Research Article
On Blow-Up Structures for a Generalized Periodic Nonlinearly Dispersive Wave Equation

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The local well-posedness for a generalized periodic nonlinearly dispersive wave equation is established. Under suitable assumptions on initial value $u_0$, a precise blow-up scenario and several sufficient conditions about blow-up results to the equation are presented.

1. Introduction

Recently, Hu and Yin [1] and Yin [2] investigate the following equation:

$$u_t - u_{xxx} + 2\omega u_x + 3uu_x = \gamma (2u_xu_{xx} + uu_{xxx}),$$  \hspace{1cm} (1)

where $\omega$ is nonnegative number and $\gamma$ is arbitrary real number. It is shown from [1, 2] that (1) has solitary wave solutions and blow-up solutions for nonperiodic case and also solutions which blow up in finite time for periodic case.

If $\gamma = 0$, (1) becomes the famous BBM equation modelling the motion of internal gravity waves in shallow channel [3]. Some results related to the equation can be found in [4, 5]. It is worthwhile to mention that the equation does not have integrability and its solitary waves are not solitons [5].

If $\gamma = 1$ in (1), attention is attracted to the well-known Camassa-Holm equation, which models the unidirectional propagation of shallow water waves over a flat bottom. Here, $u(t, x)$ represents the free surface above a flat bottom and $\omega$ is a nonnegative parameter related to the critical shallow water speed [6]. As a model to describe the shallow water motion, the Camassa-Holm equation possesses a bi-Hamiltonian structure and infinite conservation laws [7–9] and is completely integrable [10]. It is regarded as a reexpression of geodesic flow on the diffeomorphism group of circle if $\omega = 0$ [11] and on the Bott-Virasoro group if $\omega > 0$ [12]. Recently, some significant results of dynamical behaviors have been obtained for the Cauchy problem of the Camassa-Holm equation. For example, the local well-posedness of corresponding solution for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$ was given by several authors (see [13–15]). Under certain assumptions on initial data $u_0$, the equation has global strong solutions and blow-up solutions for periodic and nonperiodic case (see [13, 16–22]). The existence and uniqueness of global weak solutions in $H^1(\mathbb{R})$ for the equation were proved (see [23–25]). It is shown from [6] that the solitary waves of the equation are peakon solitons and are orbitally stable.

If $\omega = 0$ and $\gamma \in \mathbb{R}$, (1) changes into the rod equation derived by Dai [26] recently, which describes finite-length and small amplitude radial deformation waves in thin cylindrical compressible hyperelastic rods (see [26]), and $u(t, x)$ represents the radial stretch relative to a prestressed state in one-dimensional variable. The first investigation of the Cauchy problem of the rod equation on the line was done by Constantin and Strauss [27], the precise blow-up scenario, some blow-up results of strong solution, and the stability of a class of solitary waves to the rod equation are presented. In [28, 29], Zhou found the sufficient conditions to guarantee the finite blow-up of corresponding solution for periodic case. Moreover, Yin [30] discusses the rod equation on the circle and gives some interesting blow-up results.
In this paper, we consider a generalized nonlinearly dispersive wave equation on the circle

\[ u_t - u_{xxx} + 2\omega u_x + auu_x + \beta (u - u_{xxx}) = \gamma^2 \left( 2u_x u_{xx} + uu_{xxx} \right), \quad t > 0, \ x \in \mathbb{R}, \]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \]

\[ u(t, x + 1) = u(t, x), \quad t > 0, \ x \in \mathbb{R}, \]

which is equivalent to

\[ y_t + yuu_x + 2yyu_x + 2\omega u_x + \beta y \]

\[ + (a - 3\gamma) uu_x = 0, \quad t > 0, \ x \in \mathbb{R}, \]

\[ y(0, x) = u_0(x) - u_{xxx}(x), \quad x \in \mathbb{R}, \]

\[ y(t, x) = y(t, x + 1), \quad t > 0, \ x \in \mathbb{R}. \]

**Theorem 1.** Given \( u_0 \in H^r(S) \) \((r > 3/2)\), there exist a maximal \( T = T(a, b, \gamma, \omega, u_0) \) and a unique solution \( u \) to problem (2), such that

\[ u = u(\cdot, u_0) \in C([0, T); H^r(S)) \cap C^1([0, T); H^{r-1}(S)). \]

**Proof.** The proof of Theorem 1 can be finished by using Kato’s semigroup theory (see [1] or [2]). Here, we omit the detailed proof. \( \square \)

### 3. Blow-Up Solutions

**Theorem 2.** Let \( u_0 \in H^r(S), \ r > 3/2; \) the solution of \( u(\cdot, u_0) \) of problem (2) is uniformly bounded. Blow-up in finite time \( T < +\infty \) occurs if and only if

\[ \liminf_{t \uparrow T} \left\{ \inf_{x \in \mathbb{R}} \left| yu_x(t, x) \right| \right\} = -\infty. \]

Before proving the theorem, we give several useful lemmas.

**Lemma 3** (Kato and Ponce [32]). If \( r > 0 \), then \( H^r \cap L^{\infty} \) is an algebra. Moreover

\[ \|uv\|_{L^2} \leq c \left( \|u\|_{L^2} \|v\|_{L^2} + \|u\|_{L^2} \|v\|_{L^\infty} \right), \]

where \( c \) is a constant depending only on \( r \).

**Lemma 4** (Kato and Ponce [32]). Let \( r > 0 \). If \( u \in H^r \cap W^{1,\infty} \) and \( v \in H^{r-1} \cap L^{\infty} \), then

\[ \left\| [A^r, u] v \right\|_{L^2} \leq c \left( \|\partial_x u\|_{L^2} \|A^{-1}v\|_{L^2} + \|A^r u\|_{L^2} \|v\|_{L^\infty} \right). \]

**Lemma 5.** Let \( s \geq 3/2 \) and \( u(t, x) \) is the corresponding solution of (4) with initial data \( u_0(x) \in H^r(S) \); it holds that if \( q \in (0, r - 1) \), there is a constant \( c \) depending only on \( q \) such that

\[ \int_{S} (A^{q+1} u_0)^2 \, dx \leq \int_{S} (A^{q+1} u_0^0)^2 \, dx \]

\[ + c \int_0^t \left( \|u_t\|_{L^{\infty}} + 1 \right) \|u_t\|_{L^{2q+1}}^2 \, ds. \]

**Proof.** We rewrite (2) in the following equivalent form:

\[ u_t - u_{xxx} = -auu_x - 2\omega u_x + \frac{\gamma^2}{2} (u_x^2)^2 - \frac{\gamma}{2} (u_x^2) - \beta u + \beta u_x u_x. \]
For \( q \in (0, s - 1] \), applying \((\Lambda^q u)\Lambda^q\) on both sides of (11) and integrating the new equation with respect to \( x \) by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_S \left( (\Lambda^q u)^2 + (\Lambda^q u_x)^2 \right) dx
\]
\[
= -a \int_S (\Lambda^q u) \Lambda^q (uu_x) - y \int_S (\Lambda^{q+1} u) \Lambda^{q+1} (uu_x) + \frac{y}{2} \int_S (\Lambda^q u_x) \Lambda^q (u_x^2) - \beta \int_S (\Lambda^{q+1} u)^2 dx,
\]
where the Parseval equality
\[
\int_S (\Lambda^q u) \Lambda^q g^2 (u^2) dx = -2 \int_S (\Lambda^{q+1} u) \Lambda^{q+1} (uu_x) dx + 2 \int_S (\Lambda^q u) \Lambda^q (uu_x) dx
\]
\[
\text{is used. We will estimate each of the terms on the right-hand side of (11). For the first and the third terms, using integration by parts, the Cauchy-Schwartz inequality, and Lemmas 3 and 4, we have}
\]
\[
\int_S (\Lambda^q u) \Lambda^q (uu_x) dx = \int_S (\Lambda^q u) \left[ \Lambda^q (uu_x) - u \Lambda^q u_x \right] dx
\]
\[
+ \int_S (\Lambda^q u) u \Lambda^q (u_x^2) dx
\]
\[
\leq c \|u\|_{H^2(S)}^2 \|u_x\|_{L^\infty} + \frac{1}{2} \|u_x\|_{L^\infty} \|\Lambda^q u\|_{L^2}^2
\]
\[
\leq c \|u\|_{H^2(S)}^2 \|u_x\|_{L^\infty},
\]
\[
\text{(14)}
\]
where \( c \) only depends on \( q \). Using the above estimate to the second term yields
\[
\int_S (\Lambda^{q+1} u) \Lambda^{q+1} (uu_x) dx \leq c \|u\|_{H^{2q+1}} \|u_x\|_{L^\infty}.
\]
\[
\text{(15)}
\]
For the fourth term, using Lemma 3 gives rise to
\[
\int_S (\Lambda^q u_x) \Lambda^q (u_x^2) dx \leq \|\Lambda^q u_x\|_{L^2} \|\Lambda^q u_x^2\|_{L^2}
\]
\[
\leq \|u\|_{H^2(S)} \|\Lambda^q u_x\|_{H^1} \|u_x\|_{L^2}
\]
\[
\leq c \|u\|_{H^2(S)} \|u_x\|_{L^\infty},
\]
\[
\text{(16)}
\]
It follows from (12)–(16) that
\[
\frac{1}{2} \frac{d}{dt} \int_S \left( (\Lambda^q u)^2 + (\Lambda^q u_x)^2 \right) dx \leq c \left( \|u_x\|_{L^\infty} + 1 \right) \|u\|_{H^2(S)}^2,
\]
\[
\text{(17)}
\]
which leads to
\[
\int_S \left( (\Lambda^q u)^2 + (\Lambda^q u_x)^2 \right) dx - \int_S \left( (\Lambda^q u_0)^2 + (\Lambda^q u_{0,x})^2 \right) dx
\]
\[
\leq c \int_0^t \left( \|u_x\|_{L^\infty} + 1 \right) \|u\|_{H^2(S)}^2 ds.
\]
\[
\text{(18)}
\]
It completes the proof of the lemma.

**Proof of Theorem 2.** Applying (10) with \( q = r - 1 \), we have
\[
\|u\|_{H^2(S)}^2 \leq \|u_0\|_{H^2(S)}^2 + c \int_0^t \left( \|u_{x}\|_{L^\infty} + 1 \right) \|u\|_{H^2(S)}^2 ds.
\]
\[
\text{(19)}
\]
It follows from (19) and Gronwall’s inequality that
\[
\|u\|_{H^2(S)}^2 \leq \|u_0\|_{H^2(S)}^2 \exp \left( c \int_0^t \left( \|u_{x}\|_{L^\infty} + 1 \right) ds \right).
\]
\[
\text{(20)}
\]
If there is a constant \( M > 0 \) such that \( \|u_{x}\|_{L^\infty} \leq M \) on \((0, T]\), then \( \|u\|_{H^2(S)}^2 \) does not blow up. It completes the proof of Theorem 2.

**Lemma 6** (see [17]). Let \( u(t) \) be the solution to (2) on \([0, T]\) with initial data \( u_0 \in H^1(S) \), \( s > 3/2 \). Then there exists at least one point \( \xi(t) \in S \) with
\[
m(t) = \inf_{x \in S} u_x(x, t) = u_x(t, \xi(t)).
\]
\[
\text{(21)}
\]
The function \( m(t) \) is almost everywhere differentiable on \((0, t)\) with
\[
\frac{d m(t)}{dt} = u_{xx}(t, (\xi(t)))
\]
\[
\text{(22)}
\]
**Lemma 7** (see [21, 22]). (i) For every \( u \in H^1(S) \), one has
\[
\max_{x \in [0, 1]} u_x^2(x) \leq \frac{e + 1}{2(e - 1)} \|u\|_{H^1(S)}^2,
\]
\[
\text{(23)}
\]
where the constant \((e + 1)/2(e - 1)\) is sharp.

