi-Sugeno fuzzy models. This new control law, which also depends on the state variables, generalizes the results given in [8], which considered only linear plants. The proposed controller chooses a gain from a set of gains by means

of a suitable switching law that returns the smallest value of the Lyapunov function time derivative. The proposed methodology enables us to design the set of gains based on LMIs and on the parallel distributed compensation, as proposed, for instance, in [20–26].

The main advantage of this new procedure is its practical application because it eliminates the need to find the explicit expressions of the membership functions, which can often have long and/or complex expressions or may not be known due to the uncertainties. Furthermore, for certain classes of nonlinear systems, the switched controller can operate even with an uncertain reference control signal. Additionally, with the proposed methodology the closed-loop systems usually present a settling time that is smaller than those obtained with fuzzy controllers, without using switching, that are widely studied in the literature. Moreover, performance indices such as decay rate and constraints on the plant's input and output can be added in the control design procedure.

Simulation results of the control of a ball-and-beam system and of a magnetic levitator are presented to compare the performance of the proposed control law with the traditional PDC fuzzy control law [20, 22]. The computational implementations were carried out using the modeling language YALMIP [27] with the solver LMILab [28].

The paper is organized as follows. Section 2 presents the preliminary results on Takagi-Sugeno fuzzy model, fuzzy regulator design, and stability of the Takagi-Sugeno fuzzy systems via LMIs. Section 3 offers a new switching control method for some classes of nonlinear systems described by Takagi-Sugeno fuzzy models. Some examples to illustrate the performance of the new proposed method are given in Section 4. Finally, Section 5 draws the conclusions.

For convenience, in some places, the following notation is used:

$$\begin{gathered}\mathbb{K}_r = \{1, 2, \ldots, r\}, \quad r \in \mathbb{N}, \qquad x(t) = x, \\ \alpha_i(x(t)) = \alpha_i, \qquad V(x(t)) = V, \qquad \|x\|_2 = \sqrt{x^T x}, \\ (A, B, C, K)(\alpha) = \sum_{i=1}^{r} \alpha_i (A_i, B_i, C_i, K_i), \\ \alpha_i \geq 0, \quad \sum_{i=1}^{r} \alpha_i = 1, \qquad \alpha^T = [\alpha_1, \alpha_2, \ldots, \alpha_r].\end{gathered} \tag{1}$$

2. Takagi-Sugeno Fuzzy Systems and Fuzzy Regulator

Consider the Takagi-Sugeno fuzzy model as described in [29–31]:

$$\begin{aligned}&\text{Rule } i: \quad \text{IF} \quad z_1(t) \text{ is } M_1^i, \ldots, z_p(t) \text{ is } M_p^i, \\ &\qquad \text{THEN} \quad \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t), \\ y(t) = C_i x(t), \end{cases}\end{aligned} \tag{2}$$

where M_j^i is the fuzzy set j of the rule i, $i \in K_r$ and $j \in K_p$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ is the output vector, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{q\times n}$, and $z_1(t), \ldots, z_p(t)$ are premise variables that in this paper are the state variables.

From [20], $\dot{x}(t)$ given in (2) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(x(t)) \left(A_i x(t) + B_i u(t)\right), \tag{3}$$

where $\alpha_i(x(t))$ is the normalized weight of each local model system $A_i x(t) + B_i u(t)$ that satisfies (1).

Assuming that the state vector $x(t)$ is available, from the Takagi-Sugeno fuzzy model (2), the control input of fuzzy regulators via parallel distributed compensation has the following structure [20]:

$$\begin{aligned}&\text{Rule } j: \quad \text{IF} \quad z_1(t) \text{ is } M_1^j, \ldots, z_p(t) \text{ is } M_p^j, \\ &\qquad \text{THEN} \quad u(t) = -K_j x(t).\end{aligned} \tag{4}$$

Similar to (3), one can consider the control law [20]

$$u(t) = u_\alpha = -\sum_{j=1}^{r} \alpha_j(x(t)) K_j x(t). \tag{5}$$

From (5), (3), and (1), one obtains

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x(t)) \alpha_j(x(t)) \left[A_i - B_i K_j\right] x(t) \\ &= (A(\alpha) - B(\alpha) K(\alpha)) x.\end{aligned} \tag{6}$$

2.1. Stability of Takagi-Sugeno Fuzzy Systems via LMIs. The following theorem, whose proof can be seen in [20], guarantees the asymptotic stability of the origin of the system (6).

Theorem 1. *The equilibrium point $x = 0$ of the continuous-time fuzzy control system given in (6) is asymptotically stable in the large if there exist a common symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ and $M_i \in \mathbb{R}^{n\times m}$ such that, for all $i, j \in \mathbb{K}_r$, the following LMIs hold:*

$$\begin{gathered}X A_i^T + A_i X - B_i M_i - M_i^T B_i^T \prec 0, \\ \left(A_i + A_j\right) X + X\left(A_i + A_j\right)^T - B_i M_j \\ - B_j M_i - M_i^T B_j^T - M_j^T B_i^T \preceq 0, \quad i < j,\end{gathered} \tag{7}$$

excepting the pairs (i,j) such that $\alpha_i \alpha_j = 0$, for all x. If (7) are feasible, the controller gains are given by $K_i = M_i X^{-1}$, $i \in \mathbb{K}_r$.

Remark 2. In this paper, for simplicity, the new design method of the controller gains was based on Theorem 1. However, the proposed methodology does not exclude the use of other relaxed control design methods also based on LMIs, for plants described by Takagi-Sugeno fuzzy models, as those presented in [20, 22, 23, 32–35].

