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*Research Article*

# $H_\infty$ Controller Design for Asynchronous Multirate Sampled-Data Systems

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This paper considers the analysis and synthesis of a linear discrete asynchronous multirate sampled-data system. An $H_\infty$ controller based on an observer is proposed, which guarantees the stability of the closed system and makes the $H_\infty$ norm of the closed system less than a given attenuation level. To improve the performance further, a tradeoff strategy is applied. That is, the exogenous signals sampled at different rates are lifted to an appropriate signal rate, while the endogenous signals are not lifted for avoiding the causal constraint and the dimension multiplied again. The controller is obtained by solving the corresponding matrix inequality, which can be calculated by Matlab. Finally, an example is presented to demonstrate the validity of these methods.

## 1. Introduction

Sampled-data control systems, which provide digital control techniques, and emerged with the development of computers, have been long studied by many researchers [1, 2]. Multirate sampled-data control systems are required when the signals of interest are sampled at different rates compared with others. There are two primary reasons for this requirement. First, it has been shown that multirate sampling can improve the closed-loop control performance significantly. Second, technological or economic constraints in some real applications necessitate the use of control schemes where sensor measurements and control inputs have to be performed at different sampling rates. Figure 1 shows an ideal model of a standard multirate sampled-data control system. For example, in hybrid electric vehicles, a hierarchical controller is designed to decide the torque demands of motor, generator, internal combustion engine, and mechanical brake according to the driver's torque demand, speed of vehicle, and battery's state of charge (SOC), where the SOC is estimated by battery management system (BMS) and the speed of vehicle is fed by sensor [3]. As a large-scale and complex system, the sensors and control inputs of subsystems are difficult to be performed at the same sampling rates; thus a multirate sampled-data system will be a better choice for modeling.

Multirate systems have been studied since the late 1950s, resulting in the development and application of various sampling methods. Classical approaches include the lifting method, $H_2/H_\infty$ control [4, 5], LMI method [6], prediction control, LQG control, iterative control [7], and parameter estimation [8]. Among these, the lifting method [9] is a vital tool for studying multirate systems, whereby such systems can be changed into single-rate systems. This method can improve the performance of a plant to some extent [10]. However, during the lifting process the dimensions of the system will be multiplied and it is increasingly difficult to analyze the new system effectively, especially systems with very different sampling rates, which can lead to the "model explosion" problem. $H_2/H_\infty$ control is a useful method for analyzing systems and for synthesizing control problems based on the lifting technique, and LMI tools can be used to solve the inequalities [11]. By the way in real applications, fast-input and slow-output is a good way of improving the performance of the system, and this mechanism and its results have been discussed in previous studies [12, 13].

Most of lifting methods require that all signals should be sampled and held during the same time slot [1, 2, 9, 10, 14, 15].

That is, at a specific $k$,

$$S\left(k\overline{T}\right) = \operatorname{diag}\left\{\left[I_1 \;\; 0 \;\; \cdots \;\; 0\right]_{\overline{T}_1}, \left[I_2 \;\; 0 \;\; \cdots \;\; 0\right]_{\overline{T}_2}, \ldots, \left[I_p \;\; 0 \;\; \cdots \;\; 0\right]_{\overline{T}_p}\right\},$$
$$H\left(k\widetilde{T}\right) = \operatorname{diag}\left\{\left[I_1 \;\; \cdots \;\; I_1\right]'_{\widetilde{T}_1}, \left[I_2 \;\; \cdots \;\; I_2\right]'_{\widetilde{T}_2}, \ldots, \left[I_q \;\; \cdots \;\; I_q\right]'_{\widetilde{T}_q}\right\}, \tag{1}$$

where $I_i$, $i = 1, 2, \ldots, p(q)$, are unit matrices and $\overline{T}$, $\overline{T}_i$, $\widetilde{T}$ and $\widetilde{T}_i$, will be defined in Section 2. However, in most actual situations, this condition cannot be satisfied because the overall system lacks a slot when the entire signal can be sampled or held, and there is a class of multirate systems that each output of them has its own frequency of measurement and each input has its own frequency of updating [16–19]. A system with this type of sampling mechanism is referred to as asynchronous system in this paper. The traditional lifting technique is not applicable for the sampling frequencies of the sampling and holding elements are incommensurate [4, 20].

LQG control based on an observer is an effective method for reducing the dimensions of a multirate system without using the lifting technique [14, 21]. An LQG optimal controller derived is periodic and can guarantee the overall closed-loop stability. To compensate for the skewed input signal caused by slow-output feedback, only the input signal is supplemented by the observer. However, in the previous results, the controller and observer are designed separately, which may result in conservativeness.

Based on the methods mentioned above, the design of a controller for a multirate sampled-data system that does not multiply the dimensions necessitates the synthesis of various appropriate measures. In this paper, we propose an observer-based $H_\infty$ method for analysis and synthesis of linear discrete asynchronous multirate sampled-data systems. In contrast to the traditional lifting method, the control inputs and measured outputs are not lifted to avoid the dimension multiplication and causal constraints. However, the exogenous system signals are lifted rationally to change the system into a fast-input-slow-output system. The controller can be obtained by solving an inequality with LMI tools of Matlab. Although the inequality is nonlinear, it can be calculated linearly under a constraint [22]. The approach proposed in this paper is different from the observer-based LQG method in [21]. Here, the controller and observer are obtained simultaneously by solving a matrix inequality under the same optimal performance constraint, thus reducing the conservativeness and improving the system performance. In addition, the concept of asynchronism in this paper is different from those in [23, 24], and so forth, where the sampling at sensors is assumed to be asynchronized with time-varying delay and it falls into the framework of time-delay systems, not the multirate sampled-data systems.

The remainder of this paper is organized as follows. Section 2 provides a description of the multirate system and its background, including lemmas and definitions. In Section 3, the main results obtained using the synthesis methods are presented. In Section 4, a simulated numerical example is provided to demonstrate the effectiveness of the method. Finally, Section 5 concludes the paper.

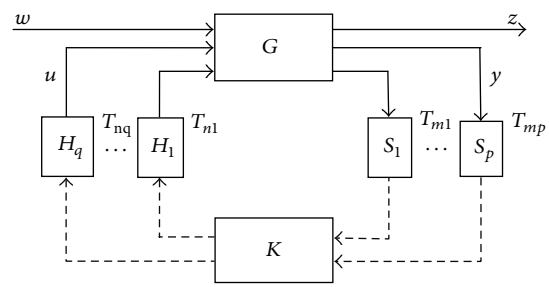


Figure 1: Multirate sampled-data system.

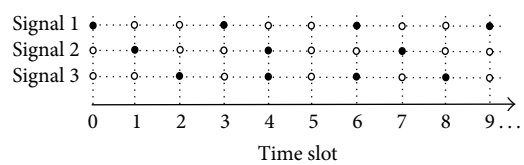


Figure 2: Asynchronous signals from the sampler or the holder.