(ii) For every \( u \in H^3(S) \), one has
\[
\max_{x \in [0, 1]} u_x^2(x) \leq c \|u\|_{H^3(S)}^2
\]
\[
\text{(24)}
\]
with the best possible constant \( c \) lying within the range \((1, 13/12)\). Moreover, the best constant \( c \) is \((e + 1)/2(e - 1)\).

**Lemma 8** (see [2]). If \( f \in H^3(S) \) is such that \( \int_S f(x) dx = a_k/2 \), then, for every \( \varepsilon > 0 \), one has
\[
\max_{x \in [0, 1]} f(x) \leq \frac{\varepsilon + 2}{24} \int_S f_x^2 dx + \frac{\varepsilon + 2}{4\varepsilon} a_0^2.
\]
\[
\text{(25)}
\]
Moreover,
\[
\max_{x \in [0, 1]} f^2(x) \leq \frac{\varepsilon + 2}{24} \|f\|_{H^3(S)}^2 + \frac{\varepsilon + 2}{4\varepsilon} a_0^2.
\]
\[
\text{(26)}
\]
**Lemma 9.** Let \( u \in H^3(S) \) and \( T > 0 \) be the maximal existence time of the solution \( u(t, x) \) to problem (4). Then it holds that
\[
(i) \int_S u(t, x) dx = \int_S u_0(x) dx = \int_S y(t, x) dx = \int_S y_0(x) dx,
\]
\[
\text{(27)}
\]
\[
(ii) \|u\|_{H^3} = \|u\|_{H^3} e^{-2\varphi}.
\]
\[
\text{(28)}
\]

Proof. The proof of (i) is similar to that of [2, lemma 3.6], so we omit it.

Multiplying \( u \) to both sides of (2) and integrating by parts, we get
\[
\frac{1}{2} \frac{d}{dt} \int_S (u^2 + u_x^2) \, dx = -\beta \int_S (u^2 + u_x^2) \, dx,
\]
which yields
\[
\|u_t\|^2_{H^1(S)} = \|u_0\|^2_{H^1(S)} e^{-2\beta t}.
\]
(28) It finishes the proof.

Lemma 10 (see [29]). Assume that a differential function \( y(t) \) satisfies
\[
y' \leq -Cy^2 (t) + K,
\]
(29) with constants \( C, K > 0 \). If the initial datum \( y(0) = y_0 < -\sqrt{K/C} \), then the solutions to (29) go to \(-\infty \) in finite time.

We now present the first blow-up result.

Theorem 11. Let \( \beta < 0 \) and \( u_0 \in H^1(S), s \geq 3 \).

(i) If \( 0 < \gamma < \alpha/3 \) and there is a \( x_0 \in S \) such that
\[
yu_0'(x_0) < -\beta - \sqrt{2} \left( \frac{\beta^2}{2} + \frac{\gamma (\alpha - 2\gamma)}{4 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{2\beta t} \\
+ 4 \gamma \omega \left( \frac{e + 1}{2 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{\beta t} \right)^{1/2},
\]
(30) then the corresponding solution to (2) blows up in finite time.

(ii) If \( \alpha/3 < \gamma < \alpha \) and there is a \( x_0 \in S \) such that
\[
yu_0'(x_0) < -\beta - \sqrt{2} \left( \frac{\beta^2}{2} + \frac{\gamma (\alpha - 2\gamma)}{8 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{2\beta t} \\
+ 4 \gamma \omega \left( \frac{e + 1}{2 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{\beta t} \right)^{1/2},
\]
(31) then the corresponding solution to (2) blows up in finite time.

(iii) If \( \gamma < 0 \) or \( \gamma > \alpha \) and there is a \( x_0 \in S \) such that
\[
yu_0'(x_0) < -\beta - \sqrt{2} \left( \frac{\beta^2}{2} + \frac{\gamma (\gamma - \alpha)}{4 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{2\beta t} \\
+ 4 \gamma |\omega| \left( \frac{e + 1}{2 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{\beta t} \right)^{1/2},
\]
(32) then the corresponding solution to (2) blows up in finite time.

Proof. Let \( T > 0 \) be the maximal time of existence of the solution \( u \) to (2) with the initial data \( u_0 \). Note that \( \partial_t^2 G \ast f = G \ast f \ast f \). Differentiating (4) with respect to \( x \) and multiplying the obtained equation by \( y \), we get
\[
yu_{xx} = -\frac{2}{2} u_x^2 - \gamma^2 uu_{xx} + \frac{\gamma (a - \gamma)}{2} u_x^2 + 2 \gamma uu - y \beta u_x \\
- G \ast \left[ \frac{(a - \gamma)}{2} u_x^2 + \frac{\gamma (a - \gamma)}{2} u_x^2 + 2 \gamma uu \right].
\]
(33) Noting that \( uu_{xx}(t, \xi(t)) = 0 \), for all \( t \in [0, T) \), and defining \( m(t) = \gamma u_{xx}(t, \xi(t)) + \beta = \inf_{x \in S} (\gamma u_{xx}(t, x) + \beta) \), we obtain
\[
m'(t) = -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{2} u^2 + 2 \gamma uu \\
- G \ast \left[ \frac{(a - \gamma)}{2} u_x^2 + \frac{\gamma (a - \gamma)}{2} u_x^2 + 2 \gamma uu \right].
\]
(34) Next, we divide (34) into three cases to prove the theorem.

