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Research Article

The Number of Spanning Trees of the Cartesian Product of Regular Graphs

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The number of spanning trees in graphs or in networks is an important issue. The evaluation of this number not only is interesting from a mathematical (computational) perspective but also is an important measure of reliability of a network or designing electrical circuits. In this paper, a simple formula for the number of spanning trees of the Cartesian product of two regular graphs is investigated. Using this formula, the number of spanning trees of the four well-known regular networks can be simply taken into evaluation.

1. Introduction

In this paper, we deal with simple undirected graphs having no self-loop or multiple edges and consider the Cartesian product of two regular graphs only. It is well known that, for designing large-scale interconnection networks, the Cartesian product is an important method to obtain large networks from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. The Cartesian product preserves many nice properties such as regularity, transitivity, super edge-connectivity, and super point-connectivity of the initial regular graphs [1–6]. In fact, many well-known networks can be constructed by the Cartesian products of simple regular graphs, for example, Boolean n-cube networks, hypercube networks, and lattice networks.

Alternatively, the study of the number of spanning trees in a graph has a long history and has been very active because the problem has different practical applications in different fields. For example, the number characterizes the reliability of a network and, in physics, designing electrical circuits, analyzing energy of masers, and investigating the possible particle transitions [7–10]. The larger degree of points a network has, the more I/O ports and edges are needed and the more cost is required.

The number of spanning trees of some special network has been taken into evaluation [11–20]. Recently, some authors derived results about the counting where the number of spanning trees can be found from [21–29]. However, the study for spanning trees of the Cartesian product of regular graphs remains an open and important invariant.

The number of spanning trees of Boolean n-cube networks, lattice networks, and generalized Boolean n-cube networks has been taken into account [13, 17, 18]; these networks belong to the class of networks Q_n with two regular graphs Q_1 and G which is defined recursively by $Q_n = Q_{n-1} \times G$ for $n \geq 2$. In this paper, we will present the formula of the number of spanning trees of the Cartesian product of regular graphs. Using this present formula, the main results in [13, 17, 18] can be obtained much more simply and will be extended.

2. The Number of Spanning Trees

Definition 1. Let G be a graph with n points labeled $1, 2, \ldots, n$. The adjacency matrix of G, $A(G)$, is an $n \times n$ matrix with the ith row and jth column entry given by

$$[A(G)]_{ij} = \begin{cases} 1 & \text{if points } i \text{ and } j \text{ are adjacent} \\ 0 & \text{ortherwise.} \end{cases} \tag{1}$$

The Kirchhoff matrix of G, $H(G)$, is equal to $D(G) - A(G)$, where $D(G)$ is an $n \times n$ diagonal matrix whose diagonal entries are the degree of point n and $A(G)$ is the adjacency matrix. Thus the ith row and jth column entry is given by

$$[H(G)]_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and points } i \text{ and } j \text{ are adjacent} \\ 0 & \text{ortherwise.} \end{cases} \tag{2}$$

Lemma 2 (see [30]). *If G is a graph on n points with Kirchhoff matrix $H(G)$ and $H_{ij}(G)$ is the submatrix of $H(G)$ obtained by removing the ith row and jth column then the number of spanning trees of G, $t(G)$, is any cofactor of $H(G)$. That is, $t(G) = (-1)^{i+j} \det(H_{ij}(G))$.*

Lemma 3. *If A is an $n \times n$ triangulable matrix, which has n eigenvalues, then the sum of product of any $n-1$ eigenvalues of A is the sum of all principal minors of A.*

Proof. Let $p(\lambda)$ be the character polynomial of A and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigenvalues of A. Then

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n). \tag{3}$$

From (3), we obtain the following.

(a) The coefficient of $\lambda = (-1)^{n-1} \cdot$ (the sum of product of any $n-1$ eigenvaules of A).

On the other hand,

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda^{(1)} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{(2)} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{21} & -a_{21} & \cdots & \lambda^{(n)} - a_{nn} \end{vmatrix}, \tag{4}$$

where $A = [a_{ij}]$ and $\lambda^{(i)} = \lambda$ for $i = 1, 2, \ldots, n$.

So we only need to prove that the coefficient of λ in $\det(\lambda I - A)$ is the sum of all principal minors of A. Let A_{ii} denote the principal minor of A obtained by removing the ith row and ith column from A.

By (4), we obtain the following:

$$\begin{gathered} \text{the coefficient of } \lambda^{(1)} = (-1)^{n-1} A_{11} \left(\text{taking } \lambda^{(2)} = \lambda^{(3)} = \cdots = \lambda^{(n)} = 0\right) \\ \text{the coefficient of } \lambda^{(2)} = (-1)^{n-1} A_{22} \left(\text{taking } \lambda^{(1)} = \lambda^{(3)} = \lambda^{(4)} \cdots = \lambda^{(n)} = 0\right) \\ \vdots \\ \text{the coefficient of } \lambda^{(n)} = (-1)^{n-1} A_{nn} \left(\text{taking } \lambda^{(1)} = \lambda^{(2)} = \cdots = \lambda^{(n-1)} = 0\right). \end{gathered} \tag{5}$$

(b) So the coefficient of $\lambda = \sum_{i=1}^{n}$ the coefficient of $\lambda^{(i)} = (-1)^{n-1}(A_{11} + A_{22} + \cdots + A_{nn})$.