3. Main Result

3.1. Case 1: Fuzzy System with Constant Matrix $B(\alpha) = B$. In this section the design of a switched controller for the Takagi-Sugeno fuzzy system (3) is proposed, assuming that $B(\alpha) = B$ is a constant matrix now given by

$$\dot{x}(t) = A(\alpha)x(t) + Bu(t). \tag{8}$$

Suppose that (7) are feasible and let $K_i = M_i X^{-1}$, $i \in \mathbb{K}_r$, be the gains of the controller given in (5), and $P = X^{-1}$ is obtained from the conditions of Theorem 1. Then, define the switched controller by

$$u(t) = u_\sigma = -K_\sigma x, \qquad \sigma = \arg\min_{i\in\mathbb{K}_r}\left(-x^T PBK_i x\right). \tag{9}$$

Therefore, from (1), the controlled system (8) and (9) can be written as follows:

$$\dot{x}(t) = A(\alpha)x(t) + Bu_\sigma = \sum_{i=1}^{r}\alpha_i\left[A_i - BK_\sigma\right]x(t). \tag{10}$$

Theorem 3. *Assume that the conditions of Theorem 1, related to the system* (8) *with the control law* (5)*, hold and obtain $K_i = M_i X^{-1}$, $i \in \mathbb{K}_r$ and $P = X^{-1}$. Then, the switched control law* (9) *makes the equilibrium point $x = 0$, of the system* (8)*, asymptotically stable in the large.*

Proof. Consider a quadratic Lyapunov candidate function $V = x^T Px$. Define $\dot{V}_{u_\alpha}$ and $\dot{V}_{u_\sigma}$ as the time derivatives of V for the system (8), with the control laws (5) and (9), respectively. Then, from (10),

$$\begin{aligned}\dot{V}_{u_\sigma} &= 2x^T P\dot{x} = 2x^T P\left(A(\alpha)x + Bu_\sigma\right)\\ &= 2x^T PA(\alpha)x + 2x^T PB\left(-K_\sigma\right)x.\end{aligned} \tag{11}$$

Thus, note that, from (1) and (9),

$$\min_{i\in\mathbb{K}_r}\left\{x^T PB\left(-K_i\right)x\right\} \le x^T PB\left(-\sum_{i=1}^{r}\alpha_i K_i\right)x. \tag{12}$$

Therefore, from (11) and the laws given in (9) and (5) observe that

$$\begin{aligned}\dot{V}_{u_\sigma} &= 2x^T PA(\alpha)x + 2\min_{i\in\mathbb{K}_r}\left\{x^T PB\left(-K_i\right)x\right\}\\ &\le 2x^T PA(\alpha)x + 2x^T PB\left(-\sum_{i=1}^{r}\alpha_i K_i\right)x\\ &= 2x^T P\left(A(\alpha) - BK(\alpha)\right)x\\ &= 2x^T P\left(A(\alpha)x + Bu_\alpha\right) = \dot{V}_{u_\alpha}.\end{aligned} \tag{13}$$

Then, $\dot{V}_{u_\sigma} \le \dot{V}_{u_\alpha}$. Furthermore, from Theorem 1 $\dot{V}_{u_\alpha} < 0$ for $x \ne 0$. Thus, the proof is concluded. □

Remark 4. Theorem 3 shows that if the conditions of Theorem 1 are satisfied, then $\dot{V}_{u_\alpha}(x(t)) < 0$ for all $x(t) \ne 0$ and thus $\dot{V}_{u_\sigma}(x(t)) < 0$ for $x(t) \ne 0$, ensuring that the equilibrium point $x = 0$ of the controlled system (8) and (9) is asymptotically stable in the large. Thus, Theorem 1 can be used to project the gains $K_1, K_2, \ldots, K_r$ and the matrix $P = X^{-1}$ of the switched control law (9). Additionally, note that the switched control law (9) does not use the membership functions α_i, $i \in \mathbb{K}_r$, which would be necessary to implement the control law (5) and may thus offer a relatively simple alternative for implementing the controller.

3.2. Case 2: Fuzzy System with Nonlinearity in the Matrix $B(\alpha)$. In this case a fuzzy system similar to (3) will be considered, with α_i, $i \in \mathbb{K}_r$, defined in (1); namely,

$$\begin{aligned}\dot{\hat{x}}(t) &= \widehat{A}(\alpha)\hat{x}(t) + \widehat{B}(\alpha)u(t),\\ \widehat{A}(\alpha) &= \sum_{i=1}^{r}\alpha_i\widehat{A}_i, \qquad \widehat{B}(\alpha) = \sum_{i=1}^{r}\alpha_i\widehat{B}_i.\end{aligned} \tag{14}$$

Let $v \in \mathbb{R}^m$ be the time derivative of the control input vector $u \in \mathbb{R}^m$. Define x_{n+l} and v_l, such that $\dot{x}_{n+l}(t) = \dot{u}_l(t) = v_l(t)$, $l \in \mathbb{K}_m$. Thus one obtains the following system:

$$\begin{aligned}\dot{\hat{x}}(t) &= \widehat{A}(\alpha)\hat{x}(t) + \widehat{B}(\alpha)u(t),\\ \dot{x}_{n+1}(t) &= v_1,\\ &\vdots\\ \dot{x}_{n+m}(t) &= v_m,\end{aligned} \tag{15}$$

or equivalently [36]

$$\dot{x}(t) = A(\alpha)x(t) + Bv(t), \tag{16}$$

where

$$\begin{aligned}x &= \left[\hat{x}^T \;\; x_{n+1} \;\; \cdots \;\; x_{n+m}\right]^T,\\ A(\alpha) &= \begin{bmatrix}\widehat{A}(\alpha) & \widehat{B}(\alpha)\\ 0_{m\times n} & 0_{m\times m}\end{bmatrix}, \qquad B = \begin{bmatrix}0_{n\times m}\\ I_{m\times m}\end{bmatrix}.\end{aligned} \tag{17}$$

After the aforementioned considerations, note that the system (16) is similar to the system (8) and therefore the control problem falls into Case 1. Thus, one can adopt the procedure stated in Case 1 for designing a switched control law $v(t) = -K_\sigma x(t)$, $K_\sigma \in \mathbb{R}^{m\times n+m}$.

3.3. Case 3: Fuzzy System with Uncertainty in the Control Signal. In this case, it is assumed that the plant given by $\dot{\overline{x}} = f(\overline{x}, \overline{u})$ has an equilibrium point $\overline{x} = x_0$ and the respective control input is $\overline{u} = u_0$, such that $f(x_0, u_0) = 0$. Suppose that x_0 is known, u_0 is uncertain, but $0 < u_0 \in [u_{0_{\min}}, u_{0_{\max}}]$, where $u_{0_{\min}}$ and $u_{0_{\max}}$ are known, and the plant can be described by the Takagi-Sugeno fuzzy system (1)–(3),

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u, \tag{18}$$

where $x(t) = \overline{x}(t) - x_0$, $\overline{x}(t)$ is the state vector of the plant and $u(t) = \overline{u}(t) - u_0$, $\overline{u}$ is the control input of the plant.