(i) The first is \( 0 < \gamma < \alpha/3 \); note that \( (a - 3\gamma)/2 \geq 0 \) and \( G \ast (u^2 + (1/2)u_0^2) \geq (1/2)u_0^2 \). Then, we have
\[
G \ast \left[ \frac{(a - \gamma)}{2} u_x^2 + \frac{\gamma (a - 2\gamma)}{2} u_x^2 + 2 \gamma uu \right] \\
= G \ast \left[ \gamma u_x^2 + \gamma^2 u_x^2 + \left( \frac{(a - \gamma)}{2} u_x^2 + \gamma^2 \right) u^2 \right] \\
\geq \gamma u_x^2 + G \ast \left( \frac{(a - 3\gamma)}{2} u_x^2 \right).
\]
Note that \( G \ast ((a - 3\gamma) u_x^2) \geq 0 \). Thus, we obtain
\[
m'(t) \leq -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - 2\gamma)}{2} u_x^2 + 2 \gamma uu - G \ast (2 \gamma uu).
\]
(36) From Lemmas 8 and 9, we have
\[
\max_{x \in S} \{u^2\} \leq \frac{e + 1}{2 (e - 1)} \|u_0\|^2_{H^1(S)} \leq \frac{e + 1}{2 (e - 1)} \|u_0\|^2_{H^1(S)} e^{-2\beta T}. 
\]
(37) Thus,
\[
\|u\|^2_{L^\infty} \leq \sqrt{\frac{e + 1}{2 (e - 1)} \|u_0\|^2_{H^1(S)} e^{-\beta T}}.
\]
(38) Note that \( G \|u_\|_{L^1(S)} = 1 \). Using Young’s inequality, we get
\[
|G \ast u| \leq \|u\|_{L^\infty} \|G\|_{L^1(S)} \leq \left( \sqrt{\frac{e + 1}{2 (e - 1)}} \right) \|u_0\|^2_{H^1(S)} e^{-\beta T}. 
\]
(39) Therefore, we deduce
\[
m'(t) \leq -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - 2\gamma)}{2} u_x^2 + \frac{\gamma}{2 (e - 1)} e + \frac{1}{2} \|u_0\|^2_{H^1(S)} e^{-2\beta T} \\
\times \|u_0\|^2_{H^1(S)} e^{-\beta T} + 4 \gamma \omega \left( \frac{e + 1}{2 (e - 1)} \right) \|u_0\|^2_{H^1(S)} e^{-\beta T},
\]
(40)
which results in
\[ m'(t) \leq -\frac{1}{2} m^2(t) + K, \]  
(41)
with
\[ K = \frac{\beta^2}{2} + \frac{\gamma (a - 2\gamma)}{2} \frac{e + 1}{(e - 1)} \| u_0 \|_{H^1}^2 e^{-2\beta T} + 4 \gamma \omega \sqrt{\frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-\beta T}}. \]  
(42)

Note that, from Lemma 10, if \( m(0) < -\sqrt{2K} \), then there exists \( T \), such that \( \lim_{t \to T} m(t) = -\infty \). Applying Theorem 2, the solution \( u \) to (2) does not exist globally in time.

(ii) The second is \( a/3 \leq \gamma \leq a \). Similar to case (i), we have \( \gamma(3\gamma - a)/4 \geq 0 \) and
\[ G \left[ \frac{(a - \gamma)}{2} u_t^2 + \frac{\gamma^2}{2} u_x^2 \right] \geq \frac{(a - \gamma)}{4} u_t^2 + G * \left( \frac{(3\gamma - a)}{4} u_x^2 \right). \]  
(43)
Therefore, we obtain
\[ m'(t) \leq -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{4} u_x^2 \]  
(44)
\[ + 2\gamma \omega u - G * (2\gamma \omega u). \]

Using (37)–(39), we get
\[ m'(t) \leq -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{4} \frac{e + 1}{2(e - 1)} \]  
(45)
\[ \times \| u_0 \|_{H^1}^2 e^{-2\beta T} + 4 \gamma \omega \sqrt{\frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-\beta T}}, \]
which results in
\[ m'(t) \leq -\frac{1}{2} m^2(t) + K, \]  
(46)
with
\[ K = \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{4} \frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-2\beta T} \]  
(47)
\[ + 4 \gamma \omega \sqrt{\frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-\beta T}}. \]

Note that, from Lemma 10, if \( m(0) < -\sqrt{2K} \), then there exists \( T \), such that \( \lim_{t \to T} m(t) = -\infty \). Repeating the proof of (i), we conclude that the solution \( u \) blows up in finite time.

(iii) The third is \( \gamma < 0 \) or \( \gamma > a \); note that \( (a - \gamma)/2 < 0 \). From Young’s inequality, we get for all \( t \in [0, T) \)
\[ |G * u|^2 \leq \| u \|_{L^\infty} \| G \|_{L^1} \leq \frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-2\beta T}. \]  
(48)
Using (37)–(39), we get
\[ m'(t) \leq -\frac{1}{2} m^2(t) + \beta^2 \frac{\gamma (a - \gamma)}{4} \frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-2\beta T} \]  
(49)
\[ + 4 \gamma \omega \sqrt{\frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-\beta T}}, \]
which results in
\[ m'(t) \leq -\frac{1}{2} m^2(t) + K, \]  
(50)
with
\[ K = \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{4} \frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-2\beta T} \]  
(51)
\[ + 4 \gamma \omega \sqrt{\frac{e + 1}{2(e - 1)} \| u_0 \|_{H^1}^2 e^{-\beta T}}. \]
Note that, from Lemma 10, if \( m(0) < -\sqrt{2K} \), then there exists \( T \), such that \( \lim_{t \to T} m(t) = -\infty \). Repeating the proof of (i), we deduce that the solution \( u \) blows up in finite time. It finishes the proof of the theorem.

Next, we give the second blow-up result.

**Theorem 12.** Let \( \beta < 0 \) and \( u_0 \in H^1(S), \ s \geq 3, \) and \( \int_S u_0 dx = \alpha_0/2. \)

(i) If \( 0 < \gamma < a/3 \) and for all \( e > 0 \) there is a \( x_0 \in S \) such that
\[ \gamma u_0'(x_0) \]
\[ < -\beta - \sqrt{2} \left( \frac{\beta^2}{2} + \gamma (a - 2\gamma) \right) \frac{e + 1}{2(e - 1)} \]  
(52)
\[ \times \left( \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-2\beta T} + \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-\beta T} + \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-\beta T} + \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-\beta T} \right)^{1/2}, \]
then the corresponding solution to (2) blows up in finite time.

