A Less Conservative Stability Criterion for Delayed Stochastic Genetic Regulatory Networks

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This paper concerns the problem of stability analysis for delayed stochastic genetic regulatory networks. By introducing an appropriate Lyapunov-Krasovskii functional and employing delay-range partition approach, a new stability criterion is given to ensure the mean square stability of genetic regulatory networks with time-varying delays and stochastic disturbances. The stability criterion is given in the form of linear matrix inequalities, which can be easily tested by the LMI Toolbox of MATLAB. Moreover, it is theoretically shown that the obtained stability criterion is less conservative than the one in W. Zhang et al., 2012. Finally, a numerical example is presented to illustrate our theory.

1. Introduction

With the further progress of gene expression, researchers find that a gene expression is affected by other genes; conversely, it also influences others. Based on this reciprocal impact relation, gene expression forms a complex network—genetic regulatory network (GRN). GRNs are dynamical systems, which consist of an interaction of genes, proteins, and small molecules. In the past two decades, scholars have established mathematical models to represent GRNs. Basically, there are four types of GRN models, that is, Petri net model [1], Bayesian network model [2], Boolean model [3, 4], and (functional) differential equation model [5, 6]. The concentrations of mRNA and protein are described as the state variables in the functional differential equation model.

As dynamical systems, stability analysis is the first priority to explore GRNs. On the one hand, time delay inevitably occurs in GRNs due to the slow process of transcription, translation, and translocation [7]. On the other hand, internal noises of cells caused by random birth and death of the individual molecules and external noises from environmental fluctuations make the gene expression be best viewed as a stochastic process [8, 9]. So, it is very necessary to analyze the stability of GRNs with time-varying delays and stochastic disturbances [6, 10–18].

Recently, for a class of GRNs with interval time-varying delays and stochastic disturbances (see (17a), (17b) below), Wu et al. [18] established several delay-range-dependent and/or rate-dependent global stochastic asymptotical stability criteria in terms of linear matrix inequalities (LMIs) by using the stochastic analysis approach, employing some free-weighting matrices and introducing a type of Lyapunov-Krasovskii functional which includes the items like \( \int_{-\tau_1}^{t-\tau_2} \int_{\theta}^{t} h_1^T(s)P h_1(s) \, ds \, d\theta \) and \( \int_{-\tau_1}^{t-\tau_2} \int_{\theta}^{t} h_3^T(s)Q h_3(s) \, ds \, d\theta \), where \( P \) and \( Q \) are real symmetric positive definite matrices. Furthermore, in [18], the item \( \int_{-\tau_1}^{t-\tau_2} \int_{\theta}^{t} h_1^T(s)P h_1(s) \, ds \) in the stochastic differential of \( \int_{-\tau_1}^{t-\tau_2} \int_{\theta}^{t} h_1^T(s)P h_1(s) \, ds \) was first estimated by employing Leibniz-Newton formula and the inequality

\[
-2\xi^T(t) M \int_{t-\tau(t)}^{t-\tau_2} h_1(s) \, ds \\
\leq (\tau(t) - \tau_1) \xi^T(t) M P^{-1} M^T \xi(t) + \int_{t-\tau(t)}^{t-\tau_1} h_1^T(s) P h_1(s) \, ds,
\]
where $M$ is a free-weighting matrix, and then $\tau(t) - \tau_1$ was enlarged to $\tau_2 - \tau_1$. Clearly, the conservatism would be produced because $\tau(t) - \tau_1$ is enlarged to $\tau_2 - \tau_1$. In order to reduce the conservatism, Zhang et al. [17] first estimated the item $\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s) P h_1(s) \, ds$ by using Leibniz-Newton formula and the inequality

$$-2k^T(t) S \int_{t-\tau(t)}^{t-\tau_1} h_1(s) \, ds$$

$$\leq (1-\alpha) (\tau(t) - \tau_1) k^T(t) S P^{-1} S^T k(t)$$

$$+ \alpha (\tau(t) - \tau_1) k^T(t) N P^{-1} N^T k(t)$$

$$+ \int_{t-\tau(t)}^{t-\tau_1} h_1^T(s) P h_1(s) \, ds,$$

where $S$ and $N$ are free-weighting matrices and $\alpha$ is an adjusting parameter with $0 < \alpha < 1$, and then a so-called convex combination technique was employed to obtain less conservative delay-range-dependent and/or rate-dependent global stochastic asymptotical stability criteria. It should be emphasized that the Lyapunov-Krasovskii functional used in [17] includes not only the items like $\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s) P h_1(s) \, ds$ and $\int_{t-\tau(t)}^{t-\tau_1} \int_{t-\tau_1}^{t} h_1^T(s) