Research Article
Parallel Algorithm with Parameters Based on Alternating Direction for Solving Banded Linear Systems

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An efficient parallel iterative method with parameters on distributed-memory multicomputer is investigated for solving the banded linear equations in this work. The parallel algorithm at each iterative step is executed using alternating direction by splitting the coefficient matrix and using parameters properly. Only it twice requires the communications of the algorithm between the adjacent processors, so this method has high parallel efficiency. Some convergence theorems for different coefficient matrices are given, such as a Hermite positive definite matrix or an $M$-matrix. Numerical experiments implemented on HP rx2600 cluster verify that our algorithm has the advantages over the multisplitting one of high efficiency and low memory space, which has a considerable advantage in CPU-times costs over the BSOR one. The efficiency for Example 1 is better than BSOR one significantly. As to Example 2, the acceleration rates and efficiency of our algorithm are better than the PEk inner iterative one.

1. Introduction

In recent years, the high-performance parallel computing technology has been rapidly developed. The large sparse banded linear systems are frequently encountered when finite difference or finite element methods are used to discretize partial differential equations in many practice scientific and engineering computing problems, especially in computational fluid dynamics (CFD). While many problems can be efficiently resolved on sequential computers but are difficult to solve on parallel computers, the communications take a significant part of the total execution time. So we need more efforts to investigate more efficient parallel algorithm to improve the experimental results.

The parallel algorithms on the large sparse linear systems have been widely investigated in [1–8]. Specifically, the multi-splitting algorithm in [1] is a popular method at present. In [3], the authors provide a method for solving block-tridiagonal linear systems in which local lower and upper triangular incomplete factors are combined into an effective approximation for global incomplete lower and upper triangular factors of coefficient matrix based on two-dimensional domain decomposition with small overlapping. The algorithm is applicable to any preconditioner of incomplete type. Duan et al. presented a parallel strategy based on the Galerkin principle for solving block-tridiagonal linear systems in [4]. In [5], a parallel direct algorithm based on Divide-and-Conquer principle and the decomposition of the coefficient matrix is investigated for solving the block-tridiagonal linear systems on distributed-memory multicomputers. The communication of the algorithm is only twice between the adjacent processors. In [7], a direct method for solving circular-tridiagonal block linear systems is presented. Some parallel algorithms for solving the linear systems can be found in [9–14]. The algorithm in this paper is discussed on the basis of the advantages of the one in [2].

The goal of this paper is to develop an efficient, stable parallel iterative method on distributed-memory multicomputer and to give some theoretical analysis. We appropriately choose the splitting matrices $W$ and $V$ to establish
the iterative scheme. Two examples have been done on the HP rx2600 cluster; the experimental results indicate that the parallel algorithm has advantages over the multisplitting one of high parallel speedup and efficiency.

The content of this paper is as follows. In Section 2, the parallel iterative algorithm is described. In Section 3, the parallel iterative process is discussed. The analysis of convergence is done in Section 4. The numerical results are shown in Section 5. In Section 6, the conclusion is presented.

2. Parallel Algorithm

Let a banded linear equation $AX = b$ be represented as

$$
\begin{pmatrix}
A_1 & B_1 \\
C_2 & A_2 & B_2 \\
& \ddots & \ddots & \ddots \\
& & C_n & A_n & B_n \\
& & & \ddots & \ddots & \ddots \\
& & & & C_{n-1} & A_{n-1} & B_{n-1} \\
& & & & & C_n \\
& & & & & & \ddots \\
& & & & & & C_n & A_n \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n \\
\end{pmatrix}
= \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_{n-1} \\
b_n \\
\end{pmatrix},
$$

where $A_i$ is a $d_i \times d_i$ matrix, $B_i$ and $C_i$ are $d_i \times d_{i+1}$ and $d_i \times d_{i-1}$ matrices, respectively, and $x_i$ and $b_i$ are $d_i$-dimensional real column vectors. In general, assuming that there are $p$ processors available and $n = 2hp$ ($h \geq 1$, $h \in \mathbb{Z}^+$), we denote the $i$th processor by $P_i$ (for $i = 1, 2, \ldots, p$) and split the coefficient matrix $A$ into $A = W + V$.

Then, we use the alternating direction iterative scheme in [2] and obtain the new iterative scheme

$$\begin{align*}
(I + \tau W)(I + \tau V)(x^{(k+1)} - x^{(k)}) &= -\alpha \tau (Ax^{(k)} - b); \\
(I + \tau V)^{-1}I + \tau W)^{-1} &= (I - 2\tau(I + \tau V)^{-1}(I + \tau W)^{-1}A)x^{(k)} \quad (3)
\end{align*}$$

From (3), let $y = (I + \tau W)^{-1}(Ax^{(k)} - b)$; we obtain

$$
(I + \tau W)y = Ax^{(k)} - b;
$$

then the detailed calculation procedure is as follows:

$$
\begin{align*}
\left(I + \tau A_{\{i-1\} \times 2h+j}\right)y_{\{i-1\} \times 2h+j} \\
&= C_{\{i-1\} \times 2h+j}(x^{(k)}_{\{i-1\} \times 2h+j-1}) + A_{\{i-1\} \times 2h+j}x^{(k)}_{\{i-1\} \times 2h+j} \\
&+ B_{\{i-1\} \times 2h+j}x^{(k)}_{\{i-1\} \times 2h+j-1} - b_{\{i-1\} \times 2h+j} \\
&\quad (i = 2, 4, \ldots, 2h);
\end{align*}
$$

Obviously, the matrices $I + \tau W$ and $I + \tau V$ should be nonsingular and the definition of $W$ and $V$ is the most important key of solving the linear systems by (3) in this paper. If $W$ and $V$ are suitable, the algorithm would have good parallelism and low CPU-times costs. So we choose $W$ and $V$ as follows
here, \( y = (y_1, y_2, \ldots, y_i, \ldots, y_p)^T \) and \( y_i \) is a 2\( h \)-dimensional row vector.

Let \( z = (1 + \tau V)^{-1} y \); then we have \((I + \tau V)z = y \), and

\[
\begin{align*}
[I + \tau (1 - \alpha) A_{(\tau - 1)2h + 1}'] z_j &= y_{(\tau - 1)2h + j}, \\
& j = (1, 3, \ldots, 2h - 1), \\
[I + \tau (1 - \beta) A_{(\tau - 1)2h + 1}'] z_j &= y_{(\tau - 1)2h + j - 1}, \\
& j = (2, 4, \ldots, 2h).
\end{align*}
\]

Let \( z = (1 + \tau V)^{-1} y \); then we have \((I + \tau V)z = y \), and according to the aforementioned formulas, we get \( x^{(k+1)} = x^{(k)} - 2\tau z \).

### 3. Process of Parallel Iterative Algorithm

Here, we show the storage method and computational procedure of the parallel algorithm as follows.

#### 3.1. Storage Method

The coefficient matrix is divided into \( A_1, A_2, \ldots, A_{k_r+1}, \ldots, A_{k_r+k_r+1} \) from left to right as banded order. Let vectors \( A_i = (a_{11}, a_{12}, \ldots, a_{22}, \ldots, a_{n_r-k_r}, a_{n_r-k_r+1}, \ldots) \), \( A_{k_r+k_r+1} = (a_{11}, a_{22}, \ldots, a_{n_r-k_r}, a_{n_r-k_r+1}, \ldots, a_{n_r-k_r}, 0, \ldots, 0)^T \).

The corresponding relationship is as follows:

\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1,k_r+1} & 0 & \cdots & \cdots & 0 \\
  a_{21} & a_{22} & \cdots & a_{2,k_r+2} & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{k_r+1,1} & a_{k_r+1,2} & \cdots & a_{k_r+1,k_r+1} & a_{k_r+1,k_r+2} & \cdots & 0 \\
  a_{k_r+2,1} & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & a_{n_r-k_r-k_r-k_r} & \cdots & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a_{n_r-k_r} & \cdots & a_{n_r-k_r} & a_{n_r-k_r-n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & a_{n_r-k_r-k_r} & \cdots & a_{n_r-k_r} & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  a_{1} & \cdots & a_{k_r+1} & \cdots & a_{k_r+k_r+1} \\
  \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
  0 & \cdots & a_{11} & \cdots & a_{1,k_r+1} \\
  \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
  0 & a_{k_r+1,1} & \cdots & a_{k_r+1,k_r+1} & a_{k_r+1,k_r+2} & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \downarrow \\
  a_{n_r-k_r} & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]

\[
A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_{k_r+1} \leftrightarrow \cdots \leftrightarrow A_{k_r+k_r+1}
\]

Then, assign \( m (m = n/p) \) rows to each processor. The processor stores the corresponding vectors \( b_i, x_i \) with \( i = 1, 2, \ldots, p \). Here \( k_r \) and \( k_l \) are upper-band width and lower-band width, respectively. In such a case, this saves much of the memory space although programming is difficult. Note that if \( n \) is not divisible by \( p \), some processors store \([n/p] + 1 \) rows-block of \( A \), sequentially, and others store \([n/p] \) rows-block; meanwhile, each processor stores the corresponding vectors of \( x^{(0)} \) and \( b \). Thereby, it makes load of each processor approach balance and shorten wait time.

#### 3.2. Cycle Process

1. \( P_i \) performs a parallel communication to obtain \( x^{(k)}_i \) and then computes

\[
y_i = -2\tau (I + \tau W_i)^{-1} (A_i x^{(k)}_i - b_i)
\]

and implements LU discretization one-step; here \( V_i \) is the \( i \)th (for \( i = 1, 2, \ldots, p \)) block of \( V \).

2. \( P_i \) performs one parallel communication to obtain \( y_{(\tau - 1)2m+1} \) and then computes

\[
x^{(k+1)}_i = x^{(k)}_i + (I + \tau V_i)^{-1} y_i
\]

and implements LU discretization one-step; here \( V_i \) is the \( i \)th (for \( i = 1, 2, \ldots, p \)) block of \( V \).

3. On the \( P_i \) processor, judge whether the inequality \( \| x^{(k+1)}_i - x^{(k)}_i \| < \epsilon \) (\( \epsilon \) is error bound, \( i = 1, 2, \ldots, p \)) holds. Stop if these inequalities hold on every processor, or return to (1) and continue cycling until all inequalities are satisfied.
4. Analysis of Convergence

To perform the theoretical analysis on convergence of the parallel algorithm, we introduce the definition and several lemmata.

Symbol and Definition

(i) $R^{nxn}$ represents the space of $n \times n$ real matrices.

(ii) $I_r$ represents the unit matrix of order $r$.

(iii) $W^H, V^H$ represent the conjugate transpose of matrices $W, V$, respectively.

(iv) $W^{-1}$ represents the inverse of matrix $W$.

Definition 1 (see [15]). Suppose $A \in R^{nxn}$ and $A = Q - S$, where $Q^{-1} \geq 0$ and $S \geq 0$; then $A = Q - S$ is called normal splitting of matrix $A$.

Definition 2 (see [15]). Suppose $A \in R^{nxn}$ and $A = Q - S$, where $Q^{-1}S \geq 0$; then $A = Q - S$ is called weak normal splitting of matrix $A$.

Definition 3 (see [15]). Suppose $A \in R^{nxn}$ and $A = Q - S$, where $Q^H + S$ is a Hermite positive definite matrix; then $A = Q - S$ is called $P$-normal splitting of matrix $A$.

Definition 4 (see [15]). Let $A = (a_{ij}) \in R^{nxn}$, if $a_{ij} \leq 0$ ($i \neq j$) and $A^{-1} \geq 0$; then the matrix $A$ is an $M$-matrix.

Here, we give some theoretical analysis for convergence of the parallel iterative algorithm.

\[
WV = \begin{pmatrix}
A_1 & (1-\beta)D_1 & E_1 \\
\beta D_1^H & A_2 & \beta F_2 \\
E_1^H & (1-\beta)F_2^H & A_3 & (1-\beta)D_3 & E_3 \\
\beta D_3^H & A_4 & \beta F_4 & E_3^H & (1-\beta)F_4^H & A_5 & (1-\beta)D_5 & E_5 \\
c & \ldots \\
\end{pmatrix},
\]

where

\[
A_1 = \alpha (1-\alpha) A_1^2 + B_1 B_1^H,
\]

\[
A_i = \begin{cases}
\beta (1-\beta) A_i^2 & (i = 2, 4, \ldots, 2n) \\
\alpha (1-\alpha) A_i^2 + B_{i-1}^H B_{i-1} + B_i B_i^H & (i = 3, 5, \ldots, 2hp - 1),
\end{cases}
\]

\[
D_i = B_i A_{i+1}, \quad E_i = B_i B_{i+1} \quad (i = 1, 2, \ldots, 2hp),
\]

\[
F_i = A_i B_i \quad (i = 2, 4, \ldots, 2hp),
\]

Lemma 5 (see [9]). Let $A \in R^{nxn}$, if the splitting $A = M - N$ is a weak normal splitting or normal splitting of coefficient matrix $A$; then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.