(ii) If \( a/3 < \gamma < a \) and for all \( e > 0 \) there is a \( x_0 \in S \) such that
\[ \gamma u_0'(x_0) < -\beta - \sqrt{2} \left( \frac{\beta^2}{2} + \gamma (a - \gamma) \right) \frac{e + 2}{24} \]  
(53)
\[ \times \left( \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-2\beta T} + \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-\beta T} + \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-\beta T} + \frac{e + 2}{24} \| u_0 \|_{H^1}^2 e^{-\beta T} \right)^{1/2}, \]
then the corresponding solution to (2) blows up in finite time.
(iii) If \( \gamma < 0 \) or \( \gamma > a \) and for all \( \epsilon > 0 \) there is a \( x_0 \in S \) such that
\[
\gamma u_0(x_0) < -\beta - \sqrt{2(\beta^2 + \epsilon + 2a^2)} + 4|\gamma|\omega \\
\times \left| \frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a_0^2} \right|^{1/2},
\]
then the corresponding solution to (2) blows up in finite time.

Proof. Similar to Theorem II, we divide (34) into three cases to prove the theorem.

(i) The first is \( 0 < \gamma < a/3 \); note that \( (a - 3\gamma)\gamma/2 \geq 0 \) and \( G * (a^2 + (1/2)\epsilon^2) \geq (1/2)a^2 \). Then, we have \( G * ((a - 3\gamma)\gamma/2a^2) \geq 0 \).

From Lemmas 8 and 9, we have
\[
\max_{x \in S} \{u^2\} \leq \frac{e + 2}{24} \|u_0\|^2_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2.
\]
Thus,
\[
\|u\|_{L^\infty} \leq \frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2.
\]
Note that \( \|G\|_{L^1(S)} = 1 \). Using Young's inequality, we get
\[
|G * u| \leq \|u\|_{L^\infty} \|G\|_{L^1(S)} \leq \frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2.
\]
Therefore, we deduce
\[
m'(t) \leq -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - 2\gamma)}{2} \\
\times \left( \frac{e + 2}{24} \|u_0\|^2_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2 \right) + 4|\gamma|\omega \sqrt{\frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2},
\]
which results in
\[
m'(t) \leq -\frac{1}{2} m^2(t) + K
\]
with
\[
K = \frac{\beta^2}{2} + \frac{\gamma (a - 2\gamma)}{2} \left( \frac{e + 2}{24} \|u_0\|^2_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2 \right) + 4|\gamma|\omega \sqrt{\frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2}.
\]
Note that, from the lemma, if \( m(0) < -\sqrt{2K} \), then there exists \( T \), such that \( \lim_{t \to T} m(t) = -\infty \). Applying Theorem 2, the solution \( u \) blows up in finite time.

(ii) The second is \( a/3 \leq \gamma \leq a \). Similar to case (i), we have
\[
G * \left[ \frac{(a - \gamma)\gamma}{2} u^2 - \frac{\gamma^2}{2} u^2_x \right] \leq \frac{(a - \gamma)\gamma}{4} u^2
\]
\[
+ G * \left( \frac{\gamma (3\gamma - a)}{4} u^2_x \right).
\]
Using (55)–(57), we get
\[
m'(t) \leq -\frac{1}{2} m^2(t) + \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{4} \\
\times \left( \frac{e + 2}{24} \|u_0\|^2_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2 \right) + 4\gamma\omega \sqrt{\frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2},
\]
which results in
\[
m'(t) \leq -\frac{1}{2} m^2(t) + K
\]
with
\[
K = \frac{\beta^2}{2} + \frac{\gamma (a - \gamma)}{4} \left( \frac{e + 2}{24} \|u_0\|^2_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2 \right) + 4\gamma\omega \sqrt{\frac{e + 2}{24} \|u_0\|_{H^1} e^{-2\beta T} + \frac{e + 2}{4a^2} a_0^2}.
\]
Following the proof of (i) in Theorem 11, we obtain that the solution \( u \) blows up in finite time. It finishes the proof of the theorem.

Next, we give the third blow-up result.

**Theorem 13.** Assume that \( \beta < 0 \) and \( u_0 \in H^s(S), s \geq 3, \) and \( \|u_0\|_{H^s} \neq 0.\)

(i) If \( 0 < \gamma < a/3 \) is such that

\[
\int_S \gamma u^3 u_t dx < -2\beta \|u_0\|_{H^s}^2 e^{-2\beta T} \left( \frac{\beta^2}{2} \|u_0\|_{H^s}^4 e^{-4\beta T} + \frac{3\gamma(a-2\gamma)(e+1)}{4(e+1)} \right)
\]

then the corresponding solution to (2) blows up in finite time.

(ii) If \( a/3 < \gamma < a \) is such that

\[
\int_S \gamma u^3 u_t dx < -2\beta \|u_0\|_{H^s}^2 e^{-2\beta T} \left( \frac{\beta^2}{2} \|u_0\|_{H^s}^4 e^{-4\beta T} + \frac{3\gamma(a-\gamma)(e+1)}{8(e+1)} \right)
\]

then the corresponding solution to (2) blows up in finite time.

(iii) If \( \gamma > a \) or \( \gamma < 0 \) is such that

\[
\int_S \gamma u^3 u_t dx < -2\beta \|u_0\|_{H^s}^2 e^{-2\beta T} \left( \frac{\beta^2}{2} \|u_0\|_{H^s}^4 e^{-4\beta T} + \frac{3\gamma(y-a)(e+1)}{8(e+1)} \right)
\]

then the corresponding solution to (2) blows up in finite time.