Now consider that $B(\alpha)$ can be written as follows:

$$B(\alpha) = Bg(x(t)), \tag{19}$$

where B is a known constant matrix and $g(x(t)) > 0$, for all x, is an uncertain nonlinear function. Thus, the system (18) can be written as follows:

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u = A(\alpha)x(t) + Bg(x(t))u. \tag{20}$$

Assume that the gains $K_i = M_i X^{-1}$, $i \in \mathbb{K}_r$, and the matrix $P = X^{-1}$ have been obtained using the vertices of the polytope of the system (18) in the LMIs (7) from Theorem 1, as proposed in [20]. Now, given a constant $\xi > 0$, define the control law as

$$u(t) = u_{(\sigma,\xi)} = \overline{u}_{(\sigma,\xi)} - u_0, \quad \text{with } \overline{u}_{(\sigma,\xi)} = -K_\sigma x + \gamma_\xi, \tag{21}$$

where

$$K_\sigma \in \{K_1, K_2, \ldots, K_r\}, \qquad \sigma = \arg\min_{i\in\mathbb{K}_r}\left\{-x^T PBK_i x\right\},$$

$$\gamma_\xi = \begin{cases} u_{0_{\max}}, & \text{if } x^T PB < -\xi, \\ \left[\left(u_{0_{\min}} - u_{0_{\max}}\right)x^T PB \right. \\ \quad \left. + \xi\left(u_{0_{\max}} + u_{0_{\min}}\right)\right] \times (2\xi)^{-1}, & \text{if } \left|x^T PB\right| \le \xi, \\ u_{0_{\min}}, & \text{if } x^T PB > \xi. \end{cases} \tag{22}$$

Within this context the following theorem is proposed.

Theorem 5. *Suppose that the conditions from Theorem 1 hold, for the system* (18) *with the control law* (5)*, and obtain* $K_i = M_i X^{-1}$, $i \in \mathbb{K}_r$, *and* $P = X^{-1}$. *Then the switched control law* (21) *and* (22) *makes the system* (18) *and* (19) *uniformly ultimately bounded.*

Proof. Consider a quadratic Lyapunov candidate function $V = x^T Px$. Define $\dot{V}_{u_\alpha}$ and $\dot{V}_{u_{(\sigma,\xi)}}$ as the time derivatives of V for the system (18), (19), with the control laws (5) and (21), and (22), respectively. Then,

$$\begin{aligned} \dot{V}_{u_{(\sigma,\xi)}} &= 2x^T P\dot{x} = 2x^T P\left(A(\alpha)x + Bg(x)u_{(\sigma,\xi)}\right) \\ &= 2x^T P\left(A(\alpha)x - B(\alpha)K(\alpha)x\right) \\ &\quad + 2x^T PBg(x)\left[\overline{u}_{(\sigma,\xi)} - u_0 + K(\alpha)x\right] \\ &= \dot{V}_{u_\alpha} + 2x^T PBg(x)\left[-K_\sigma x + \gamma_\xi - u_0 + K(\alpha)x\right] \\ &= \dot{V}_{u_\alpha} + 2g(x)\min_{i\in\mathbb{K}_r}\left\{-x^T PBK_i x\right\} \\ &\quad + 2g(x)x^T PB\left[\gamma_\xi - u_0 + K(\alpha)x\right]. \end{aligned} \tag{23}$$

Remembering that $\alpha_i > 0$, $i \in \mathbb{K}_r$ and $\sum_{i=1}^r \alpha_i = 1$, $g(x) > 0$, and $g(x)B = B(\alpha)$ and noting that $\min_{i\in\mathbb{K}_r}\{-x^T PBK_i x\} \le -x^T PB(\sum_{i=1}^r \alpha_i K_i)x$, from (23)

$$\begin{aligned} \dot{V}_{u_{(\sigma,\xi)}} &\le \dot{V}_{u_\alpha} - 2x^T PB(\alpha)\left(\sum_{i=1}^r \alpha_i K_i\right)x \\ &\quad + 2g(x)x^T PB\left[\gamma_\xi - u_0 + K(\alpha)x\right] \\ &= \dot{V}_{u_\alpha} + 2g(x)x^T PB\left(\gamma_\xi - u_0\right). \end{aligned} \tag{24}$$

Now, if $|x^T PB| \ge \xi$, then from (22), $g(x)x^T PB(\gamma_\xi - u_0) \le 0$. Thus, from (24) $\dot{V}_{u_{(\sigma,\xi)}} \le \dot{V}_{u_\alpha} < 0$ for $x \ne 0$, since the system (18) with the control law (5) is globally asymptotically stable. Otherwise, if $|x^T PB| < \xi$, one obtains from (24) and (22)

$$\begin{aligned} \dot{V}_{u_{(\sigma,\xi)}} &\le \dot{V}_{u_\alpha} + 2g_{\max}\left|x^T PB\right| \cdot \left|\gamma_\xi - u_0\right| \\ &\le -\epsilon\|x\|_2^2 + 2g_{\max}\left|\gamma_\xi - u_0\right|\xi \\ &\le -\epsilon\|x\|_2^2 + 2g_{\max}\left(\left|\gamma_\xi\right| + \left|u_0\right|\right)\xi \\ &\le -\epsilon\|x\|_2^2 + 4g_{\max}\cdot u_{0_{\max}}\cdot\xi \\ &\le -\epsilon\|x\|_2^2 + \epsilon_1, \end{aligned} \tag{25}$$

where $-\epsilon < 0$ denotes the maximum eigenvalue of $P(A(\alpha) - B(\alpha)K(\alpha)) + (A(\alpha) - B(\alpha)K(\alpha))^T P$ for all α defined in (1), $g_{\max} = \max\{g(x)\}$, and $\epsilon_1 = 4g_{\max}\cdot u_{0_{\max}}\cdot\xi$. Therefore, $\dot{V}_{u_{(\sigma,\xi)}} < 0$, if $\|x\|_2 > \sqrt{\epsilon_1/\epsilon}$. Thus, according to [37] the controlled system is uniformly ultimately bounded and the proof is concluded. □

Remark 6. Observe that the function γ_ξ given in (22) is important to ensure the uniform ultimate boundedness of the system and smoothness of the control input. Note that when ξ is equal to zero, the function γ_ξ is a discontinuous function and therefore the control input can also be discontinuous, as can be seen in [8]. Thus, the designer must choose ξ according to the requirements.