Lemma 6 (see [10]). Let $A$ be an $M$-matrix. If any element of $A$ increases while outside elements of the main diagonal keep nonpositive, then the transformation matrix $B$ is also an $M$-matrix and $B^{-1} \leq A^{-1}$.

Lemma 7 (see [15]). Let $A \in C^{nxn}$ be a nonsingular Hermite matrix. If $A = M - N$ is a $P$-normal splitting of the matrix $A$, then $\rho(M^{-1}N) < 1$ if and only if $A$ is a positive definite matrix.

Theorem 8. Let $A \in R^{nxn}$ be a Hermite positive definite matrix. If $r > 0$, $\beta = 1/2$, and $0 \leq \alpha \leq 1$, then the iterative scheme (3) is convergent for all vector $x^{(0)}$.

Proof. Since $(I + \tau W)(I + \tau V)(x^{(k+1)} - x^{(k)}) = -2\tau(Ax^{(k)} - b)$ and

\[
B_k = (I - 2\tau(I + \tau V)^{-1}(I + \tau W)^{-1} A) \\
= (I + \tau V)^{-1}(I + \tau W)^{-1}(I - \tau W)(I - \tau V),
\]

we have $A = M - N$; here $M = (I + \tau V)(I + \tau W)$, $N = (I - \tau W)(I - \tau V)$,

\[
M^H + N = (I + \tau V)^H(I + \tau V)W^H + (I - \tau W)(I - \tau V) \\
= 2I + \tau A - \tau^2 V^H W^H + I - \tau A + \tau^2 VW \\
= 2I + \tau^2 (WV + V^H W^H).
\]

Since

\[
A = \begin{pmatrix}
0 & B_1 \\
\frac{1}{2} A_2 & C_3 & 0 \quad B_3 \\
& \hspace{0.5cm} \ddots \\
& & C_{2hp-1} & 0 \quad B_{2hp-1} \\
& & & \frac{1}{2} A_{2hp}
\end{pmatrix},
\]

and let

\[
U = \begin{pmatrix}
0 & B_1 \\
\frac{1}{2} A_2 & C_3 & 0 \quad B_3 \\
& \hspace{0.5cm} \ddots \\
& & C_{2hp-1} & 0 \quad B_{2hp-1} \\
& & & \frac{1}{2} A_{2hp}
\end{pmatrix};
\]
then we have

\[ WV + V^H W^H - 2U^H U = \begin{pmatrix}
2\Lambda_1 - Q_1 & 2\Lambda_2 - Q_2 & \cdots & 2\Lambda_{2n} - Q_{2n}
\end{pmatrix};
\]

(16)

here

\[ Q_1 = 2B_1 B_1^H, \]
\[ Q_i = \begin{cases}
\frac{1}{2} A_i^2 & (i = 2, 4, \ldots, 2hp) \\
2B_{i-1}^H B_{i-1} + 2B_i B_i^H & (i = 3, 5, \ldots, 2hp - 1),
\end{cases} \]
\[ 2\Lambda_i - Q_i = \begin{cases}
2\alpha (1 - \alpha) A_i^2 & (i = 1, 3, \ldots, 2hp - 1) \\
2\beta (1 - \beta) - \frac{1}{2} A_i^2 & (i = 2, 4, \ldots, 2hp).
\end{cases}
\]

(17)

Obviously, \( WV + V^H W^H - 2U^H U \) is a semipositive definite matrix or a positive definite matrix. Hence the matrix

\[ M^H + N = 2I + \tau^2 \left( WV + V^H W^H \right) \]
\[ = 2I + \tau^2 \left( WV + V^H W^H - 2U^H U \right) + 2\tau^2 U^H U \]

(18)

is a Hermite positive definite matrix.

Therefore, \( A = M - N \) is a \( P \)-normal splitting of the matrix \( A \), and then \( \rho(M^{-1} N) < 1 \) by Lemma 7; we know that our algorithm iterative scheme is convergent.

\[ \square \]

By the theorem, we know that the parallel algorithm is convergent if \( A \) is a Hermite positive definite matrix.