**Proof.** Let \( T > 0 \) be the maximal time of existence of the solution \( u \) to (2) with the initial data \( u_0. \) Applying \( \gamma u^2 \partial_x \) to both sides of (2) and integrating by parts, we get

\[
\frac{d}{dt} \int_S \gamma u^3 u_t dx = -\frac{3\gamma^2}{2} \int_S u^4 \partial_x u dx - 3\beta \int_S u^3 u_t dx + \frac{(a-\gamma)\gamma}{2} \int_S u^2 u_t dx + 3\gamma \int_S u^2 u_t dx + 6\omega \gamma \int_S u^2 u_t dx + 3 \int_S u^2 G \left( \frac{(a-\gamma)\gamma}{2} u^2 + \frac{\gamma^2}{2} u^2 + 2\omega \gamma u \right).
\]

(72)

Since

\[
\int_S \gamma u^3 u_t dx \leq \left( \int_S u^4 u_t dx \right)^{1/2} \left( \int_S u^2 u_t dx \right)^{1/2},
\]

(73)

thus

\[
\int_S \gamma u^3 u_t dx \geq \left( \int_S u^4 u_t dx \right)^{1/2} \left( \int_S u^2 u_t dx \right)^{1/2} \geq \frac{\int_S u^4 u_t dx}{\int_S u^2 u_t dx}.
\]

(74)

Therefore, we obtain

\[
\frac{d}{dt} \int_S \gamma u^3 u_t dx \leq -\frac{3\gamma^2}{2} \|u_0\|_{H^s}^2 e^{-2\beta T} \int_S u^4 u_t dx - 3\beta \int_S u^3 u_t dx + 3(a-\gamma)\gamma \int_S u^2 u_t dx + 6\omega \gamma \int_S u^2 u_t dx + \frac{\gamma^2}{2} \int_S u^2 u_t dx + 2\omega \gamma u dx.
\]

(75)

Next, we divide (75) into three cases to prove the theorem.

(i) The first is \( 0 < \gamma \leq a/3; \) from case (i) of Theorem 11, we know that \( \gamma(a-3\gamma) \geq 0 \) and \( G \left( ((a-3\gamma)\gamma/2)u^2 \right) \geq 0. \) From Holder's inequality and Young's inequality, we have

\[
\left\| u_t \right\|_{L^4}^2 \int_S u^2 u_t dx \leq \frac{e+1}{2(e-1)} \|u_0\|_{H^s}^3 e^{-3\beta T},
\]

(71)
\[
\left| \int_S u^2 \mathcal{G} \ast u \, dx \right| \leq \| \mathcal{G} \ast u \|_{L^\infty} \int_S u^2 \, dx \leq \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3.
\]

(76)

Using (76) and (75), it yields

\[
\frac{d}{dt} \int_S \gamma u^3 \, dx \leq - \frac{3}{2\| u \|_{H^1}^2} e^{-2\beta t} x \int_S u^2 \, dx + 3 \beta^2 \left( \int \gamma u^3 \, dx + \beta \| u \|_{H^1}^2 e^{-2\beta t} \right)^2 + \frac{3 \beta^2}{2}
\]

\[
\times \| u \|_{H^1}^2 e^{-2\beta t} + 3 \frac{(a - 2\gamma)}{2(e - 1)} \| u \|_{H^1}^4 e^{-4\beta t}
\]

\[
\times \| u \|_{H^1}^4 e^{-4\beta t} + 12 \omega y \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3 e^{-3\beta t}.
\]

(77)

Setting

\[
m(t) = \int_S \gamma u^3 \, dx + \beta \| u \|_{H^1}^2 e^{-2\beta t},
\]

(78)

\[
K = \frac{3 \beta^2}{2} \| u \|_{H^1}^2 e^{-2\beta t} + 3 \frac{(a - 2\gamma)}{2(e - 1)} \| u \|_{H^1}^4 e^{-4\beta t}
\]

\[
+ 12 \omega y \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3 e^{-3\beta t}.
\]

(79)

we have

\[
\frac{d}{dt} m(t) \leq - \frac{3}{2\| u \|_{H^1}^2} e^{-2\beta t} m^2(t) + K.
\]

(80)

Note that, from the lemma, if \( m(0) < -\sqrt{2K/3}\| u \|_{H^1} e^{-\beta t} \), similar to the proof in case (i) of Theorem II, we derive that the corresponding solution will blow up in finite time.

(ii) The second is \( a/3 < \gamma \leq a \); from case (ii) of the theorem, we have \( \gamma(3\gamma - a) \geq 0 \) and \( G \ast ((\gamma(3\gamma - a)/4)u^2) \geq 0 \).

Using (76) and (75), it yields

\[
\frac{d}{dt} \int_S \gamma u^3 \, dx \leq - \frac{3}{2\| u \|_{H^1}^2} e^{-2\beta t} x \int_S u^2 \, dx + 3 \beta^2 \left( \int \gamma u^3 \, dx + \beta \| u \|_{H^1}^2 e^{-2\beta t} \right)^2 + \frac{3 \beta^2}{2}
\]

\[
\times \| u \|_{H^1}^2 e^{-2\beta t} + 3 \frac{(a - 2\gamma)}{4} \| u \|_{H^1}^4 e^{-4\beta t}
\]

\[
\times e^{-4\beta t} + 12 \omega y \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3 e^{-3\beta t}.
\]

(81)

Setting

\[
m(t) = \int_S \gamma u^3 \, dx + \beta \| u \|_{H^1}^2 e^{-2\beta t},
\]

(82)

\[
K = \frac{3 \beta^2}{2} \| u \|_{H^1}^2 e^{-2\beta t} + 3 \frac{(a - \gamma)}{4} \| u \|_{H^1}^4 e^{-4\beta t}
\]

\[
+ 12 \omega y \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3 e^{-3\beta t},
\]

(83)

we have

\[
\frac{d}{dt} m(t) \leq - \frac{3}{2\| u \|_{H^1}^2} e^{-2\beta t} m^2(t) + K.
\]

(84)

Note that, from the lemma, if \( m(0) < -\sqrt{2K/3}\| u \|_{H^1} e^{-\beta t} \), similar to the proof in case (i) of Theorem II, we derive that the corresponding solution will blow up in finite time.