4. Examples

4.1. Example of Case 1. To illustrate this case, presented is the control design of a ball-and-beam system, in Figure 1, whose mathematical model [38, page 26] is given by the following equations:

$$\ddot{r}(t) = \beta r(t)\dot{\theta}^2(t) - \beta g\sin(\theta(t)), \qquad \ddot{\theta}(t) = u(t), \tag{26}$$

where r is the position of the ball; θ is the angle of the beam relative to the ground; u is the torque applied to the beam and the control input; $g = 9.81\ \text{m/s}^2$ is the acceleration of the gravity; and $\beta = mR^2/(J_b + mR^2)$ is an uncertain parameter of the system which depends on the mass m, the radius R, and the moment of inertia J_b of the ball.

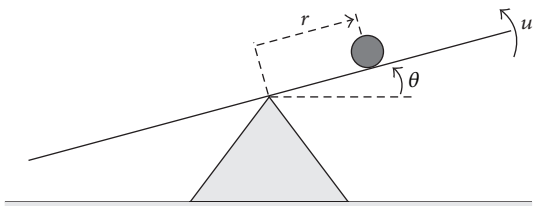

Figure 1: Ball-and-beam system.

Define the state variables $x_1 = r(t)$, $x_2 = \dot{r}(t)$, $x_3 = \theta(t)$, and $x_4 = \dot{\theta}(t)$. Then, by defining the state vector $x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T$, the system (26) can be written as follows:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \beta x_1 x_4^2 - \beta g \sin(x_3), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= u, \end{aligned} \tag{27}$$

or equivalently

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & f_{23}(x,\beta) & f_{24}(x,\beta) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{28}$$

where

$$f_{23}(x,\beta) = -\frac{\beta g \sin(x_3)}{x_3}, \qquad f_{24}(x,\beta) = \beta x_1 x_4. \tag{29}$$

Note that, for implementing the switched controller (9), the controller gains will be designed using the generalized form proposed in [22], and therefore the following domain will be considered for the system (28) and (29):

$$\begin{aligned} D_1 = \Big\{ & x \in \mathbb{R}^4 : -1 \le x_1 \le 1, -\frac{\pi}{12} \le x_3 \le \frac{\pi}{12}, \\ & -2 \le x_4 \le 2,\ 0.60 \le \beta \le 0.7143 \Big\}. \end{aligned} \tag{30}$$

After the calculations the following maximum and minimum values of the functions f_{23} and f_{24} were obtained:

$$\begin{aligned} a_{23_1} &= \max_{(x,\beta)\in D_1} \{f_{23}(x,\beta)\} = -4.8492, \\ a_{23_2} &= \min_{(x,\beta)\in D_1} \{f_{23}(x,\beta)\} = -5.0053, \\ a_{24_1} &= \max_{(x,\beta)\in D_1} \{f_{24}(x,\beta)\} = 1.0204, \\ a_{24_2} &= \min_{(x,\beta)\in D_1} \{f_{24}(x,\beta)\} = -1.0204. \end{aligned} \tag{31}$$

Thus, the nonlinear function f_{23} can be represented by a Takagi-Sugeno fuzzy model, considering that there exists a convex combination with membership functions $\sigma_{23_1} = \sigma_{23_1}(x,\beta)$ and $\sigma_{23_2} = \sigma_{23_2}(x,\beta)$ and constant values a_{23_1} and a_{23_2} given in (31) such that [22]

$$f_{23}(x,\beta) = \sigma_{23_1}(x,\beta)\, a_{23_1} + \sigma_{23_2}(x,\beta)\, a_{23_2}, \tag{32}$$

with

$$0 \le \sigma_{23_1},\ \sigma_{23_2} \le 1, \quad \sigma_{23_1} + \sigma_{23_2} = 1. \tag{33}$$

Therefore, from (32) note that

$$\begin{aligned} \sigma_{23_1}(x,\beta) &= \frac{f_{23}(x,\beta) - a_{23_2}}{a_{23_1} - a_{23_2}}, \\ \sigma_{23_2}(x,\beta) &= 1 - \sigma_{23_1}(x,\beta). \end{aligned} \tag{34}$$

Similarly, from (31), there exist $\xi_{24_1} = \xi_{24_1}(x,\beta)$ and $\xi_{24_2} = \xi_{24_2}(x,\beta)$ such that

$$f_{24}(x,\beta) = \xi_{24_1}(x,\beta)\, a_{24_1} + \xi_{24_2}(x,\beta)\, a_{24_2}, \tag{35}$$

with

$$0 \le \xi_{24_1},\ \xi_{24_2} \le 1, \quad \xi_{24_1} + \xi_{24_2} = 1. \tag{36}$$

Hence, from (35) observe that

$$\begin{aligned} \xi_{23_1}(x,\beta) &= \frac{f_{24}(x,\beta) - a_{24_2}}{a_{24_1} - a_{24_2}}, \\ \xi_{24_2}(x,\beta) &= 1 - \xi_{24_1}(x,\beta). \end{aligned} \tag{37}$$

Recall that $\xi_{24_1}(x,\beta) + \xi_{24_2}(x,\beta) = 1$ and $\sigma_{23_1}(x,\beta) + \sigma_{23_2}(x,\beta) = 1$, from (33) and (36), respectively. Then, it follows that

$$\begin{aligned} f_{23}(x,\beta) &= \sigma_{23_1}\xi_{24_1}a_{23_1} + \sigma_{23_1}\xi_{24_2}a_{23_1} \\ &\quad + \sigma_{23_2}\xi_{24_1}a_{23_2} + \sigma_{23_2}\xi_{24_2}a_{23_2}, \\ f_{24}(x,\beta) &= \sigma_{23_1}\xi_{24_1}a_{24_1} + \sigma_{23_1}\xi_{24_2}a_{24_2} \\ &\quad + \sigma_{23_2}\xi_{24_1}a_{24_1} + \sigma_{23_2}\xi_{24_2}a_{24_2}. \end{aligned} \tag{38}$$

Now, define

$$\begin{aligned} \alpha_1(x,\beta) &= \sigma_{23_1}\xi_{24_1}, \qquad \alpha_2(x,\beta) = \sigma_{23_1}\xi_{24_2}, \\ \alpha_3(x,\beta) &= \sigma_{23_2}\xi_{24_1}, \qquad \alpha_4(x,\beta) = \sigma_{23_2}\xi_{24_2}, \end{aligned} \tag{39}$$

as the membership functions of the system (28) and (29), and their local models:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23_1} & a_{24_1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23_1} & a_{24_2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23_2} & a_{24_1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23_2} & a_{24_2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ B_1 &= B_2 = B_3 = B_4 = [0 \;\; 0 \;\; 0 \;\; 1]^T. \end{aligned} \tag{40}$$