Hence the theorem is proved due to (a) and (b). □

Since a real symmetric matrix is with the property that the sum of its rows (and its columns) is zero, the rank of $H(G) \leq n-1$. So 0 is the smallest eigenvalue. We write the eigenvalues of $H(G)$ as an ordered list:

$$0 = \lambda_0(G) \leq \lambda_1(G) \leq \lambda_2(G) \cdots \leq \lambda_{n-1}(G). \tag{6}$$

The main result in Kelmans and Chelnokov [31] can also be obtained by the following method.

Lemma 4 (see [31]). *If the eigenvalues of the Kirchhoff matrix $H(G)$ of the n points graph G are $0 = \lambda_0(G) \leq \lambda_1(G) \leq \lambda_2(G) \cdots \leq \lambda_{n-1}(G)$ then $t(G)$, the number of spanning trees of G, is given by*

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i(G). \tag{7}$$

Proof. By Lemmas 2 and 3,

$$\begin{aligned} &\prod_{i=1}^{n-1} \lambda_i(G) \\ &\quad = \text{the sum of product of any } n-1 \text{ eigenvaules of } H(G) \\ &\quad = \text{the sum of all principal minors of } H(G) = n \cdot t(G). \end{aligned} \tag{8}$$

Hence $t(G) = (1/n)\prod_{i=1}^{n-1} \lambda_i(G)$. □

Lemma 5. *Let the eigenvalues of the adjacency matrix $A(G)$ of the regular graph G be written by $u_1 \leq u_2 \cdots \leq u_{n-1} \leq u_n = n$, where r is the degree of the regular graph G; then, the number of spanning trees of G is given by*

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} (r - u_i). \tag{9}$$

Proof. We know $H(G) = rI_n - A(G)$, where I_n is the identity $n \times n$ matrix. Since u_i is the eigenvalue of $A(G)$ for $i = 1, 2, \ldots, n-1$, there exists eigenvector x_i for $i = 1, 2, \ldots, n-1$, such that $A(G)x_i = u_i x_i$. So $(rI_n - H(G))x_i = u_i x_i$, $rI_n x_i - H(G)x_i = u_i x_i$, $rx_i - H(G)x_i = u_i x_i$, and $H(G)x_i = rx_i - u_i x_i$.

We obtain $H(G)x_i = (r-u_i)x_i$. Thus $r-u_i$ is the eigenvalue of $H(G)$ for $i = 1, 2, \ldots, n-1$.

Hence the lemma is proved by Lemma 4. □

3. Cartesian Product and Kronecker Product

Definition 6. Let $G = (N, E)$ denote a connected graph with N set of all points and E set of all edges in G and let $\{u, v\}$ denote edge joining points u and v. Let $G_i = (N_i, E_i)$ for $i = 1, 2$; the Cartesian product of G_1 and G_2 is defined by $G_1 \times G_2 = (N, E)$, where $N = N_1 \times N_2$, $E = E_1 \times E_2$, and $\{(u_1, v_1), (u_2, v_2)\} \in E$ if and only if $u_1 = u_2$ and $\{v_1, v_2\} \in E_2$ or $v_1 = v_2$ and $\{u_1, u_2\} \in E_1$.

Definition 7 (see [32]). Let $B = [b_{ij}]$ be an $n \times n$ matrix and C an $m \times m$ matrix; then, the Kronecker product $B \times_K C$ is defined as the $mn \times mn$ matrix with block description

$$\begin{bmatrix} b_{11}C & b_{12}C & \cdots & b_{1n}C \\ b_{21}C & b_{22}C & \cdots & b_{2n}C \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}C & b_{n2}C & \cdots & b_{nn}C \end{bmatrix}. \tag{10}$$

The Kronecker sum is defined by $B +_K C = B \times_K I_m + I_n \times_K C$, where I_k is the $k \times k$ identity matrix for $k = m, n$. Let M be an $mn \times mn$ matrix. M can be partitioned into n^2 blocks which are denoted by $B_{\alpha\beta}$ for $\alpha = 1, 2, \ldots, n$ and $\beta = 1, 2, \ldots, n$. That is,

$$M = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix}, \tag{11}$$

where $B_{\alpha\beta}$ is the $m \times m$ matrix for $\alpha = 1, 2, \ldots, n$ and $\beta = 1, 2, \ldots, n$. M is called an $n \times n$ $(m \times m)$ block matrix.

Lemma 8. *If A is an $n \times n$ matrix and B, C are $m \times m$ matrices then*

(1) $A \times_K (B + C) = A \times_K B + A \times_K C$,

(2) $(B + C) \times_K A = B \times_K A + C \times_K A$.