(iii) The third is \( \gamma > a \) or \( \gamma < 0 \); note that \( (a - \gamma)/2 < 0 \). Using (76) and (75), it yields

\[
\frac{d}{dt} \int_S \gamma u^3 \, dx \leq - \frac{3}{2\| u \|_{H^1}^2} e^{-2\beta t} x \int_S u^2 \, dx + 3 \beta^2 \left( \int \gamma u^3 \, dx + \beta \| u \|_{H^1}^2 e^{-2\beta t} \right)^2 + \frac{3 \beta^2}{2}
\]

\[
\times \| u \|_{H^1}^2 e^{-2\beta t} + 3 \frac{(a - 2\gamma)}{4} \| u \|_{H^1}^4 e^{-4\beta t}
\]

\[
\times e^{-4\beta t} + 12 \omega y \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3 e^{-3\beta t}.
\]

(85)

Setting

\[
m(t) = \int_S \gamma u^3 \, dx + \beta \| u \|_{H^1}^2 e^{-2\beta t},
\]

(86)

\[
K = \frac{3 \beta^2}{2} \| u \|_{H^1}^2 e^{-2\beta t} + 3 \frac{(a - \gamma)}{4} \| u \|_{H^1}^4 e^{-4\beta t}
\]

\[
+ 12 \omega y \sqrt{\frac{e + 1}{2(e - 1)}} \| u \|_{H^1}^3 e^{-3\beta t},
\]

we have

\[
\frac{d}{dt} m(t) \leq - \frac{3}{2\| u \|_{H^1}^2} e^{-2\beta t} m^2(t) + K.
\]

(87)

Note that, from the lemma, if \( m(0) < -\sqrt{2K/3}\| u \|_{H^1} e^{-\beta t} \), similar to the proof in case (i) of Theorem II, we deduce that the corresponding solution will blow up in finite time. It completes the proof of Theorem 13.
Finally, we give the fourth blow-up result.

**Theorem 14.** Let $\beta \geq 0$ and $u_0 \in H^s$, $s \geq 3$. Assume that $u_0 \neq 0$ and $\int_S u_0 \, dx = 0$. If $\gamma$ and $\omega$ satisfy one of the following conditions:

(i) $a \sinh(1/2)/(6 + \sinh(1/2)) < \gamma < a$ and $\omega < (6\sqrt{3}/169)(3\gamma/\sinh(1/2) - (a - \gamma)/2)\|u_0\|_{H^s} e^{-4\beta t}$,

(ii) $\gamma > a$ and $\omega < (6\sqrt{3}/169)(3\gamma/\sinh(1/2) + (a - \gamma)/2)\|u_0\|_{H^s} e^{-4\beta t}$,

(iii) $\gamma < a \sinh(1/2)/\sinh(1/2) - 6$ and $\omega < (6\sqrt{3}/169)\|\gamma(\sinh(1/2))/\gamma(\sinh(1/2) + \gamma(\sinh(1/2))/2)\|u_0\|_{H^s} e^{-4\beta t}$,

then the corresponding solution of (2) blows up in finite time.

**Proof.** Supposing that the statement is not correct, then the solution of (2) exists globally in time.

Thus, applying $\gamma u_0^2 \Delta u$ to both sides of (4) and integrating by parts, we get

$$
\frac{d}{dt} \int_S \gamma u_0^3 \, dx = \frac{3\gamma^2}{2} \int_S u_0^4 - 3\beta \int_S \gamma u_0^3 \, dx + \frac{3(a - \gamma)\gamma}{2} \int_S u_0^2 \, dx
$$

$$
\times \int_S u_0^2 u_0^2 \, dx + 6 \omega \gamma \int_S u_0^2 \, dx
$$

$$
- 3 \int_S u_0^2 G \ast (a - \gamma) \gamma u_0^2 \, dx
$$

$$
- 3 \int_S u_0^2 G \ast \frac{\gamma^2}{2} u_0^2 \, dx - 6 \omega \gamma \int_S u_0^2 G \ast u \, dx.
$$

(87)

Due to $\|u\|_{L^\infty}^2 \leq (1/12) \int_S u_0^2 \, dx \leq (1/12)\|u_0\|_{H^s}^2$, and $\int_S u_0^2 \, dx \geq (12/13)\|u_0\|_{H^2}^2$ (see [29]), we get

$$
\int_S u_0^2 \, dx \leq \max \{u^2\} \int_S u_0^2 \, dx \leq \frac{1}{12} \left( \int_S u_0^2 \, dx \right)^2,
$$

$$
\int_S uu_0^2 \, dx \leq \max \{|u|\} \int_S u_0^2 \, dx \leq \frac{\sqrt{3}}{6} \left( \int_S u_0^2 \, dx \right)^{3/2},
$$

$$
\int_S u_0^2 G \ast \left( a^2\right) \, dx \leq \max \{u^2\} \left( \int_S u_0^2 \, dx \right)^2,
$$

$$
\int_S u_0^2 G \ast u \, dx \leq \max \{|u|\} \int_S u_0^2 \, dx \leq \frac{\sqrt{3}}{6} \left( \int_S u_0^2 \, dx \right)^{3/2}.
$$

(88)

Since $1/2 \sinh(1/2) \leq G(x) \leq \cosh(1/2)/2 \sinh(1/2)$, we have

$$
\int_S u_0^2 G \ast \left( u_0^2 \right) \, dx \geq \frac{1}{2 \sinh(1/2)} \left( \int_S u_0^2 \, dx \right)^2
$$

$$
\geq \frac{72}{169 \sinh(1/2)} \|u_0\|_{H^s}^4.
$$

(89)

Therefore, we deduce

$$
\frac{d}{dt} \int_S \gamma u_0^3 \, dx = -\frac{3\gamma^2}{2} \int_S u_0^4 - 3\beta \int_S \gamma u_0^3 \, dx + 2\sqrt{3}\omega \gamma
$$

$$
\times \left( \int_S u_0^2 \, dx \right)^{3/2}
$$

$$
+ \frac{(a - \gamma)\gamma}{8} \left[ -\frac{3\gamma^2}{4 \sinh(1/2)} \right] \left( \int_S u_0^2 \, dx \right)^2.
$$

(90)

Next, we divide (90) three cases to prove the theorem.