## 2. System Descriptions

*Definition 1.* A multirate sampled-data system is called an asynchronous system if the signals from its sampler or holder cannot be obtained simultaneously during any time slot.

For example, as shown in Figure 2, Signal 1 and Signal 2 are the signals of the sampler and the holder with period 3, respectively, while Signal 3 is the signal with period 2. Thus, they cannot be obtained simultaneously during any time slot because the odd period signals appear alternately. In Figure 2, the filled circles indicate that the signal is sampled or held in the slot, whereas the hollow cylinders indicate that the signal is not sampled or held. Therefore, this system cannot satisfy the assumption mentioned in most previous studies where all signals can be sampled or held during the same time slot.

We consider a linear discrete asynchronous multirate sampled-data system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + M_1 w(k), \\ z(k) &= Cx(k) + M_2 w(k), \\ y(k) &= S(k) z(k), \end{aligned} \tag{2}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^q$, $z(k) \in \mathbb{R}^p$, $w(k) \in \mathbb{R}^l$, $x(0) = x_0$, $|M_2| \neq 0$, $S(k)$ is a sampling device, $y(k)$ is the output measured vector, and $x_0$ is the initial state.

Assume that the $i$th component $u_i(k)$, $i = 1, 2, \ldots, q$, of the input vector $u(k)$ can be modified every $\widetilde{T}_i$ time instant

$\widetilde{T}_i \in \mathbb{Z}^+$. Then, $u(k)$ is the output of the following discrete-time period system, which is defined as the input-holding mechanism:

$$\begin{aligned} v(k+1) &= (I - H(k))\, v(k) + H(k)\, r(k), \\ u(k) &= (I - H(k))\, v(k) + H(k)\, r(k), \end{aligned} \tag{3}$$

where

$$\begin{aligned} H(k) &= \operatorname{diag}\left\{h_1(k), h_2(k), \ldots, h_q(k)\right\}, \\ h_i(k) &= \begin{cases} 1 & k = j\widetilde{T}_i + \widetilde{\tau}_i, \\ 0 & k \neq j\widetilde{T}_i + \widetilde{\tau}_i, \end{cases} \quad j \text{ integer}, \end{aligned} \tag{4}$$

where $v(k) := \left[v_1(k)\ \ v_2(k)\ \ \cdots\ \ v_q(k)\right]'$ is a new state variable and $r(k) := \left[r_1(k)\ \ r_2(k)\ \ \cdots\ \ r_q(k)\right]'$ is a new input variable. The integer $0 \le \widetilde{\tau}_i < \widetilde{T}_i$ $(i = 1, 2, \ldots, q)$ describes the skew of the inputs-holding mechanism.

Then, matrix $H(\cdot)$ and system (2) have a period of $\widetilde{T}$, and

$$\widetilde{T} := \underset{i=1,2,\ldots,q}{\text{l.c.m.}} \left\{\widetilde{T}_i\right\}. \tag{5}$$

Similarly, the output sampling mechanism is given as

$$S(k) = \operatorname{diag}\left\{v_1(k), v_2(k), \ldots, v_p(k)\right\}, \tag{6}$$

where

$$v_i(k) = \begin{cases} 1 & k = j\overline{T}_i + \overline{\tau}_i, \\ 0 & k \neq j\overline{T}_i + \overline{\tau}_i, \end{cases} \quad j \text{ integer}, \tag{7}$$

where $0 \le \overline{\tau}_i < \overline{T}_i$ $(i = 1, 2, \ldots, p)$ indicates the skew of the outputs mechanism.

Matrix $S(\cdot)$ has a $\overline{T}$-period, and

$$\overline{T} := \underset{i=1,2,\ldots,p}{\text{l.c.m.}} \left\{\overline{T}_i\right\}. \tag{8}$$

After combining (2) and (3), we obtain a new sampling-holding system:

$$\begin{aligned} \xi(k+1) &= \Phi(k)\,\xi(k) + \Gamma(k)\, r(k) + \Psi w(k), \\ z(k) &= \Upsilon\xi(k) + M_2 w(k), \\ y(k) &= \Delta(k)\,\xi(k) + \Omega(k)\, w(k), \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\xi(k) := \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}, \qquad \Phi(k) := \begin{bmatrix} A & B(I - H(k)) \\ 0 & I - H(k) \end{bmatrix}, \\ &\Gamma(k) := \begin{bmatrix} B \\ I \end{bmatrix} H(k), \qquad \Psi := \begin{bmatrix} M_1 \\ 0 \end{bmatrix}, \\ &\Upsilon := \left[C \ \ 0\right], \qquad \Delta(k) := \left[S(k)\, C \ \ 0\right], \\ &\Omega(k) := S(k)\, M_2. \end{aligned} \tag{10}$$

Based on the $\widetilde{T}$ of matrix $H(\cdot)$ and the $\overline{T}$ of matrix $S(\cdot)$, system (9) has the period of

$$T := \text{l.c.m.}\left\{\widetilde{T}, \overline{T}\right\}. \tag{11}$$

The state observer of system (2) is

$$\widehat{x}(k+1) = A\widehat{x}(k) + Bu(k) + L(k) \cdot \left[y(k) - S(k)\, C\widehat{x}(k)\right], \tag{12}$$

where $L(k)$ asymptotically stabilizes system (12).

The observer-based controller is chosen as

$$r(k) = K(k)\,\widehat{\xi}(k), \tag{13}$$

where $\widehat{\xi}(k) := \left[\begin{smallmatrix} \widehat{x}(k) \\ v(k) \end{smallmatrix}\right]$.

The state error is defined as

$$e(k) := x(k) - \widehat{x}(k). \tag{14}$$

According to (2), (12) and (14), the following equation can be obtained:

$$\begin{aligned} e(k+1) = {} & \left[A - L(k)\, S(k)\, C\right] e(k) \\ & + \left(M_1 - L(k)\, S(k)\, M_2\right) w(k). \end{aligned} \tag{15}$$

Combining (9), (13), and (15) yields

$$\begin{aligned} \begin{bmatrix} \xi(k+1) \\ e(k+1) \end{bmatrix} &= \widetilde{A} \begin{bmatrix} \xi(k) \\ e(k) \end{bmatrix} + \widetilde{B} w(k), \\ z(k) &= \widetilde{C} \begin{bmatrix} \xi(k) \\ e(k) \end{bmatrix} + \widetilde{D} w(k), \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\widetilde{A} := \begin{bmatrix} \Phi(k) + \Gamma(k)\, K(k) & -\Gamma(k)\, K(k) \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \\ 0 & A - L(k)\, S(k)\, C \end{bmatrix}, \\ &\widetilde{B} := \begin{bmatrix} \Psi \\ M_1 - L(k)\, S(k)\, M_2 \end{bmatrix}, \\ &\widetilde{C} := \left[\Upsilon \ \ 0\right], \qquad \widetilde{D} := M_2. \end{aligned} \tag{17}$$

**Lemma 2** (see [21, 25] ). *If*

(i) *the pair* $(A, B)$ *is stabilizable,*

(ii) *the pair* $(A, M_1)$ *is stabilizable,*

(iii) *the pair* $(A, C)$ *is detectable,*

(iv) *for any pair of distinct eigenvalues of* $A$, $\lambda_i$ *and* $\lambda_j$, $|\lambda_i| \ge 1$, $|\lambda_j| \ge 1$, *it follows that* $\lambda_i^{\widetilde{T}} \neq \lambda_j^{\widetilde{T}}$,

(v) *there is no eigenvalue* $\lambda$ *of* $A$, $\lambda \neq 1$, *such that* $\lambda^{\widetilde{T}} = 1$,

*then the pairs* $(\Phi(k), \Gamma(k))$ *and* $(\Phi(k), \Psi)$ *are controllable, and the pair* $(\Phi(k), \Upsilon)$ *is detectable.*

**Lemma 3** (see [10] (the Bounded Real Lemma) ). *For a given scalar $\gamma > 0$, if there is a matrix $X = X' > 0$ that satisfies the matrix inequality*

$$\begin{bmatrix} X & \widetilde{A}X & \widetilde{B} & 0 \\ X\widetilde{A}' & X & 0 & X\widetilde{C}' \\ \widetilde{B}' & 0 & \gamma I & \widetilde{D}' \\ 0 & \widetilde{C}X & \widetilde{D} & \gamma I \end{bmatrix} > 0, \tag{18}$$

*then the system* (16) *is asynchronously stable and it achieves a specified attenuation level $\gamma > 0$, such that $\|T_{zw}\|_\infty < \gamma$.*

## 3. Main Results

For the linear discrete asynchronous multirate sampled-data system (9), the exogenous parts of the system are lifted to improve its performance. To avoid multiplying the dimensions of the control inputs and measured outputs, we do not lift them, thereby avoiding causal constraints of the traditional lifting method.

If we assume that there is an underlying clock with a base period of $\tau$, then system (9) represents a fast discrete system with a subperiod of $\tau/m$ for the discrete process described in [9]. The sampler $S_a$ samples the continuous signal $w$ for a discrete $a$-period and the holder $H_b$ holds the output signal $z$ during period $b$. The rates of the sampler and the holder can be set at different flexible rates; that is, $a \neq b$. If we select (13) as the new input, $K(k)$ is the controller of system (9).

These concepts are described in Figure 3, where

$$m := \text{l.c.m.}\,\{a, b\}\,. \tag{19}$$

**Theorem 4.** *Consider the sampled-data system* (9) *with the lifting process shown in Figure 3. For a given attenuation level $\gamma > 0$, if there exist matrices $X_1(k) = X_1'(k) > 0$, $X_2(k) = X_2'(k) > 0$, $K(k)$, and $L(k)$ ($k = 0, \ldots, N$ denotes the iteration time), which are suboptimal solutions of the inequalities below, then the closed system of* (9) *with the controller* (13) *is asynchronously stable and satisfies the $H_\infty$ norm constraint $\|T_{zw}\|_\infty < \gamma$:*

$$\begin{bmatrix} X_1(k) & 0 & A_d + B_{2d}K(k) & -B_{2d}K(k)\begin{bmatrix} I_2 \\ 0 \end{bmatrix} & \underline{B}_1 & 0 \\ * & X_2(k) & 0 & A - L(k)S(k)C & (M_1 - L(k)S(k)M_2)\,[I_l \;\; 0 \;\; \cdots \;\; 0]_m & 0 \\ * & * & X_1^{-1}(k) & 0 & 0 & \underline{C}_1' + K'(k)\,\underline{D}_{12}' \\ * & * & * & X_2^{-1}(k) & 0 & -[I_2 \;\; 0]\,K'(k)\,\underline{D}_{12}' \\ * & * & * & * & \gamma I & \underline{D}_{11}' \\ * & * & * & * & * & \gamma I \end{bmatrix} > 0. \tag{20}$$

*Proof.* After lifting system (9) according to Figure 3, the transform function of a completely discrete system with a period of $\tau/m$, which is partly contained within the dot-dash line, is

$$\begin{aligned} \underline{G} &= \begin{bmatrix} L_m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S_f & 0 \\ 0 & S(k) \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} H_f & 0 \\ 0 & H(k) \end{bmatrix} \begin{bmatrix} L_m^{-1} & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} L_m S_f G_{11} H_f L_m^{-1} & L_m S_f G_{12} H(k) \\ S(k) G_{21} H_f L_m^{-1} & S(k) G_{22} H(k) \end{bmatrix}. \end{aligned} \tag{21}$$

Based on the above, we can derive minimal state-space representations $\widehat{g}(\lambda)$ for $\underline{G}$ as follows based on the transfer function theory given in [9, 15]. However, in contrast to the theory, $H$ and $S$ are not blocks of unit matrices because sampling and holding are not synchronous; that is, the entire signal cannot be sampled or held at the same time:

$$\widehat{g}(\lambda) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{22}$$

Next, we calculate the minimal state-space realization of (22) as follows.

*(a) Transfer Function for $P_{11}$.* Note that $P_{11} = L_m S_f G_{11} H_f L_m^{-1}$, which comes directly from the theory in [8], and its corresponding state model is

$$\widehat{P}_{11}(\lambda) = \left[\begin{array}{c|c} A_d & \underline{B}_1 \\ \hline \underline{C}_1 & \underline{D}_{11} \end{array}\right], \tag{23}$$

where

$$A_d := \begin{bmatrix} A^m & \sum_{j=1}^{m} A^{m-j} B\,(I - H(k)) \\ 0 & I - H(k) \end{bmatrix},$$

$$\underline{B}_1 := \begin{bmatrix} A^{m-1}M_1 & A^{m-2}M_1 & \cdots & M_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\underline{C}_1 := \begin{bmatrix} C & 0 \\ CA & 0 \\ \vdots & \vdots \\ CA^{m-1} & 0 \end{bmatrix},$$

$$\underline{D}_{11} := \begin{bmatrix} M_2 & 0 & \cdots & 0 \\ CM_1 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{m-2}M_1 & CA^{m-3}M_1 & \cdots & M_2 \end{bmatrix}. \tag{24}$$

*(b) Transfer Function for $P_{12}$.* As $H(k)$ in $P_{12} = L_m S_f G_{12} H(k)$ is not an $m$-blocks unit matrix, the theory cannot be applied directly. However, note that

$$S_f H(k) = L_m^{-1} \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}_m H(k). \tag{25}$$