Proof. The lemma is easily obtained. □

Lemma 9. *If the products AB and CD are defined then $(AB) \times_K (CD) = (A \times_K C)(B \times_K D)$.*

Proof. Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times p$ matrix

$$\begin{aligned} (A \times_K C)(B \times_K D) &= \begin{bmatrix} a_{11}C & \cdots & a_{1n}C \\ \vdots & \ddots & \vdots \\ a_{m1}C & \cdots & a_{mn}C \end{bmatrix} \begin{bmatrix} b_{11}D & \cdots & b_{1p}D \\ \vdots & \ddots & \vdots \\ b_{n1}D & \cdots & b_{np}D \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}b_{k1}CD & \cdots & \sum_{k=1}^{n} a_{1k}b_{kp}CD \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1}CD & \cdots & \sum_{k=1}^{n} a_{mk}b_{kp}CD \end{bmatrix} \\ &= (AB) \times_K (CD). \end{aligned} \tag{12}$$

□

Lemma 10. *If P and Q are invertible then $(P \times_K Q)^{-1} = P^{-1} \times_K Q^{-1}$.*

Proof. Consider

$$\begin{aligned} (P \times_K Q)(P^{-1} \times_K Q^{-1}) &= (PP^{-1}) \times_K (QQ^{-1}) \\ &= I_m \times_K I_n = I_{mn}, \end{aligned} \tag{13}$$

where P is $m \times m$ and Q is $n \times n$. □

4. The Number of Spanning Trees of the Cartesian Product of Regular Graphs

Lemma 11. *If the points of G_1 and G_2 are labeled by $u_1, u_2, \ldots u_n$ and $v_1, v_2, \ldots, v_m$, respectively, and points of $G_1 \times G_2$ are ordered lexicographically, that is, the label of (u_i, v_j) is smaller than that of (u_k, v_l) if and only if $i < k$ and $j < l$, then $A(G_1 \times G_2) = A(G_1) +_K A(G_2)$.*

Proof. Since $A(G_1 \times G_2)$ is an $mn \times mn$ matrix, $A(G_1 \times G_2)$ is an $n \times n$ $(m \times m)$ block matrix. By the definition of $G_1 \times G_2$, we describe

$$A(G_1 \times G_2) = \begin{bmatrix} A(G_2) & [A(G_1)]_{12}I_m & [A(G_1)]_{13}I_m & \cdots & [A(G_1)]_{1,n-1}I_m & [A(G_1)]_{1,n}I_m \\ [A(G_1)]_{21}I_m & A(G_2) & [A(G_1)]_{23}I_m & \cdots & [A(G_1)]_{2,n-1}I_m & [A(G_1)]_{2,n}I_m \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ [A(G_1)]_{n-1,1}I_m & [A(G_1)]_{n-1,2}I_m & [A(G_1)]_{n-1,3}I_m & \cdots & A(G_2) & [A(G_1)]_{n-1,n}I_m \\ [A(G_1)]_{n,1}I_m & [A(G_1)]_{n,2}I_m & [A(G_1)]_{n,3}I_m & \cdots & [A(G_1)]_{n,n-1}I_m & A(G_2) \end{bmatrix}, \tag{14}$$

where $[A(G_1)]_{ij}$ is the (i, j) entry of the adjacent matrix $A(G_1)$ of G_1 and $A(G_2)$ is the $m \times m$ adjacent matrix of G_2. We know $I_n \times_K A(G_2)$ is an $mn \times mn$ matrix; it can be described as an $n \times n$ $(m \times m)$ block matrix

$$I_n \times_K A(G_2) = \begin{bmatrix} A(G_2) & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times m} & A(G_2) & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & 0_{m\times m} & \cdots & A(G_2) & 0_{m\times m} \\ 0_{m\times m} & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & A(G_2) \end{bmatrix}, \tag{15}$$

where $0_{m\times m}$ is the $m \times m$ zero matrix. Since $A(G_1) \times_K I_m$ is an $mn \times mn$ matrix, it can be described as an $n \times n$ $(m \times m)$ block matrix

$$A(G_1) \times_K I_m = \begin{bmatrix} 0_{m\times m} & [A(G_1)]_{12} I_m & [A(G_1)]_{13} I_m & \cdots & [A(G_1)]_{1,n-1} I_m & [A(G_1)]_{1,n} I_m \\ [A(G_1)]_{21} I_m & 0_{m\times m} & [A(G_1)]_{23} I_m & \cdots & [A(G_1)]_{2,n-1} I_m & [A(G_1)]_{2,n} I_m \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ [A(G_1)]_{n-1,1} I_m & [A(G_1)]_{n-1,2} I_m & [A(G_1)]_{n-1,3} I_m & \cdots & 0_{m\times m} & [A(G_1)]_{n-1,n} I_m \\ [A(G_1)]_{n,1} I_m & [A(G_1)]_{n,2} I_m & [A(G_1)]_{n,3} I_m & \cdots & [A(G_1)]_{n,n-1} I_m & 0_{m\times m} \end{bmatrix}. \tag{16}$$

Clearly $A(G_1 \times G_2) = A(G_1) \times_K I_m + I_n \times_K A(G_2) = A(G_1) +_K A(G_2)$. □