(i) The first is $a \sinh(1/2)/(6 + \sinh(1/2)) < \gamma < a$. Note that $\gamma(\sinh(1/2))/\gamma(\sinh(1/2) + \gamma(\sinh(1/2))/2) < 0$. From the condition of Theorem 14, we have

$$
\frac{d}{dt} \int_S \gamma u_0^3 \, dx \leq -\frac{3\gamma^2}{2} \int_S u_0^4 - 3\beta \int_S \gamma u_0^3 \, dx + 2\sqrt{3}\omega \gamma
$$

$$
\times \|u_0\|_{H^s}^3 + \frac{36}{169} \left( \frac{(a - \gamma)\gamma}{2} - \frac{3\gamma^2}{4 \sinh(1/2)} \right) \|u_0\|_{H^s}^4 e^{-4\beta t} > 0.
$$

From Holder’s inequality, we obtain

$$
\int_S \gamma u_0^3 \, dx \leq \gamma \left( \int_S u_0^3 \, dx \right)^{3/4}.
$$

(92)

Thus, we have

$$
\frac{d}{dt} \int_S \gamma u_0^3 \, dx \leq -\frac{3\gamma^2/3}{2} \left( \int_S \gamma u_0^3 \, dx \right)^{4/3} - 3\beta \int_S \gamma u_0^3 \, dx - \delta.
$$

(93)

From (93), we have

$$
\frac{d}{dt} \int_S \gamma u_0^3 \, dx \leq -3\beta \int_S \gamma u_0^3 \, dx - \delta.
$$

(94)

Solving (94), we obtain

$$
\int_S \gamma u_0^3 \, dx \leq -\frac{\delta}{3\beta} + Ce^{-3\beta t},
$$

(95)

where $C < 0$ is constant which implies that there is a $t_0 \geq 0$ such that

$$
\int_S \gamma u_0^3 \, dx < 0, \quad t \geq t_0.
$$

(96)

On the other hand,

$$
\frac{d}{dt} \int_S \gamma u_0^3 \, dx \leq -\frac{3\gamma^2/3}{2} \left( \int_S \gamma u_0^3 \, dx \right)^{4/3} - 3\beta \int_S \gamma u_0^3 \, dx.
$$

(97)
Solving (97), we obtain
\[ 0 > \left( \left( \int_{S} \gamma u_{x}^{3} dx \right)^{-1} \right)^{-1} \geq \left( -\frac{\gamma^{2/3}}{2} + C_{1} e^{\beta t} \right)^{3}, \quad (98) \]
where \( C_{1} \geq 0 \) is constant, which leads to a contradiction as \( t \) becomes large enough. Therefore, case (i) is proved.

(ii) The second is \( \gamma \geq a \). Note that \( \gamma(a - \gamma)/8 < 0 \) and \(-\gamma(a - \gamma)/8 - 3\gamma^{2}/4 \sinh(1/2) < 0\). From the condition of the theorem, we have
\[ \frac{d}{dt} \int_{S} \gamma u_{x}^{3} dx \leq -\frac{3\gamma^{2}}{2} \int_{S} u_{x}^{4} - 3\beta \int_{S} \gamma u_{x}^{3} dx + 2\sqrt{3}\omega \gamma \times \| u_{t} \|_{H}^{3} + \frac{36}{169} \left( \frac{a - \gamma}{2} \right) - \frac{3\gamma^{2}}{2} \sinh(1/2) \right) \times \| u_{t} \|_{H}^{3} e^{-4\beta t} \leq -\frac{3\gamma^{2}}{2} \int_{S} u_{x}^{4} - 3\beta \int_{S} \gamma u_{x}^{3} dx - \delta, \quad (99) \]
where \( \delta = 2\sqrt{3}\omega |\gamma|\| u_{t} \|_{H}^{3} + (36/169)(-\gamma)/2 - 3\gamma^{2}/4 \sinh(1/2) > 0 \). Repeating the arguments in the proof of case (i), we deduce that the corresponding solution blows up.

(iii) The third is \( \gamma < a \sinh(1/2)/(\sinh(1/2) - 6) \). Note that \( \gamma(a - \gamma)/8 < 0 \) and \(-\gamma(a - \gamma)/8 - 3\gamma^{2}/4 \sinh(1/2) < 0\). From the condition of the theorem, we have
\[ \frac{d}{dt} \int_{S} \gamma u_{x}^{3} dx \leq -\frac{3\gamma^{2}}{2} \int_{S} u_{x}^{4} - 3\beta \int_{S} \gamma u_{x}^{3} dx + 2\sqrt{3}\omega |\gamma| \times \| u_{t} \|_{H}^{3} + \frac{36}{169} \left( \frac{a - \gamma}{2} \right) - \frac{3\gamma^{2}}{2} \sinh(1/2) \right) \times \| u_{t} \|_{H}^{3} e^{-4\beta t} \leq -\frac{3\gamma^{2}}{2} \int_{S} u_{x}^{4} - 3\beta \int_{S} \gamma u_{x}^{3} dx - \delta, \quad (100) \]
where \( \delta = 2\sqrt{3}\omega |\gamma|\| u_{t} \|_{H}^{3} + (36/169)(-\gamma)/2 - 3\gamma^{2}/4 \sinh(1/2) > 0 \). Repeating the arguments in the proof of case (i), we derive that the corresponding solution blows up. This finishes the proof of the theorem. \( \square \)

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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