Then,

$$P_{12} = L_m S_f G_{12} H_f L_m^{-1} \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}_m H(k). \tag{26}$$

The corresponding state model can be obtained as

$$\widehat{P}_{12}(\lambda) = \left[\begin{array}{c|c} A_d & B_{2d} \\ \hline \underline{C}_1 & \underline{D}_{12} \end{array}\right], \tag{27}$$

where $A_d$ and $\underline{C}_1$ have been defined previously and

$$B_{2d} := \begin{bmatrix} \sum_{i=1}^{m} A^{m-i} B \\ I \end{bmatrix} H(k),$$

$$\underline{D}_{12} := \begin{bmatrix} 0 \\ 0 + \sum_{j=m}^{m} CA^{m-j} B \\ \vdots \\ 0 + \sum_{j=2}^{m} CA^{m-j} B \end{bmatrix} H(k). \tag{28}$$

*(c) Transfer Function for $P_{21}$.* Similarly, because $S(k)$ is also not an $m$-blocks unit matrix, $P_{21}$ can be changed to

$$P_{21} = S(k) \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}_m L_m S_f G_{21} H_f L_m^{-1}. \tag{29}$$

Thus, the state model can be derived as

$$\widehat{P}_{21}(\lambda) = \left[\begin{array}{c|c} A_d & \underline{B}_1 \\ \hline C_2 & \underline{D}_{21} \end{array}\right], \tag{30}$$

where $A_d$ and $\underline{B}_1$ were defined previously and

$$C_2 := \Delta(k),$$
$$\underline{D}_{21} := \begin{bmatrix} \Omega(k) & 0 & \cdots & 0 \end{bmatrix}. \tag{31}$$

*(d) Transfer Function for $P_{22}$.* Like to parts $b$ and $c$, the state description of $P_{22}$ is

$$\widehat{P}_{22}(\lambda) = \left[\begin{array}{c|c} A_d & B_{2d} \\ \hline C_2 & D_{22} \end{array}\right], \tag{32}$$

where $A_d$, $B_{2d}$, and $C_2$ were defined previously, while

$$D_{22} := 0. \tag{33}$$

Finally, formula (23) is represented using the state-space form as follows:

$$\begin{aligned} \xi(k+1) &= A_d \xi(k) + \underline{B}_1 \underline{w}(k) + B_{2d} r(k), \\ \underline{z}(k) &= \underline{C}_1 \xi(k) + \underline{D}_{11} \underline{w}(k) + \underline{D}_{12} r(k), \\ y(k) &= C_2 \xi(k) + \underline{D}_{21} \underline{w}(k), \end{aligned} \tag{34}$$

where the underlining below $z$ and $w$ indicates that the signals are lifted.

According to Lemma 2, the controller $K(k)$ exists and can stabilize the system.

Let $\wp(k) := \begin{bmatrix} \xi'(k) & e'(k) \end{bmatrix}'$, and after combining (13), (15), and (34), we obtain

$$\begin{aligned} \wp(k+1) &= A_c \wp(k) + B_c \underline{w}(k), \\ \underline{z}(k) &= C_c \wp(k) + D_c \underline{w}(k), \end{aligned} \tag{35}$$

where

$$\begin{aligned} A_c &:= \begin{bmatrix} A_d + B_{2d} K(k) & -B_{2d} K(k) \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \\ 0 & A - L(k) S(k) C \end{bmatrix}, \\ B_c &:= \begin{bmatrix} \underline{B}_1 \\ (M_1 - L(k) S(k) M_2) \begin{bmatrix} I_l & 0 & \cdots & 0 \end{bmatrix}_m \end{bmatrix}, \\ C_c &:= \begin{bmatrix} \underline{C}_1 + \underline{D}_{12} K(k) & -\underline{D}_{12} K(k) \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \end{bmatrix}, \\ D_c &:= \underline{D}_{11}. \end{aligned} \tag{36}$$

In the light of [5], we set

$$X(k) := \begin{bmatrix} X_1(k) & 0 \\ 0 & X_2(k) \end{bmatrix}. \tag{37}$$

By substituting $X$ in Lemma 3 with $X(k)$, we can obtain

$$\begin{bmatrix} X_1(k) & 0 & (A_d + B_{2d}K(k))X_1(k) & -B_{2d}K(k)\begin{bmatrix} I_2 \\ 0 \end{bmatrix} X_2(k) & \underline{B}_1 & 0 \\ * & X_2(k) & 0 & (A - L(k)S(k)C)X_2(k) & (M_1 - L(k)S(k)M_2)[I_l \ \ 0 \ \cdots \ 0]_m & 0 \\ * & * & X_1(k) & 0 & 0 & X_1(k)(\underline{C}_1' + K'(k)\underline{D}_{12}') \\ * & * & * & X_2(k) & 0 & -X_2(k)[I_2 \ \ 0]K'(k)\underline{D}_{12}' \\ * & * & * & * & \gamma I & \underline{D}_{11}' \\ * & * & * & * & * & \gamma I \end{bmatrix} > 0. \tag{38}$$

Then, according to [26], after multiplying the left- and right-hand sides of inequality (38) by $\operatorname{diag}\{I, I, X_1^{-1}(k), X_2^{-1}(k), I, I\}$, we can obtain inequality (20).

According to quadratic stability theory, the system should be stable at each step $k$. Thus, for a discrete linear time-invariant system, the eigenvalues of the closed system should lie within a unit circle.

From Theorem 4, if matrix inequalities (20) can be solved, there exists a state feedback $H_\infty$ controller that can asymptotically stabilize closed system (35) of enlarged system (34), which satisfies the performance index. Since system (34) is lifted from the original system (9) and we know that the lifting operator is an isometric isomorphism, that is, the feedback stability and system norm can be preserved, then system (16), which is the closed system of (9), is asynchronously stable and it satisfies the $H_\infty$ norm constraint $\|T_{zw}\|_\infty < \gamma$. □

The synchronous case can be regarded as a special example of the asynchronous case. In this case, the elements of sampling and holder are unit matrices with the assumption that the whole signals of sampling and holding can be obtained at the beginning of every period; then the system (9) turns into a linear discrete synchronous multirate sampled-data system with period $\tau$ as follows:

$$\begin{aligned} \xi(k+1) &= \Phi\xi(k) + \Gamma r(k) + \Psi w(k), \\ z(k) &= \Upsilon\xi(k) + M_2 w(k), \\ y(k) &= \Delta\xi(k) + \Omega w(k). \end{aligned} \tag{39}$$

As $H$ and $S$ are unit matrices, that is, constant now, $\Phi$, $\Gamma$, $\Delta$, and $\Omega$ related to $H$ and $S$ are also constant. Theorem 4 can be applied to system (39) straightforwardly.