Lemma 12. *Let G_i be the regular graph of degree r_i for $i = 1, 2$; then, the degree of $G_1 \times G_2$ is $r_1 + r_2$. If the number of the points of G_1 (resp., G_2) is n (resp., m) and the points of $G_1 \times G_2$ are ordered lexicographically then $H(G_1 \times G_2) = H(G_1) +_K H(G_2)$.*

Proof. By Lemmas 8 and 11,

$$\begin{aligned} H(G_1 \times G_2) &= (r_1 + r_2) I_{mn} - A(G_1 \times G_2) \\ &= (r_1 + r_2) I_{mn} - (A(G_1) +_K A(G_2)) \\ &= (r_1 + r_2) I_{mn} - (A(G_1) \times_K I_m + I_n \times_K A(G_2)) \\ &= (r_1 I_{mn} - I_n \times_K A(G_2)) \\ &\quad + (r_2 I_{mn} - A(G_1) \times_K I_m) \\ &= (I_n \times_K r_1 I_m - I_n \times_K A(G_2)) \\ &\quad + (r_2 I_n \times_K I_m - A(G_1) \times_K I_m) \\ &= I_n \times_K (r_1 I_m - A(G_2)) + (r_2 I_n - A(G_1)) \times_K I_m \\ &= I_n \times_K H(G_2) + H(G_1) \times_K I_m \\ &= H(G_1) +_K H(G_2), \end{aligned} \tag{17}$$

where I_{mm} is the $mn \times mn$ identity matrix. □

Lemma 13. *If A and B are triangulable matrices then the eigenvalues of $A +_k B$ are given by $\alpha + \beta$, respectively, as α and β vary through the eigenvalues of A and B.*

Proof. Since A and B are triangulable, there exist invertible matrices Q and P such that $A_1 = QAQ^{-1}$ and $B_1 = PBP^{-1}$ are upper triangular. If A and B are $n \times n$ and $m \times m$ matrices, respectively, by Lemmas 9 and 10,

$$\begin{aligned} A_1 +_k B_1 &= A_1 \times_K I_m + I_n \times_K B_1 \\ &= \left(QAQ^{-1}\right) \times_K \left(PI_mP^{-1}\right) \\ &\quad + \left(QI_nQ^{-1}\right) \times_K \left(PBP^{-1}\right) \\ &= (Q \times_K P)\left(AQ^{-1} \times_K I_m P^{-1}\right) \\ &\quad + (Q \times_K P)\left(I_n Q^{-1} \times_K BP^{-1}\right) \\ &= (Q \times_K P)(A \times_K I_m)\left(Q^{-1} \times_K P^{-1}\right) \\ &\quad + (Q \times_K P)(I_n \times_K B)\left(Q^{-1} \times_K P^{-1}\right) \\ &= (Q \times_K P)\left[(A \times_K I_m) + (I_n \times_K B)\right]\left(Q^{-1} \times_K P^{-1}\right) \\ &= (Q \times_K P)\left[(A \times_K I_m) + (I_n \times_K B)\right](Q \times_K P)^{-1} \\ &= (Q \times_K P)(A +_k B)(Q \times_K P)^{-1}. \end{aligned} \tag{18}$$

So $A +_k B$ is similar to $A_1 +_k B_1$ and they have the same eigenvalues. Obviously $A_1 +_k B_1 = A_1 \times_K I_m + I_n \times_K B_1$ is upper triangular with diagonal entries given by $\alpha + \beta$, respectively, as α and β vary through the eigenvalues of A_1 and B_1. Hence the eigenvalues of $A +_k B$ are $\alpha + \beta$, respectively, as α and β vary through the eigenvalues of A and B. □

Theorem 14. *Let G_1 and G_2 be the regular graphs with degrees m and n, respectively. If the eigenvalues of the adjacency matrix $A(G_1)$ are written as $\alpha_1 \le \alpha_2 \cdots \le \alpha_{n-1} \le \alpha_m = m$ and the eigenvalues of the adjacency matrix $A(G_2)$ are written as $\beta_1 \le \beta_2 \cdots \le \beta_{n-1} \le \beta_n = n$, then the number of spanning trees of the Cartesian product $G_1 \times G_2$ is*

$$t(G_1 \times G_2) = \frac{1}{mn}\prod_{i,j}\left[(m+n) - \left(\alpha_i + \beta_j\right)\right], \tag{19}$$

where i and j satisfy $(i,j) \in \{1,2,\ldots,m\} \times \{1,2,\ldots,n\} - \{(m,n)\}$.