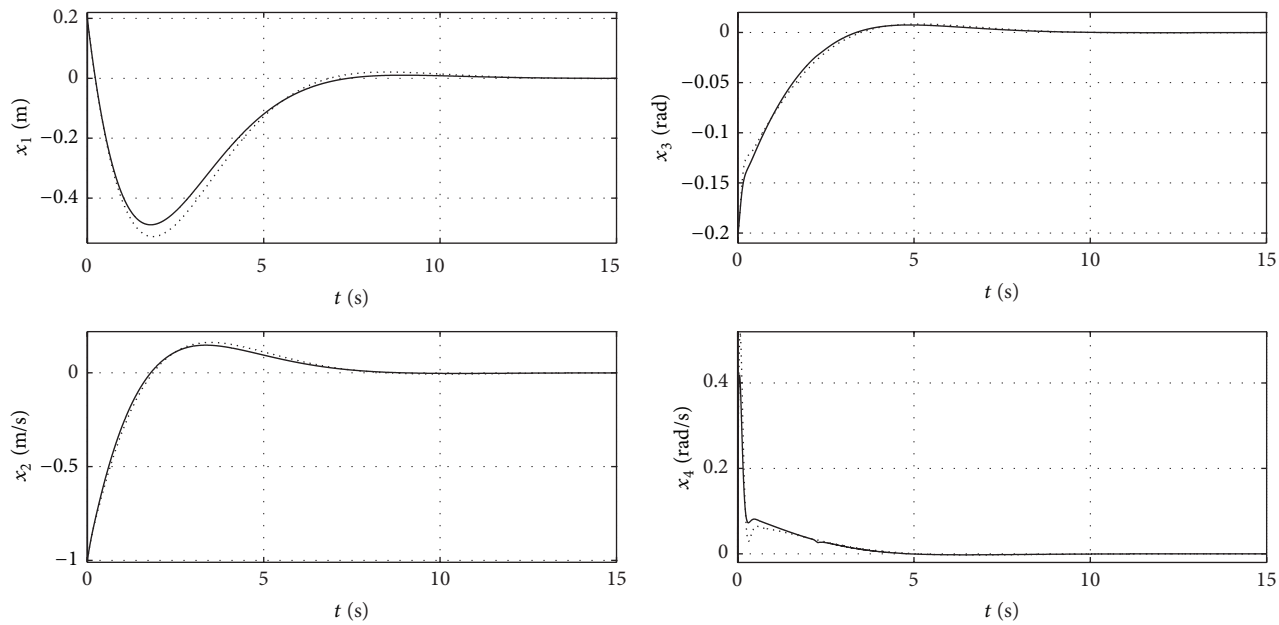

Figure 2: State variables of the ball-and-beam system (28) using the switched controller (9) (solid line) and the fuzzy controller (5) (dotted line), considering $\beta = 0.7$ and the initial condition $x(0) = [0.2 \quad -1 \quad -0.2 \quad 0]^T$.

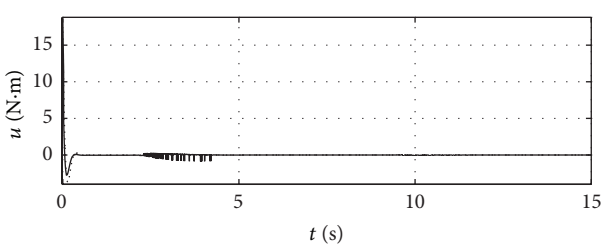

Figure 3: Control signal of the ball-and-beam system (28) using the switched controller (9) (solid line) and the fuzzy controller (5) (dotted line), considering $\beta = 0.7$ and the initial condition $x(0) = [0.2 \quad -1 \quad -0.2 \quad 0]^T$.

Thus, using the LMIs (7) from Theorem 1, one obtains the following controller gains and symmetric positive definite matrix:

$$\begin{aligned} K_1 &= [-17.6809 \quad -35.4033 \quad 252.2903 \quad 18.5316], \\ K_2 &= [-26.9467 \quad -62.7180 \quad 360.9879 \quad 26.9307], \\ K_3 &= [-17.6320 \quad -35.2593 \quad 251.7173 \quad 18.4873], \\ K_4 &= [-26.8979 \quad -62.5740 \quad 360.4149 \quad 26.8865], \\ P &= \begin{bmatrix} 0.0022 & 0.0032 & -0.0170 & -0.0012 \\ 0.0032 & 0.0095 & -0.0380 & -0.0029 \\ -0.0170 & -0.0380 & 0.2302 & 0.0170 \\ -0.0012 & -0.0029 & 0.0170 & 0.0020 \end{bmatrix}. \end{aligned} \tag{41}$$

The goal of the simulation is to keep the ball at the origin $(r, \theta) = (0, 0)$. Considering, $\beta = 0.7$, the initial condition $x_0 = [0.2 \quad -1 \quad -0.2 \quad 0]^T$, and the equilibrium point $x_e = [0 \quad 0 \quad 0 \quad 0]^T$, the simulation of the controlled systems (28), (29), (9), (41) and (28), (29), (5), (31)–(41) presented the responses shown in Figures 2 and 3.

Note that the controller gains have been found using the generalized form proposed in [22]. However, the switched controller u_σ given in (9) does not use the membership functions and therefore it is not necessary to find and implement such functions. Thus, an advantage of this new methodology is that one can eliminate all the steps of the project given in (32)–(39) that are needed to find the membership functions, which can sometimes have long and/or complex expressions or may not be known due to the uncertainties and so their practical implementations are not possible, as is the case of this example.

4.2. Example of Case 2. To illustrate this case, consider the control system design of a magnetic levitator presented in Figure 4, whose mathematical model [38, page 24] is given by

$$m\ddot{y} = -k\dot{y} + mg - \frac{\lambda \mu i^2}{2(1 + \mu y)^2}, \tag{42}$$

where $m = 0.05$ Kg is the mass of the ball; $g = 9.8\,\mathrm{m/s^2}$ is the gravity acceleration; $\lambda = 0.460$ H, $\mu = 2\,\mathrm{m^{-1}}$, and $k = 0.001$ Ns/m are positive constants; i is the electric current; and y is the position of the ball.