**Theorem 9.** Let \( A \in \mathbb{R}^{n \times n} \) be an \( M \)-matrix. If \( 0 < \tau \leq \min\{1/\alpha, 1/\beta, 1/(1 - \alpha), 1/(1 - \beta)\} \min(1/a_{ii}) \) for \( i = 1, 2, \ldots, 2hp \), here \( 0 < \alpha, \beta < 1 \) and \( a_{ii} \) is the diagonal element of \( A \); then the iterative scheme (3) is convergent for all vector \( x^{(0)} \).

**Proof.** Since \( M = (I + \tau V)(I + \tau W), N = (I - \tau W)(I - \tau V) \), and

\[ I + \tau V^{-1} = \begin{pmatrix}
I + (1 - \alpha) \tau A_1 & \tau C_2 & I + (1 - \beta) \tau A_2 & \tau B_2 \\
\tau C_2 & I + (1 - \alpha) \tau A_3 & \tau B_3 & \cdots \\
I + (1 - \beta) \tau A_3 & \tau B_3 & \cdots \\
\tau C_4 & I + (1 - \beta) \tau A_4 & \tau B_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\tau C_{2hp} & I + (1 - \beta) \tau A_{2hp-1} & \tau B_{2hp} & \cdots \\
\end{pmatrix},
\]

(19)

we have

\[ \left( I + \tau V^{-1} \right)^{-1} = \begin{pmatrix}
(I + (1 - \alpha) \tau A_1)^{-1} & Q_2 & (I + (1 - \beta) \tau A_2)^{-1} & F_3 \\
Q_2 & (I + (1 - \alpha) \tau A_3)^{-1} & Q_4 & (I + (1 - \beta) \tau A_4)^{-1} \cdots \cdots \\
(I + (1 - \alpha) \tau A_3)^{-1} & Q_4 & (I + (1 - \beta) \tau A_4)^{-1} & \cdots \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
(I + (1 - \alpha) \tau A_{2hp-1})^{-1} & \cdots & \cdots & \cdots \\
Q_{2hp} & (I + (1 - \beta) \tau A_{2hp})^{-1} & \cdots & \cdots \\
\end{pmatrix}.\]

(20)
Here
\[
Q_i = -\tau (I + (1 - \beta) \tau A_i)^{-1} C_i (I + (1 - \alpha) \tau A_{i+1})^{-1}, \\
F_i = -\tau (I + (1 - \beta) \tau A_i)^{-1} B_i (I + (1 - \alpha) \tau A_{i+1})^{-1},
\]
\[(i = 2, 4, \ldots, 2hp), \quad (i = 2, 4, \ldots, 2hp - 2). \]

Hence, we know that \((I + (1 - \alpha) \tau A_{i+1}), (I + (1 - \alpha) \tau A_{i+1})^{-1}, (I + (1 - \beta) \tau A_i)^{-1}, (I + (1 - \beta) \tau A_i)^{-1}, (I + (1 - \beta) \tau A_i)^{-1} \geq 0, (I + (1 - \beta) \tau A_i)^{-1} \geq 0, Q_i \geq 0, F_i \geq 0; \) we obtain \((I + \tau V)^{-1} \geq 0.\) Similarly, we can obtain \((I + \tau V)^{-1} \geq 0, and M^{-1} \geq 0.\)

By the theorem, we know that the parallel algorithm is convergent if \(A\) is an \(M\)-matrix and \(0 < \tau \leq \min\{1/\alpha, 1/\beta, 1/(1 - \alpha), 1/(1 - \beta)\} \min(1/a_{ij})\) for \(i = 1, 2, \ldots, 2hp.\)

5. Numerical Examples

We performed two numerical experiments on the HP rx2600 cluster. The results are shown as follows.

Example 1. Consider a banded linear system \(AX = b;\) here

\[
A = \begin{pmatrix}
A_1 & B_1 \\
C_2 & A_2 & B_2 \\
& \ddots & \ddots & \ddots \\
& C_{m-1} & A_{m-1} & B_{m-1} \\
& & C_m & A_m
\end{pmatrix},
\]

\[
A_i = \begin{pmatrix}
15.1 & -3.5 & -6.9 \\
-2.7 & 20.1 & -4.8 \\
-15.7 & -5.3 & 25.1
\end{pmatrix},
\]

\[
B_i = C_i = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad b_i = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}.
\]

Let initialization value \(x_0 = (0 \ 0 \ 0)^T\) and \(m = 80000.\) We apply this algorithm with the optimal relaxation factor, the multisplitting method, and BSOR method to the systems on the HP rx2600 cluster. Here \(P\) is the number of processor, \(T\) is the run times (seconds), the \(S\) is speedup \((T\) of one processor/\(T\) of all processors), \(L\) is iteration times, \(E\) is the efficiency \((E = S/P),\) and the error \(\epsilon = 1 \times 10^{-10}.\) See Tables 1, 2, and 3 and Figures 1 and 2.