**Corollary 5.** *Consider the sampled-data system* (39) *with the lifting process shown in Figure 3. For a given attenuation level $\gamma > 0$, if there exist matrices $X_1 = X_1' > 0$, $X_2 = X_2' > 0$, $K$, and $L$, which are suboptimal solutions of the inequalities below, then the closed system of* (39) *with the controller* (13) *is synchronously uniformly stable and it satisfies the $H_\infty$ norm constraint $\|T_{zw}\|_\infty < \gamma$:*

$$\begin{bmatrix} X_1 & 0 & A_d + B_{2d}K & -B_{2d}K\begin{bmatrix} I_2 \\ 0 \end{bmatrix} & \underline{B}_1 & 0 \\ * & X_2 & 0 & A - LC & (M_1 - LM_2)[I_l \ \ 0 \ \cdots \ 0]_m & 0 \\ * & * & X_1^{-1} & 0 & 0 & \underline{C}_1' + K'\underline{D}_{12}' \\ * & * & * & X_2^{-1} & 0 & -[I_2 \ \ 0]K'\underline{D}_{12}' \\ * & * & * & * & \gamma I & \underline{D}_{11}' \\ * & * & * & * & * & \gamma I \end{bmatrix} > 0. \tag{40}$$

*Remark 6.* Let us compare the designed observer (12) with that in [21], where the observer (predictor) is

$$\begin{aligned} \hat{z}(k+1 \mid k) &= \overline{A}\hat{z}(k \mid k-1) + \overline{B}u(k) \\ &\quad + L(k)\cdot\left[\overline{\zeta}(k) - \overline{C}(k)\hat{z}(k \mid k-1)\right], \end{aligned}$$

$$\overline{\zeta}(k) = \sum_{i=1}^{\overline{T}^*} y(k-i), \quad \overline{T}^* = \max_{i=1,\ldots,p}\left\{\overline{T}_i\right\}. \tag{41}$$

The sum of outputs is used to avoid the singularity but may cause more noise. The observer applied in this paper is

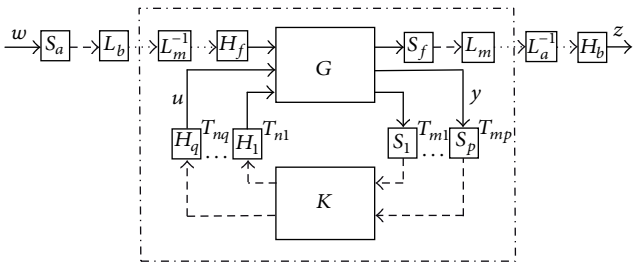


FIGURE 3: System lifting and controller designing.

simpler; that is, just the current output is introduced in the observer. This will result in better tracking performance and smaller state ripple than [21], where the past outputs other than current output are used for feedback.

*Remark 7.* In Theorem 4 and Corollary 5, due to the existence of inverse matrices $X_1^{-1}$ and $X_2^{-1}$, inequalities (20) and (40) are not LMIs and cannot be directly solved by the LMI tool. However, notice that the matrices and their inverse matrices appear in pairs; we can solve the inequalities by iteration on LMIs according to Cone Complementarity Linearization Algorithm [22].

## 4. Numerical Example with Simulation

Consider the original plant described by

$$\begin{aligned} \dot{x}^c(t) &= A^c x^c(t) + B^c u^c(t) + M_1^c w^c(t), \\ z^c(t) &= C^c x^c(t) + M_2^c w^c(t), \end{aligned} \tag{42}$$

where

$$\begin{gathered} A^c = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \qquad B^c = \begin{bmatrix} 0 & 1 \\ 1 & 1.2 \end{bmatrix}, \qquad M_1^c = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \\ C^c = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \qquad M_2^c = 0.01. \end{gathered} \tag{43}$$

If the fast discretization formulae follow $A = e^{A^c(\tau/m)}$, $B = \int_0^{\tau/m} e^{A^c\sigma} d\sigma B^c$, $M_1 = \int_0^{\tau/m} e^{A^c\sigma} d\sigma M_1^c$, $C = C^c$, and $M_2 = M_2^c$, then system (2) can be obtained. The initial state is $x_0 = [30 \;\; 20]$.

If $\widetilde{T}_1 = \widetilde{T}_2 = 2$, the input-holding mechanism can be described as

$$H(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad H(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{44}$$

If $\overline{T}_1 = 3$ and $\overline{T}_2 = 1$, the output-sampling matrices are assumed to be

$$S(0) = 1, \qquad S(1) = 0, \qquad S(2) = 0. \tag{45}$$

Therefore, this is a linear discrete asynchronous dual-rate sampled-data system [18], which applies a fast-input-slow-output mechanism for $\widetilde{T} < \overline{T}$.

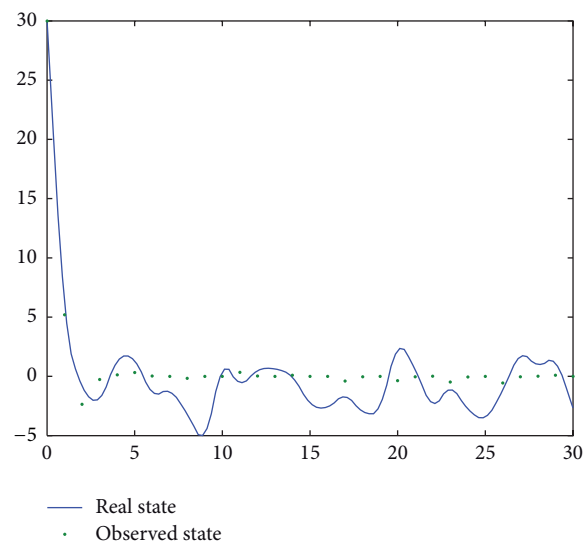


FIGURE 4: Plant state $x_1(k)$.

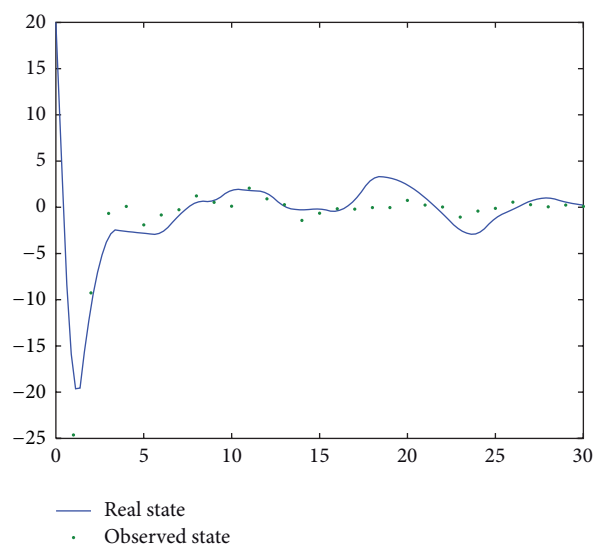


FIGURE 5: Plant state $x_2(k)$.