Proof. We know $G_1 \times G_2$ has mn points and the degree of $G_1 \times G_2$ is $m+n$. By Lemma 11, $A(G_1 \times G_2) = A(G_1) +_K A(G_2)$. By Lemma 13, the eigenvalues of $A(G_1 \times G_2)$ are $\alpha_i + \beta_j$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. The result follows by Lemma 5. □

Theorem 15. *If G_1 and G_2 are the regular graph of degrees m and n, respectively, the eigenvalues of the Kirchhoff matrix $H(G_1)$ are written as $0 = \lambda_0 \le \lambda_1 \cdots \le \lambda_{m-1}$, and the eigenvalues of the Kirchhoff matrix $H(G_2)$ are written as $0 = \gamma_0 \le \gamma_1 \cdots \le \gamma_{n-1}$, then the number of spanning trees of the Cartesian product of G_1 and G_2 is*

$$t(G_1 \times G_2) = \frac{1}{mn}\prod_{i,j}\left(\lambda_i + \gamma_j\right), \tag{20}$$

where i and j satisfy $(i,j) \in \{0,1,\ldots,m-1\} \times \{0,1,\ldots,n-1\} - \{(0,0)\}$.

Proof. By Lemma 12, $H(G_1 \times G_2) = H(G_1) +_K H(G_2)$. By Lemma 13, the eigenvalues of $H(G_1 \times G_2)$ are $\lambda_i + \gamma_j$ for $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n-1$. Hence by Lemma 4, the result follows. □

5. The Number of Spanning Trees of the r^n-Lattice Network

Definition 16 (see [17, 18]). The r^n-lattice networks $R(r,n)$ are defined as

$$R(r,n) = \begin{cases} K_r & \text{for } n = 1 \\ R(r,n-1) \times K_r & \text{for } n \ge 2, \end{cases} \tag{21}$$

where K_r is a complete graph of r points.

When $r = 2$, $R(2,n)$ is well known, the Boolean n-cube network.

We denote $R(r,n) = K_r^{(1)} \times K_r^{(2)} \times \cdots \times K_r^{(n)}$.

Lemma 17. *The eigenvalues of $H(K_r)$ are r with multiplicity $r-1$ and 0 with multiplicity 1.*

Proof. Since $H(K_r) = rI_r - J_r$, where J_r is the matrix of all ones, letting $P_{K_r}(\lambda)$ be the character polynomial of $H(K_r)$, we obtain by Gaussian elimination

$$\begin{aligned} P_{K_r}(\lambda) &= \lambda \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \lambda - r & -1 & \cdots & \cdots \\ 0 & 0 & \lambda - r & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \lambda - r \end{vmatrix} \\ &= \lambda(\lambda - r)^{r-1}. \end{aligned} \tag{22}$$

Hence the result follows. □

Lemma 18. *If the distinct eigenvalues of the Kirchhoff matrix $H(R(r,n))$ are $0 = \lambda_0 \le \lambda_1 \cdots \le \lambda_{n-1} \le \lambda_n$ then*

$$\begin{aligned} \lambda_0 &= \text{ with multiplicity } C(n,n) = 1 \\ \lambda_1 &= r \text{ with multiplicity } (r-1)C(n,n-1) \\ \lambda_2 &= 2r \text{ with multiplicity } (r-1)^2 C(n,n) \\ &\vdots \\ \lambda_{n-1} &= (n-1)r \text{ with multiplicity } (r-1)^{n-1} C(n,1) \\ \lambda_n &= nr \text{ with multiplicity } (r-1)^n C(n,0), \end{aligned} \tag{23}$$

where $C(n,r) = n!/r!(n-r)!$.

Proof. Since $R(r,n) = K_r^{(1)} \times K_r^{(2)} \times \cdots \times K_r^{(n)}$ and by Lemma 13, we obtain each of eigenvalues of $H(R(r,n)) = \sum_{i=1}^{n}$ one of eigenvalues of $H(K_r^{(i)})$.

Hence if we take the eigenvalue 0 of $H(K_r^{(i)})$ for $i = 1,2,\ldots,n$ then $\lambda_0 = 0$ with multiplicity $C(n,n) = 1$. If we take the eigenvalue r of $H(K_r^{(l)})$ for some one $l \in \{1,2,\ldots,n\}$ and the eigenvalue 0 of $H(K_r^{(i)})$ for each $i \in \{1,2,\ldots,n\} - \{l\}$ then $\lambda_1 = r$ with multiplicity $(r-1)C(n,n-1)$. If we take the eigenvalue r of $H(K_r^{(l)})$ and $H(K_r^{(m)})$, respectively, for $l,m \in \{1,2,\ldots,n\}$ and the eigenvalue 0 of $H(K_r^{(i)})$ for each $i \in \{1,2,\ldots,n\} - \{l,m\}$, then $\lambda_2 = 2r$ with multiplicity $(r-1)^2 C(n,n)$. We keep performing the same process. Hence the result follows. □

The main theorem in [17, 18] can be obtained much more simply by Theorem 19 as follows.