### Table 1: The results for model 1 (the algorithm in the paper \((\tau = 0.9, \alpha = \beta = 1/2)\)).

<table>
<thead>
<tr>
<th>(P)</th>
<th>(1)</th>
<th>(2)</th>
<th>(4)</th>
<th>(8)</th>
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<tbody>
<tr>
<td>(T)</td>
<td>2.9921</td>
<td>1.5179</td>
<td>0.8028</td>
<td>0.6492</td>
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<tr>
<td>(S)</td>
<td>1.9712</td>
<td>3.7271</td>
<td>4.6089</td>
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</tr>
<tr>
<td>(E)</td>
<td>0.9856</td>
<td>0.9318</td>
<td>0.5761</td>
<td></td>
</tr>
<tr>
<td>(L)</td>
<td>39</td>
<td>39</td>
<td>39</td>
<td>39</td>
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</tbody>
</table>

### Table 2: The results for model 1 (the multisplitting method).

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<tbody>
<tr>
<td>(T)</td>
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<td>3.3874</td>
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<tr>
<td>(S)</td>
<td>1.1148</td>
<td>2.3482</td>
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<tr>
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<td>(L)</td>
<td>160</td>
<td>224</td>
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### Table 3: The results for model 1 (BSOR method \((\omega = 1.85)\)).

<table>
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<tr>
<td>(E)</td>
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<tr>
<td>(L)</td>
<td>495</td>
<td>532</td>
<td>532</td>
<td>532</td>
</tr>
</tbody>
</table>

Example 2. Consider an elliptic partial differential equation

\[
C_x \frac{\partial^2 u}{\partial x^2} + C_y \frac{\partial^2 u}{\partial y^2} + (C_1 \sin 2\pi x + C_2) \frac{\partial u}{\partial x} + (D_1 \sin 2\pi x + D_2) \frac{\partial u}{\partial y} + Eu = 0,
\]

\[(0 \leq x, y \leq 1),\]

equipped with the boundary conditions \(u|_{x=0} = u|_{x=1} = 10 + \cos \pi y, u|_{y=0} = u|_{y=1} = 10 + \cos \pi x;\) here \(C_x, C_y, C_1, C_2, D_1, D_2,\) and \(E\) are all constants.

We denote \(C_x = C_y = E = 1, C_1 = C_2 = D_1 = D_2 = 0.\) Using the finite difference method, we obtain two block-tridiagonal linear systems on condition that the step sizes \(h = 1/100.\) Then, we apply this algorithm with the optimal relaxation factor, BSOR method, PEK method, and the multisplitting algorithm to the systems on the HP rx2600 cluster. The numerical results are shown in Tables 4, 5, 6, and 7 and Figures 3 and 4.

6. Results Analysis

From Table 1 to Table 7, we can get the following conclusion.

(i) It can be known that the results of the parallel algorithm verify the results of the theoretical analysis. The conditions in the theorems are only sufficient conditions.

(ii) By the numerical results, it can be known that the parallel one has good parallelism.
(iii) As to Examples 1 and 2, the results of the examples show that the efficiency of the algorithm is better than the multisplitting ones. Our algorithm has good parallel speedup the same as BSOR methods to the examples. As to Example 2, the efficiency of the algorithm is also better than PEk methods.

(iv) The parallel algorithm is easily implemented on parallel computer and more flexible and simple than [1] in practice.

7. Conclusions

An efficient parallel iterative method on a distributed-memory multicomputer has been presented for solving the large banded linear systems. We make full use of the decomposition of the coefficient matrix to choose $W$ and $V$ to save computational cost. The storage strategy can save memory space. Only twice it requires the communications of the algorithm between the adjacent processors. Theoretical analysis and experiment show that the algorithm in this paper has good parallelism and high efficiency. The results also confirm correctness of convergence theorems. When the coefficient
matrix is a Hermite positive definite matrix or an $M$-matrix, we know that the parallel algorithm is convergent if the given conditions are established. Our algorithm has an advantage over the multisplitting one of high efficiency.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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