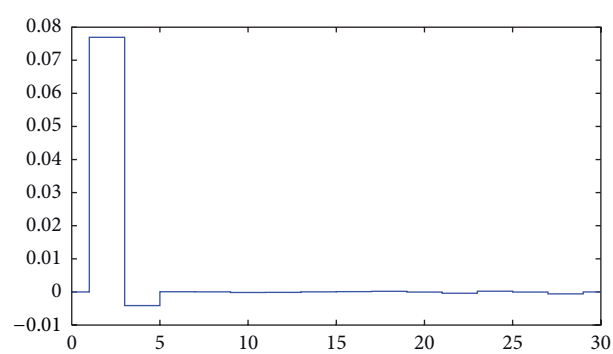


FIGURE 6: Control signal $u_1(k)$.

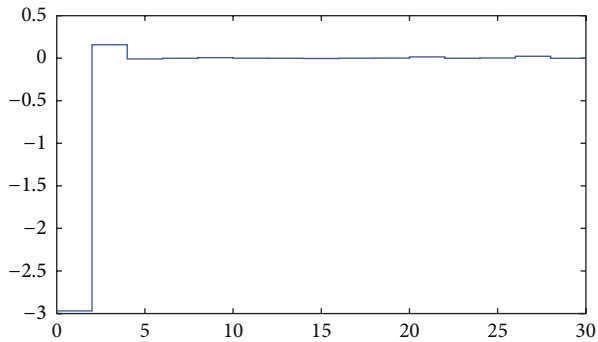


FIGURE 7: Control signal $u_2(k)$.

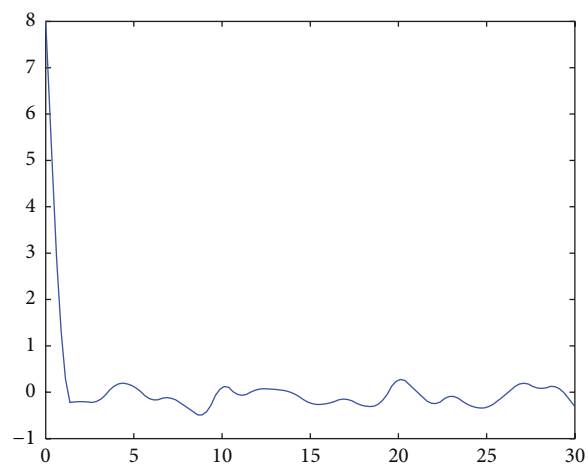


FIGURE 8: Output.

For comparison, we introduce a previous method [21], where the system is not lifted according to the pure LQG theory, and also let $w_1 = w_2 = w$ be Gaussian white noise; amplitude is increased to 10, which is much bigger to the system. The results obtained using this method are shown in Figures 4, 5, 6, 7, and 8, where $\tau = 1.2$ s.

Next, the synthesis method described in Theorem 4 is applied to the same plant. The simulations obtain the transients of the system shown in Figures 9, 10, 11, 12, and 13, where $\gamma = 0.2$, $a = 1$, and $b = 1$, while $m = 1$ (monorate).

A comparison of Figures 4–8 with Figures 9–13 shows clearly that the $H_\infty$ method performs better than LQG; the controlled states are consistent with the observed states with smaller ripples, especially state $x_2(k)$, the biggest ripple of which is almost half of the LQG method. And the output of $H_\infty$ method has better performance of disturbance attenuation than LQG method.

When $\tau = 0.6$ s and $m = 1$ (monorate), applying Theorem 4 again, we can get Figures 14, 15, 16, 17, and 18. Then the lifting number is modified to $m = 2$ (multirate); the results shown in Figures 19, 20, 21, 22, and 23 are obtained.

Comparing Figures 14–18 with Figures 19–23, when the lifting amplitude increases from 1 to 2, the controlled states are almost consistent with the observed states with smaller ripples; here the biggest ripple of state $x_2(k)$ is almost half of

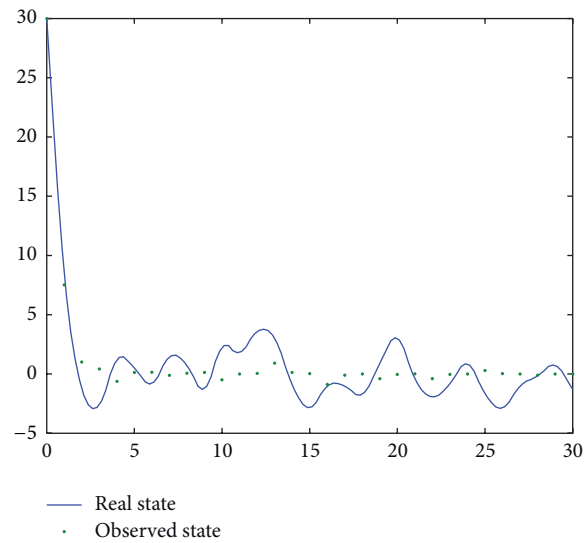


FIGURE 9: Plant state $x_1(k)$.

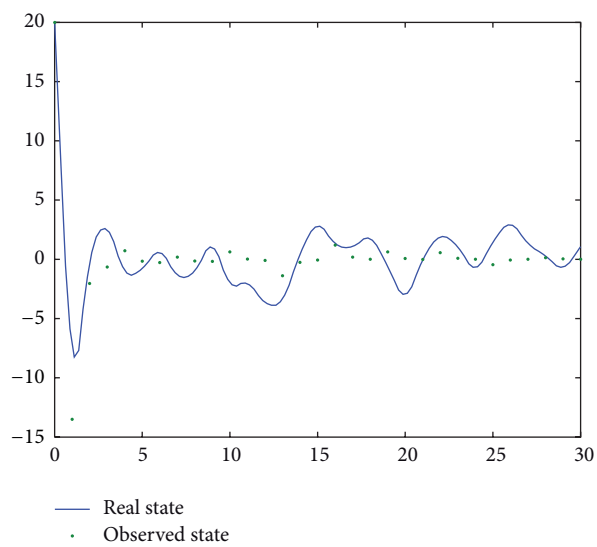


FIGURE 10: Plant state $x_2(k)$.

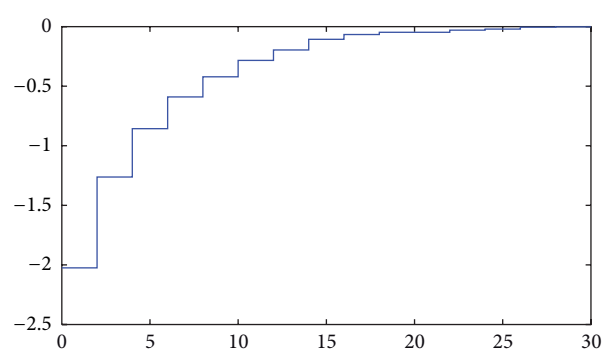


FIGURE 11: Plant control signal $u_1(k)$.