Theorem 19 (see [17]). *The number of spanning trees of $R(r,n)$ is*

$$t(R(r,n)) = r^{r^n-n-1}\prod_{i=2}^{n} i^{C(n,i)(r-1)^i}. \tag{24}$$

Proof. Since the degree of $R(r,n)$ is $n(r-1)$, the number of points of $R(r,n)$ is r^n. By Lemma 18 and Theorem 15, we obtain

$$\begin{aligned} t(R(r,n)) &= \frac{1}{r^n}\cdot r^{C(n,n-1)(r-1)}\cdot (2r)^{C(n,n-2)(r-1)^2} \\ &\quad \cdot (3r)^{C(n,n-3)(r-1)^3}\cdots (nr)^{C(n,0)(r-1)^n} = \frac{1}{r^n} \\ &\quad \cdot r^{C(n,n-1)(r-1)+C(n,n-2)(r-1)^2+C(n,n-3)(r-1)^3+\cdots+C(n,0)(r-1)^n} \\ &\quad \cdot \prod_{i=1}^{n} i^{C(n,n-i)(r-1)^i} \\ &= \frac{1}{r^n}\cdot r^{r^n-1}\cdot \prod_{i=1}^{n} i^{C(n,n-i)(r-1)^i} \\ &= r^{r^n-n-1}\prod_{i=2}^{n} i^{C(n,i)(r-1)^i}. \end{aligned} \tag{25}$$

□

Corollary 20 (see [18]). *The number of spanning trees of the Boolean n-cube network B_n is*

$$t(B_n) = 2^{2^n-n-1}\prod_{i=2}^{n} i^{C(n,i)}. \tag{26}$$

Proof. Since $B_n = R(2,n)$, by Theorem 15, the result follows. □

6. The Number of Spanning Trees of the $2\times 3\cdots\times n$-Lattice Network

Definition 21. The $2\times 3\cdots\times n$ lattice network Q_n can be defined recursively by $Q_2 = K_2$ and $Q_n = K_n\times Q_{n-1}$.

Thus Q_n has $n!$ points. We denote $Q_n = K_2\times K_3\times\cdots\times K_n$.

Theorem 22. *The number of spanning trees of Q_n is*

$$t(Q_n) = \frac{1}{n!}\prod_{i=1}^{n-1}\prod_{2\le r_1<r_2<\cdots<r_i\le n}\left(\sum_{j=1}^{i} r_j\right)^{\prod_{j=1}^{i}(r_j-1)}. \tag{27}$$

Proof. Since the eigenvalues of $H(K_r)$ are r with multiplicity $r-1$ and 0 with multiplicity 1, the distinct $\sum_{i=0}^{n-1} C(n-1,i) = 2^{n-1}$ eigenvalues of Q_n are 0 and $\lambda_{i,r_1,r_2,\ldots,r_i} = \sum_{j=1}^{i} r_j$ with multiplicity $(r_1-1)(r_2-1)\cdots(r_i-1)$, where $r_1,r_2,\ldots,r_i$, satisfying $2\le r_1<r_2<\cdots<r_i\le n$, $i=1,2,\ldots,n-1$, are nonzero eigenvalues of $H(K_{r_1})$, $H(K_{r_2}),\ldots,H(K_{r_i})$, respectively, and take zero eigenvalues for the remaining $H(K_r)$, where $r\ne r_1,r_2,\ldots,r_i$, $r = 2,3,\ldots,n-1$. By Lemma 13 and Theorem 15, the result follows. □

Example 23. The number of spanning trees of Q_3 and Q_4 is as shown in Figure 1, where $t(Q_3) = (1/3!)2^{(2-1)}3^{(3-1)}(2+3)^{(2-1)(3-1)} = 75$ and $t(Q_4) = (1/4!)2^{(2-1)}3^{(3-1)}4^{(4-1)}(2+3)^{(2-1)(3-1)}(2+4)^{(2-1)(4-1)}(3+4)^{(3-1)(4-1)}(2+3+4)^{(2-1)(3-1)(4-1)} = 1620609272381440$.

7. The Number of Spanning Trees of the Generalized Boolean n-Cube Network

Definition 24. The generalized Boolean n-cube network $BR(r,n)$ can be defined by

$$BR(r,n) = \begin{cases} C_r & \text{for } n=1 \\ BR(r,n-1)\times K_2 & \text{for } n\ge 2, \end{cases} \tag{28}$$

where C_r is a cycle with r points. One denotes $BR(r,n) = C_r\times K_2\times\cdots\times K_2$.

Setting E_n is the $n\times n$ matrix by

$$E_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$E_n^i = \begin{array}{c} \begin{array}{ccccccccc} 1 & & & & i+1 & & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ \begin{array}{ccccccccc} & & & i & & & & & \end{array} \end{array}. \tag{29}$$

Lemma 25. *The eigenvalues of the adjacent matrix $A(K_r)$ are -1 with multiplicity $r-1$ and $r-1$ with multiplicity 1.*

Lemma 26 (see [33]). *If B is a sequence matrix, λ is an eigenvalue of B, and f is a polynomial then $f(\lambda)$ is the eigenvalue of $f(B)$.*

Lemma 27. *The eigenvalues of the adjacent matrix $A(C_n)$ are $2\cos(2\pi k/n)$ for $k = 0,1,2,\ldots,n-1$.*