Define the state variables $\overline{x}_1 = y$ and $\overline{x}_2 = \dot{y}$. Then, (42) can be written as follows [39]:

$$\dot{\overline{x}}_1 = \overline{x}_2, \quad \dot{\overline{x}}_2 = g - \frac{k}{m}\overline{x}_2 - \frac{\lambda \mu i^2}{2m(1 + \mu \overline{x}_1)^2}. \tag{43}$$

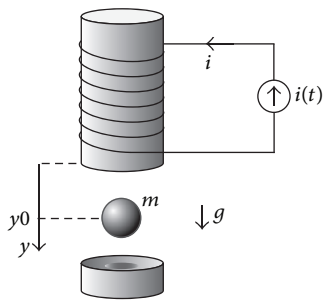

Figure 4: Magnetic levitator.

Consider that during the required operation, $[\overline{x}_1 \;\; \overline{x}_2]^T = (\overline{x}_1, \overline{x}_2) \in D_2$, where

$$D_2 = \left\{(\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2 : 0 \le \overline{x}_1 \le 0.15\right\}. \tag{44}$$

The objective of the paper is to design a controller that keeps the ball in a desired position $y = \overline{x}_1 = y_0$, after a transient response. Thus, the equilibrium point of the system (43) is $\overline{x}_e = [\overline{x}_{1_e} \;\; \overline{x}_{2_e}]^T = [y_0 \;\; 0]^T$.

From the second equation in (43), observe that, in the equilibrium point, $\dot{\overline{x}}_2 = 0$ and $i = i_0$, where

$$i_0^2 = \frac{2mg}{\lambda\mu}(1 + \mu y_0)^2. \tag{45}$$

Note that the equilibrium point is not in the origin $[\overline{x}_1 \;\; \overline{x}_2]^T = [0 \;\; 0]^T$. Thus, for the stability analysis the following change of coordinates is necessary:

$$x_1 = \overline{x}_1 - y_0, \qquad x_2 = \overline{x}_2, \qquad u = i^2 - i_0^2; \tag{46}$$

that is,

$$\overline{x}_1 = x_1 + y_0, \qquad \overline{x}_2 = x_2, \qquad i^2 = u + i_0^2. \tag{47}$$

Therefore, $\dot{x}_1 = \dot{\overline{x}}_1$ and $\dot{x}_2 = \dot{\overline{x}}_2$ and from (45), $i^2 = u + 2mg(1 + \mu y_0)^2/\lambda\mu$.

Hence, the system (43) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{g\mu(\mu x_1 + 2\mu y_0 + 2)}{(1 + \mu(x_1 + y_0))^2} x_1 - \frac{k}{m} x_2 \\ &\quad - \frac{\lambda\mu}{2m(1 + \mu(x_1 + y_0))^2} u. \end{aligned} \tag{48}$$

Finally, from (48) it follows that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f_{21} & -\dfrac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g_{21} \end{bmatrix} u, \tag{49}$$

where

$$\begin{aligned} f_{21} &= f_{21}(x_1, y_0) = \frac{g\mu(\mu x_1 + 2\mu y_0 + 2)}{(1 + \mu(x_1 + y_0))^2}, \\ g_{21} &= g_{21}(x_1, y_0) = \frac{-\lambda\mu}{2m(1 + \mu(x_1 + y_0))^2}. \end{aligned} \tag{50}$$

Now, define x_3 and v such that $\dot{x}_3 = \dot{u} = v$; that is, $x_3 = u$. Thus, considering (50), the system (49) can be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ f_{21} & -\dfrac{k}{m} & g_{21} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v. \tag{51}$$

After this adjustment it is seen that the problem falls into Case 1. Thus, the procedure stated in Case 1 can be used for designing a switched control law $v(t) = -K_\sigma x(t)$, $K_\sigma \in \mathbb{R}^3$.

Thus, to find the local models, the maximum and minimum values of functions f_{21} and g_{21} must be obtained. In this case the methodology proposed in [39] will be used. Then, suppose that the desired position is known and belongs to the set $y_0 \in [0.04, 0.11]$ and consider y_0 as a new variable for the specification of the domain D_3 of the nonlinear functions f_{21} and g_{21} [39]:

$$\begin{aligned} D_3 = \{&(x_1, x_2, y_0) \in \mathbb{R}^3 : -0.11 \le x_1 \le 0.11, \\ &0.04 \le y_0 \le 0.11\}. \end{aligned} \tag{52}$$

As expected, after the calculations, considering (50) and (52), one obtains

$$\begin{aligned} a_{21_1} &= \max_{(x_1, y_0) \in D_3} \{f_{21}(x_1, y_0)\} = 51.4116, \\ a_{21_2} &= \min_{(x_1, y_0) \in D_3} \{f_{21}(x_1, y_0)\} = 25.1427, \\ b_{21_1} &= \max_{(x_1, y_0) \in D_3} \{g_{21}(x_1, y_0)\} = -4.4367, \\ b_{21_2} &= \min_{(x_1, y_0) \in D_3} \{g_{21}(x_1, y_0)\} = -12.4392. \end{aligned} \tag{53}$$

Therefore, from (53) one has the following local models:

$$\begin{gathered} A_1 = \begin{bmatrix} 0 & 1 & 0 \\ a_{21_1} & -0.02 & b_{21_1} \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ a_{21_1} & -0.02 & b_{21_2} \\ 0 & 0 & 0 \end{bmatrix}, \\ A_3 = \begin{bmatrix} 0 & 1 & 0 \\ a_{21_2} & -0.02 & b_{21_1} \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 1 & 0 \\ a_{21_2} & -0.02 & b_{21_2} \\ 0 & 0 & 0 \end{bmatrix}, \\ B_1 = B_2 = B_3 = B_4 = [0 \;\; 0 \;\; 1]^T. \end{gathered} \tag{54}$$

Using the LMIs (7) from Theorem 1, the following controller gains and symmetric positive definite matrix were obtained:

$$\begin{aligned} K_1 &= \left[-636.9216 \quad -109.3352 \quad 15.6269\right], \\ K_2 &= 10^3\left[-2.2199 \quad -0.3851 \quad 0.0535\right], \\ K_3 &= \left[-784.7978 \quad -135.0930 \quad 19.1611\right], \\ K_4 &= 10^3\left[-2.3678 \quad -0.4108 \quad 0.0570\right], \\ P &= \begin{bmatrix} 5.7404 & 0.8178 & -0.1420 \\ 0.8178 & 0.1424 & -0.0195 \\ -0.1420 & -0.0195 & 0.0077 \end{bmatrix}. \end{aligned} \tag{55}$$