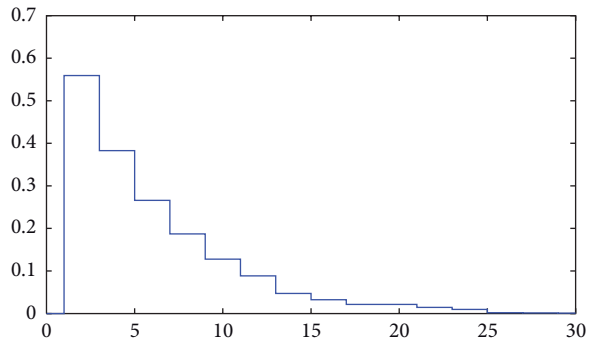


FIGURE 12: Plant control signal $u_2(k)$.

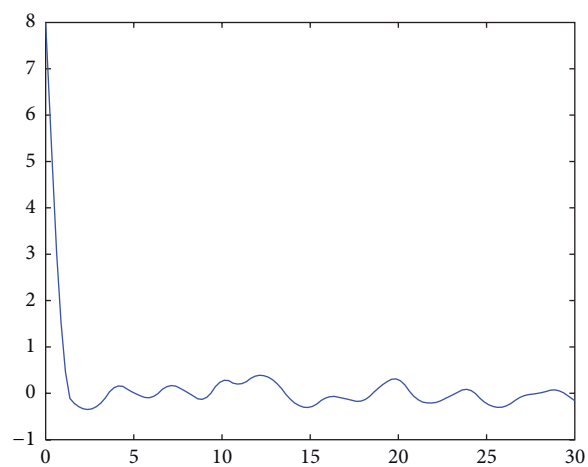


FIGURE 13: Output.

TABLE 1: Minimum $H_\infty$ attenuation level.

| $\tau$ | $m = 1$ | $m = 2$ | $m = 4$ |
|---|---|---|---|
| 1.2 sec | 0.0974 | 0.0959 | 0.0942 |
| 0.6 sec | 0.0904 | 0.0863 | 0.0856 |

the monorate. And the output of the multirate case has better performance of disturbance attenuation than the monorate case.

Furthermore, we change the lifting amplitude to $m = 4$, adjust the underlying period of system (2) to $\tau = 0.6$ and can draw the similar pictures; here we omit them but only list their minimum $H_\infty$ attenuation levels to compare them with different cases; see Table 1.

The inequality acquired using the theorem is almost linear, except in a few terms. Although these terms are just the inverses of their corresponding terms, they could easily be calculated by LMI tool of Matlab under a constraint. Thus, it is far easier to solve the controller and observer under the same $H_\infty$ constraint comparing with most of the existing methods, which use highly complex formulae.

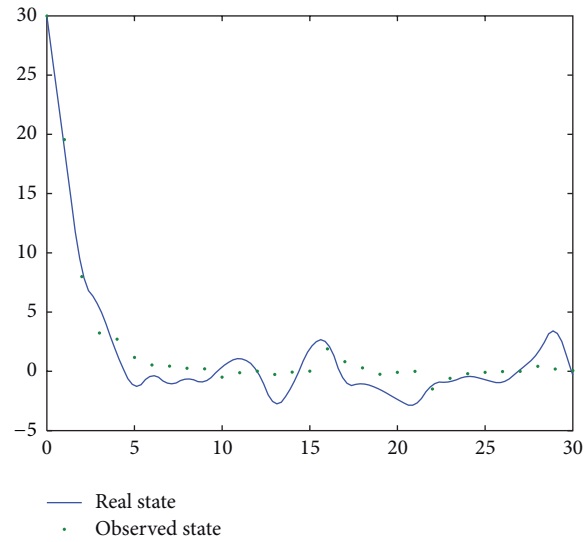


FIGURE 14: Plant state $x_1(k)$.

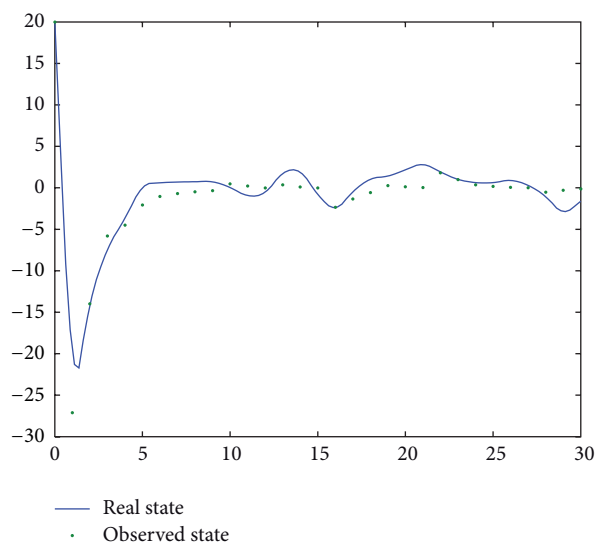


FIGURE 15: Plant state $x_2(k)$.

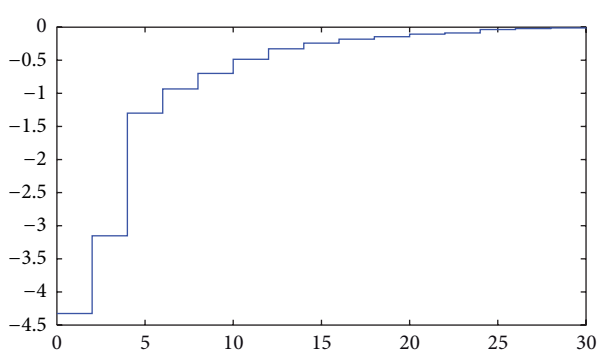


FIGURE 16: Plant control signal $u_1(k)$.

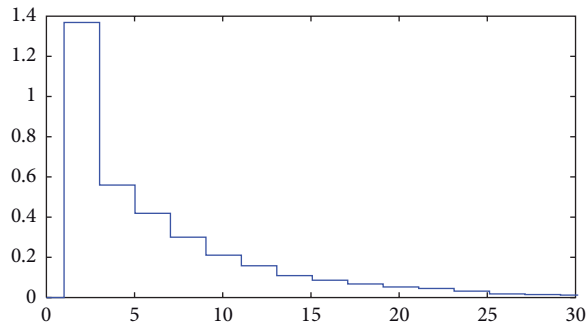


FIGURE 17: Plant control signal $u_2(k)$.