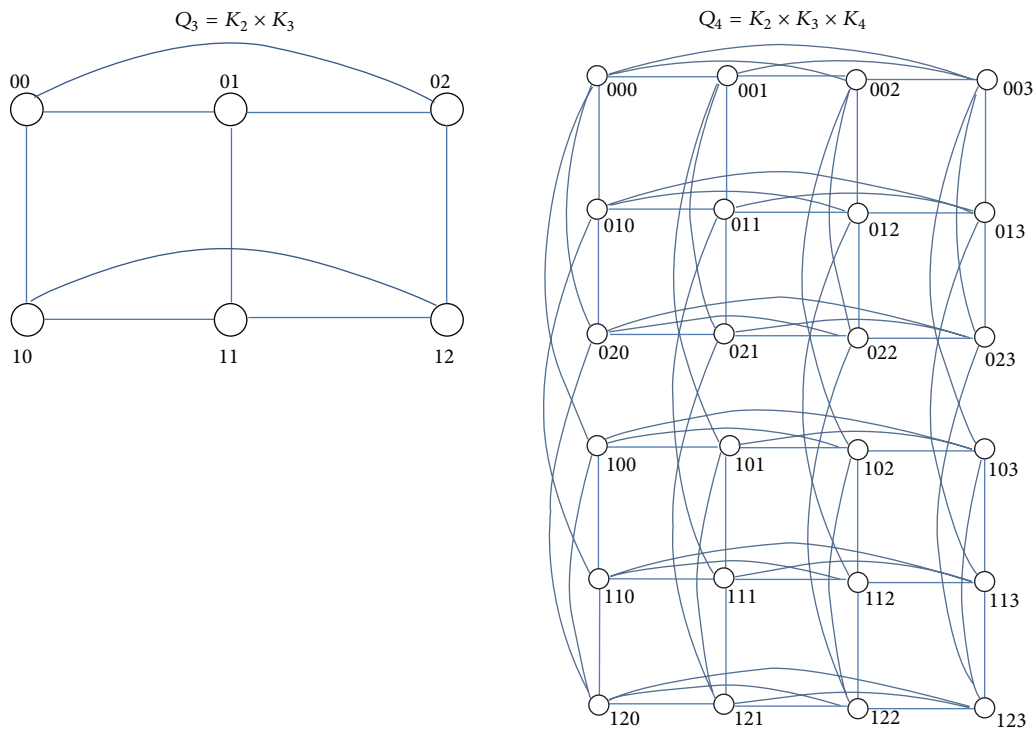

Figure 1

Proof. Since

$$\det(\lambda I_n - E_n) = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & -1 \\ -1 & 0 & \cdots & 0 & \lambda \end{vmatrix} \tag{30}$$
$$= \lambda \cdot \lambda^{n-1} + (-1)(-1)^{n+1}(-1)^{n-1}$$
$$= \lambda^n - 1,$$

the eigenvalues of E_n are $e^{(2\pi k/n)i}$ for $k = 0, 1, 2, \ldots, n-1$. It follows that $A(C_n) = E_n + E_n^{n-1}$. By Lemma 26, the eigenvalues of $A(C_n)$ are $e^{(2\pi k/n)i} + (e^{(2\pi k/n)i})^{n-1} = e^{(2\pi k/n)i} + (e^{2\pi k i-(2\pi k/n)i}) = e^{(2\pi k/n)i} + e^{-(2\pi k/n)i} = 2\cos(2\pi k/n)$ for $k = 0, 1, 2, \ldots, n-1$. □

The main theorem in [13] can be obtained much more simply as follows.

Theorem 28 (see [13]). *The number of spanning trees of $BR(r, n)$ is*

$$t(BR(r,n)) = r2^{r(2^{n-1}-1)-n+1} \times \prod_{i=1}^{n-1}\left(i\prod_{k=1}^{r-1}\left(i+1-\cos\frac{2\pi k}{r}\right)\right)^{C(n-1,i)}. \tag{31}$$

Proof. It follows that the points of $BR(r, n)$ are $r \cdot 2^{n-1}$ and the degree of any edge of $BR(r, n)$ is $n + 1$. By Lemma 25, the eigenvalues of the adjacent matrix $A(K_2)$ are -1 and 1. By Lemmas 13 and 27, the distinct eigenvalues of the adjacent matrix $A(BR(r, n))$ are

$$(n-2i-1) + 2\cos\frac{2\pi k}{r} \quad k = 0, 1, \ldots, r-1 \quad \text{with multiplicity } C(n-1, i), \tag{32}$$

where $i = 0, 1, \ldots, n-1$.