For numerical simulation, at $t = 0$ s the initial condition was $\overline{x}(0) = \left[0.04 \quad 1\right]^T$ and $y_0 = 0.1$ m. Since $x_3 = u = i^2 - i_0^2$ and $i_0^2 = 1.5339\,\mathrm{A}^2$ (assuming that $i^2(0) = 0$, the initial condition for the system (51) is $x_0 = \left[0.04 \quad 1 \quad 0\right]^T - \left[0.1 \quad 0 \quad 1.5339\right]^T = \left[-0.06 \quad 1 \quad -1.5339\right]^T$; that is, $x_3(0) = -1.5339$), at $t = 1$ s, from Figure 5, the system is practically at the point $\overline{x}(1) = \left[\overline{x}_1(1) \quad \overline{x}_2(1)\right]^T = \left[0.1 \quad 0\right]^T$ and $x_3(1) = 0$. After changing y_0 from 0.1 m to 0.04 m at $t = 2$ s, one can see that the system is practically at the point $\overline{x}(2) = \left[0.04 \quad 0\right]^T$ and $x_3(2) = 0$, which will be the new initial condition. Finally, y_0 changes from 0.04 m to 0.08 m at $t \geq 2$ s. Thus, as shown in Figure 5, $\overline{x}(\infty) = \left[0.08 \quad 0\right]$ and $x_3(\infty) = 0$. Figures 5 and 6 illustrate the system response.

4.3. Example of Case 3. Consider the magnetic levitator from Section 4.2 given in (43) and (49)-(50), where the mass m is uncertain and define D_4 as the operation domain [39]:

$$\begin{aligned} D_4 = \big\{&(x_1, y_0, m) \in \mathbb{R}^3 : -0.08 \leq x_1 \leq 0.1, \\ &0.05 \leq y_0 \leq 0.08,\ 0.08 \leq m \leq 0.1\big\}. \end{aligned} \tag{56}$$

Thus, as described in Section 4.2, the system (49) can be written as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g_{21} \end{bmatrix} u, \tag{57}$$

where

$$\begin{aligned} f_{21} &= f_{21}(x_1, y_0) = \frac{g\mu\left(\mu x_1 + 2\mu y_0 + 2\right)}{\left(1 + \mu\left(x_1 + y_0\right)\right)^2}, \\ f_{22} &= f_{22}(m) = -\frac{k}{m}, \\ g_{21} &= g_{21}(x_1, y_0, m) = \frac{-\lambda\mu}{2m\left(1 + \mu\left(x_1 + y_0\right)\right)^2}. \end{aligned} \tag{58}$$

Observe that the system (57) can be rewritten as in (20); that is, $\dot{x} = A(\alpha)x + Bg(x)u$, where $B = \left[0 \quad -1\right]^T$ and $g(x) = g(x_1, y_0, m) = \lambda\mu/2m(1 + \mu(x_1 + y_0))^2$. Note that $g(x) > 0$, for all $x \in D_4$.

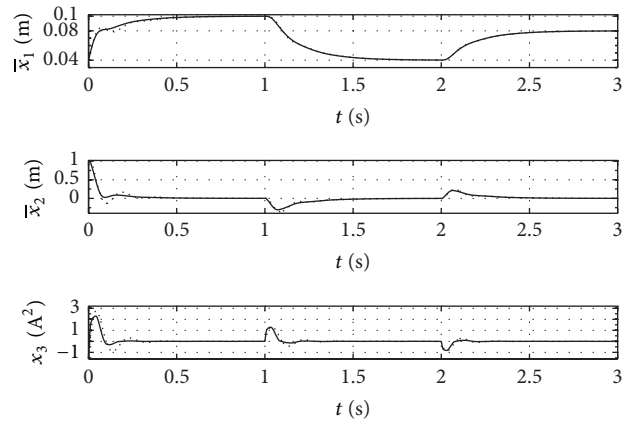

Figure 5: State variables of the magnetic levitator (43) and state x_3 given in (51) using the switched controller (9) (solid line) and the fuzzy controller (5) (dotted line), considering $y_0 = 0.1$ m, $y_0 = 0.04$ m, and $y_0 = 0.08$ m, for $t \in [0, 1)$, $t \in [1, 2)$, and $t \geq 2$, respectively.

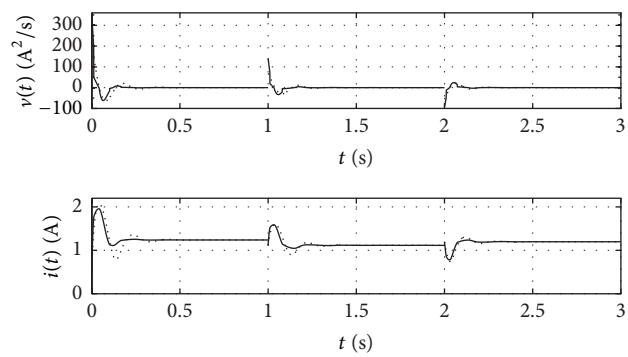

Figure 6: Switched control signal $v(t)$ (9) and electric current $i(t)$ (solid line) and fuzzy control signal (5) and electric current $i(t)$ (dotted line), for $y_0 \in [0.04, 0.11]$, considering $y_0 = 0.1$ m, $y_0 = 0.04$ m, and $y_0 = 0.08$ m, for $t \in [0, 1)$, $t \in [1, 2)$, and $t \geq 2$, respectively.

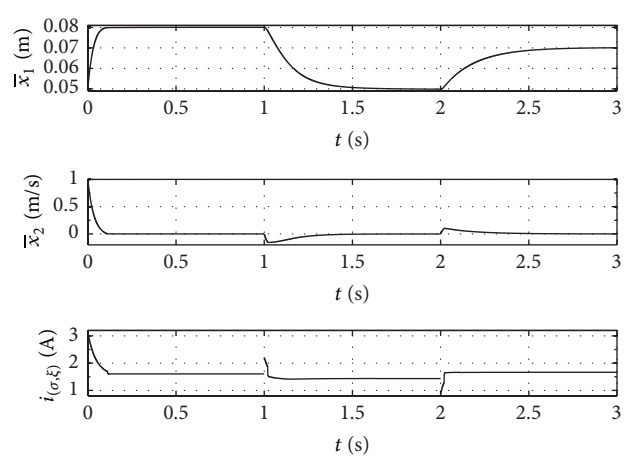

Figure 7: Position ($y(t) = \overline{x}_1(t)$), velocity ($\overline{x}_2(t)$), and electric current ($i_{(\sigma,\xi)}(t)$) of the controlled system, considering $y_0 = 0.08$ m and $m = 0.09$ Kg, $y_0 = 0.05$ m and $m = 0.08$ Kg, and $y_0 = 0.07$ m and $m = 0.1$ Kg, for $t \in [0, 1)$, $t \in [1, 2)$, and $t \geq 2$, respectively.