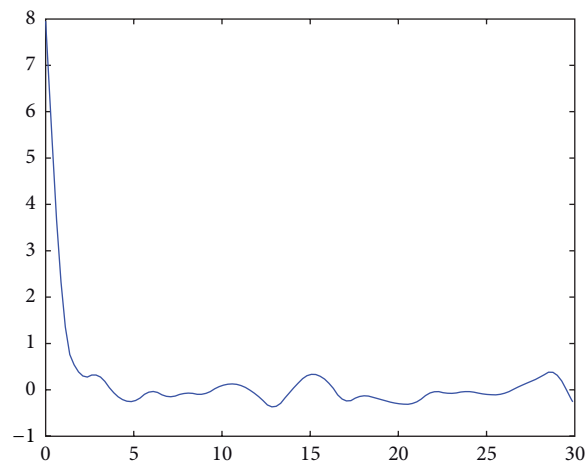


FIGURE 18: Output.

## 5. Conclusion

Motivated by a desire to provide a better solution for the design of a controller for a discrete-time asynchronous multirate sampled-data system, we developed synthesis methods in this paper to improve performance and guarantee closed-loop transient behavior via observer-based $H_\infty$ control. Furthermore, an extended lifting technique was adopted to change the original system into a fast-input-slow-output discrete system, thereby improving its performance. Finally, we presented an example that demonstrates the effectiveness of these methods in controlling a discrete-time asynchronous multirate sampled-data system, whose performance is thus improved considerably.

In this paper, we focus on the standard $H_\infty$ control of asynchronous multirate sampled-data system, and the input noise discussed here is Gaussian white for simplicity. As is well known, practical input signals often have finite frequency (FF), so it is reasonable to consider the $H_\infty$ performance in finite frequency range for sampled-data systems. As in [27–31], where Kalman-Yakubovic-Popov (KYP) lemma is used to establish the equivalence between a frequency domain inequality (FDI) and a linear matrix inequality, it is a valuable work to extend the results in this paper to the FF framework based on KYP lemma. This will be the direction for future development of this paper.

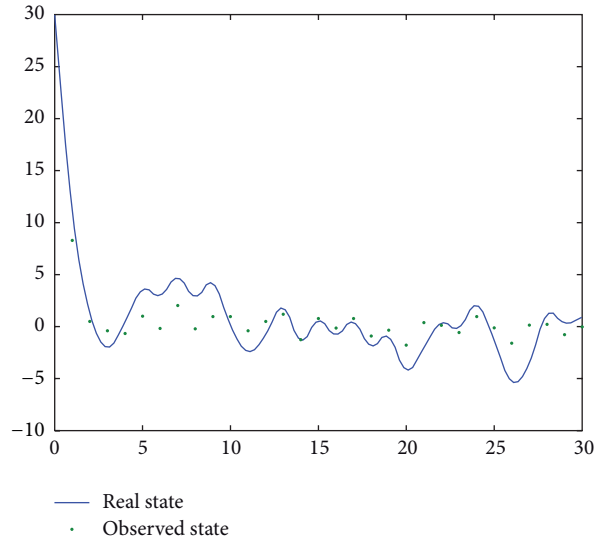


FIGURE 19: Plant state $x_1(k)$.

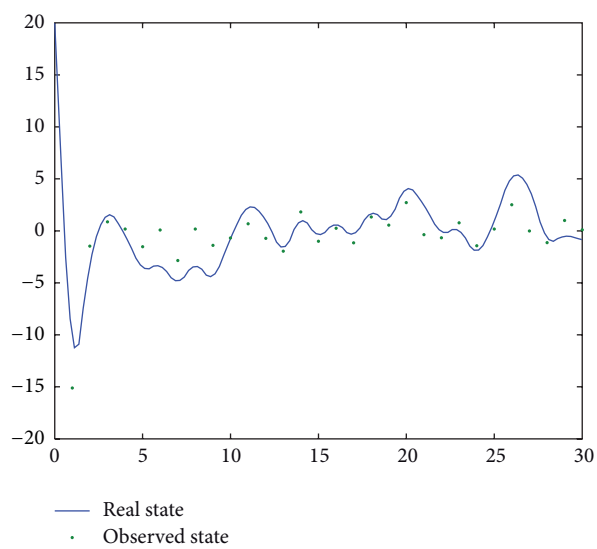


FIGURE 20: Plant state $x_2(k)$.

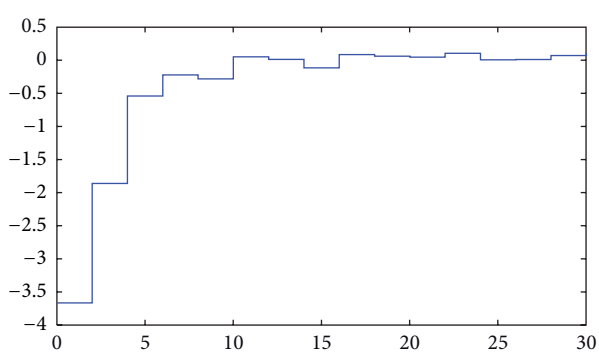


FIGURE 21: Plant control signal $u_1(k)$.

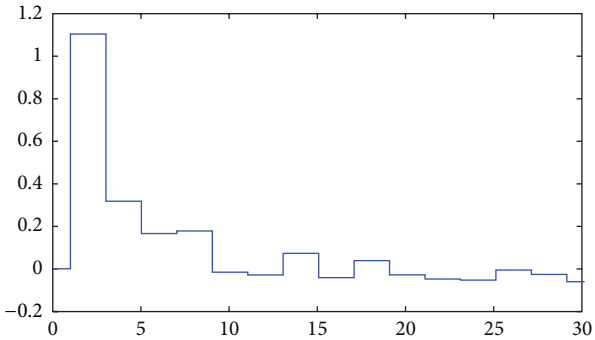


Figure 22: Plant control signal $u_2(k)$.

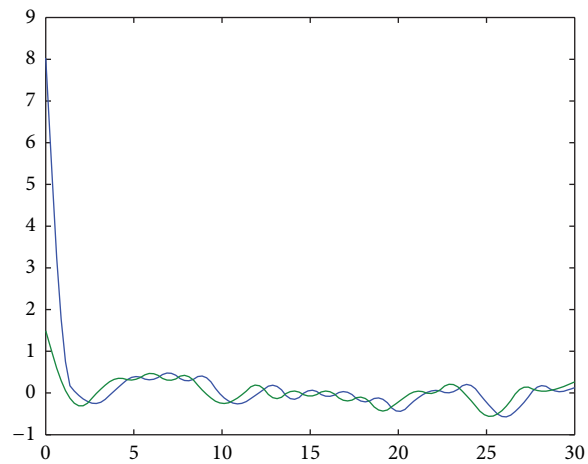


Figure 23: Output.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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