When $k = 0$ and $i = 0$, the eigenvalue is $n+1$. When $k = 0$ and $i = 0, 1, \ldots, n-1$, the eigenvalues are $(n-2i-1)+2$. By Theorem 14,

$$\begin{aligned} t(BR(r,n)) &= \frac{1}{r\cdot 2^{n-1}}\prod_{i=1}^{n-1}((n+1)-((n-2i-1)+2))^{C(n-1,i)} \\ &\quad \cdot \prod_{k=1}^{r-1}\prod_{i=0}^{n-1}\left((n+1)-\left((n-2i-1)+2\cos\frac{2\pi k}{r}\right)\right)^{C(n-1,i)} \\ &= \frac{1}{r\cdot 2^{n-1}}\prod_{i=1}^{n-1}(2i)^{C(n-1,i)} \\ &\quad \cdot \prod_{k=1}^{r-1}\prod_{i=0}^{n-1}\left(2\left(i+1-\cos\frac{2\pi k}{r}\right)\right)^{C(n-1,i)}. \end{aligned} \tag{33}$$

Since $\prod_{i=0}^{n-1} 2^{C(n-1,i)} = 2^{\sum_{i=0}^{n-1} C(n-1,i)} = 2^{2^{n-1}}$ and $\prod_{i=1}^{n-1} 2^{C(n-1,i)} = 2^{2^{n-1}-1}$, hence

$$t(BR(r,n)) = \frac{1}{r \cdot 2^{n-1}} 2^{2^{n-1}-1} 2^{(r-1)2^{n-1}} \prod_{i=1}^{n-1} i^{C(n-1,i)} \cdot \prod_{k=1}^{r-1} \prod_{i=0}^{n-1} \left(i + 1 - \cos\frac{2\pi k}{r}\right)^{C(n-1,i)}. \tag{34}$$

Since $\prod_{k=1}^{r-1} \sin^2(\pi k/r) = r^2$ and $\prod_{k=1}^{r-1}(1 - \cos(2\pi k/r)) = \prod_{k=1}^{r-1} 2\sin^2(\pi k/r) = r^2/2^{r-1}$ as $i = 1$,

$$\begin{aligned} t(BR(r,n)) &= r 2^{r(2^{n-1}-1)-n+1} \prod_{i=1}^{n-1} i^{C(n-1,i)} \cdot \prod_{k=1}^{r-1} \prod_{i=1}^{n-1} \left(i + 1 - \cos\frac{2\pi k}{r}\right)^{C(n-1,i)} \\ &= r 2^{r(2^{n-1}-1)-n+1} \times \prod_{i=1}^{n-1} \left(i \prod_{k=1}^{r-1} \left(i + 1 - \cos\frac{2\pi k}{r}\right)\right)^{C(n-1,i)}. \end{aligned} \tag{35}$$

□

8. The Number of Spanning Trees of the Hypercube Network

Definition 29. The hypercube network $H(r,m)$ can be defined by

$$H(r,m) = \begin{cases} C_r & \text{for } m = 1 \\ H(r,m-1) \times C_r & \text{for } m \geq 2, \end{cases} \tag{36}$$

where C_r is a cycle with r points.

Theorem 30. *The number of spanning trees of $H(r,m)$ is*

$$t(H(r,m)) = \frac{2^{r^m-1}}{r^m} \prod_{l_1,l_2,\ldots,l_m} \left(m - \sum_{i=1}^{m} \cos\frac{2\pi l_i}{r}\right), \tag{37}$$

where $l_1, l_2, \ldots, l_m$ satisfy $(l_1, l_2, \ldots, l_m) \in \{0, 1, \ldots, r-1\} \times \{0, 1, \ldots, r-1\} \times \cdots \times \{0, 1, \ldots, r-1\} - \{(0, 0, \ldots, 0)\}$.

Proof. It follows that the points of $H(r,m)$ are r^m and the degree of any edge of $H(r,m)$ is $2m$. By Lemma 26 and Theorem 14,

$$\begin{aligned} t(H(r,m)) &= \frac{1}{r^m} \prod_{l_1,l_2,\ldots,l_m} \left(2m - 2\sum_{i=1}^{m} \cos\frac{2\pi l_i}{r}\right) \\ &= \frac{2^{r^m-1}}{r^m} \prod_{l_1,l_2,\ldots,l_m} \left(m - \sum_{i=1}^{m} \cos\frac{2\pi l_i}{r}\right), \end{aligned} \tag{38}$$

where $l_1, l_2, \ldots, l_m$ satisfy $(l_1, l_2, \ldots, l_m) \in \{0, 1, \ldots, r-1\} \times \{0, 1, \ldots, r-1\} \times \cdots \times \{0, 1, \ldots, r-1\} - \{(0, 0, \ldots, 0)\}$. □

9. Conclusion

Due to the high dependence of the network design and reliability problem, electrical circuits designing issue are on the graph theory. For example, the larger degree of points a network has, the more I/O ports and edges are needed and the more cost is required. The evaluation of this number not only is interesting from a mathematical (computational) perspective but also is an important issue on practical applications. However, the study for spanning trees of the Cartesian product of regular graphs remains an open and important invariant. In this paper, the eigenvalues of the Kirchhoff matrix of Cartesian product of two regular graphs, G_1 and G_2, are given by $\lambda + \gamma$ as λ and γ vary through the eigenvalues of the Kirchhoff matrices $H(G_1)$ and $H(G_2)$, respectively. By this result, the formula for the number of spanning trees of the four regular networks can be simply obtained. Using this formula, the main results in [13, 17, 18] can be obtained much more simply and will be extended.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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