After the calculations, the maximum and minimum values of the functions f_{21}, f_{22}, g_{21}, and i_0^2, in the domain D_4, were obtained as follows:

$$\begin{aligned}
a_{21_1} &= \max_{(x_1,y_0)\in D_4} \{f_{21}(x_1,y_0)\} = 45.2512,\\
a_{21_2} &= \min_{(x_1,y_0)\in D_4} \{f_{21}(x_1,y_0)\} = 26.7042,\\
a_{22_1} &= \max_{m\in D_4} \{f_{22}(m)\} = -0.0100,\\
a_{22_2} &= \min_{m\in D_4} \{f_{22}(m)\} = -0.0125,\\
b_{21_1} &= \max_{(x_1,y_0,m)\in D_4} \{g_{21}(x_1,y_0,m)\} = -2.4870,\\
b_{21_2} &= \min_{(x_1,y_0,m)\in D_4} \{g_{21}(x_1,y_0,m)\} = -6.5075,\\
\max u_0 &= \max_{y_0,m\in D_4} \{i_0^2(x(t))\} = 2.8667,\\
\min u_0 &= \max_{y_0,m\in D_4} \{i_0^2(x(t))\} = 2.0623.
\end{aligned} \tag{59}$$

From (59), define the following local models of the plant (57) and (58):

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 1 \\ a_{21_1} & a_{22_1} \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 1 \\ a_{21_1} & a_{22_2} \end{bmatrix},\\
A_5 &= \begin{bmatrix} 0 & 1 \\ a_{21_2} & a_{22_1} \end{bmatrix}, \qquad A_7 = \begin{bmatrix} 0 & 1 \\ a_{21_2} & a_{22_2} \end{bmatrix},\\
B_1 &= \begin{bmatrix} 0 \\ b_{21_1} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ b_{21_2} \end{bmatrix},
\end{aligned} \tag{60}$$

where $A_1 = A_2, A_3 = A_4, A_5 = A_6, A_7 = A_8, B_1 = B_3 = B_5 = B_7$, and $B_2 = B_4 = B_6 = B_8$.

Thus, using the LMIs (7) from Theorem 1, the following controller gains and symmetric positive definite matrix were obtained:

$$\begin{aligned}
K_1 &= [-67.1269 \quad -8.6056],\\
K_2 &= [-39.6153 \quad -4.9426],\\
K_3 &= [-67.1260 \quad -8.6051],\\
K_4 &= [-39.6157 \quad -4.9423],\\
K_5 &= [-65.5197 \quad -8.7998],\\
K_6 &= [-34.0771 \quad -4.6633],\\
K_7 &= [-65.5169 \quad -8.7990],\\
K_8 &= [-34.0824 \quad -4.6634],\\
P &= \begin{bmatrix} 3.0219 & 0.4500 \\ 0.4500 & 0.0827 \end{bmatrix}.
\end{aligned} \tag{61}$$

Setting $\xi = 10^{-4}$, the control law (21) for the levitator is given by

$$\begin{aligned}
&u(t) = u_{(\sigma,\xi)}(t) = i_{(\sigma,\xi)}^2(t) - i_0^2,\\
&\text{with} \quad i_{(\sigma,\xi)}^2(t) = -K_\sigma x(t) + \gamma_\xi,
\end{aligned} \tag{62}$$

where K_i, $i \in \mathbb{K}_8$, are presented in (61).

Consider

$$K_\sigma \in \{K_1, K_2, \ldots, K_8\}, \qquad \sigma = \arg\min_{i\in\mathbb{K}_r} \{-x^T PBK_i x\},$$

$$\gamma_\xi = \begin{cases} 2.8667, & \text{if } x^T PB \le -\xi,\\ -4022.3x^T PB + 2.4645, & \text{if } |x^T PB| \le \xi,\\ 2.0623, & \text{if } x^T PB \ge \xi. \end{cases} \tag{63}$$

For the simulation illustrated in Figure 7, the initial condition was $\overline{x}(0) = [0.05 \quad 1]^T$ and, at $t = 0$ s, $y_0 = 0.08$ m and $m = 0.09$ Kg. In $t = 1$ s, from Figure 7, the system is practically at the point $\overline{x}(1) = [0.08 \quad 0]^T$. After changing y_0 from 0.08 m to 0.05 m and m from 0.09 Kg to 0.08 Kg at $t = 2$ s, one can see that the system is practically at the point $\overline{x}(2) = [0.05 \quad 0]^T$, which will be the new initial condition. Finally, the last changes occur at $t = 2$ s: y_0 from 0.05 m to 0.07 m and m from 0.08 Kg to 0.01 Kg. Thus, observe that in Figure 7, $x(\infty) = [0.07 \quad 0]^T$.

Note that in this case it is not possible to obtain the membership functions, since the mass is uncertain, but the proposed method overcomes this problem, because it does not depend on such functions. Observe also that even with uncertainty in the reference control signal (because $u = i^2 - i_0^2$ and i_0^2 given in (45) is uncertain considering that m is uncertain), the proposed methodology was efficient and provided an appropriate transient response, as shown in Figure 7.

Remark 7. In a control design it is important to assure stability and usually other indices of performance for the controlled system, such as the settling time (related to the decay rate), constraints on input control and output signals. The proposed methodology allows specifying these performance indices, without changing the LMIs given in [40] or their relaxations as presented, for instance, in [20, 23], by adding a new set of LMIs.

5. Conclusions

This paper proposed a new switched control design method for some classes of uncertain nonlinear plants described by Takagi-Sugeno fuzzy models. The proposed controller is based on LMIs and the gain is chosen by a switching law that returns the smallest time derivative value of the Lyapunov function. An advantage of the proposed methodology is that it does not change the LMIs given in the control design methods commonly used for plants described by Takagi-Sugeno fuzzy models as proposed, for instance, in [20, 22, 23, 34]. Furthermore, it eliminates the need to obtain

the explicit expressions of the membership functions, to implement the control law. This fact is relevant in cases where the membership functions depend on uncertain parameters or are difficult to implement. Simulating the implementation of this new procedure in the control design of a ball-and-beam system and of a magnetic levitator, the controlled system presented an appropriate transient response, as seen in Figures 2, 3, 5, 6, and 7. Thus, the authors consider that the proposed method can be useful in practical applications for the control design of uncertain nonlinear systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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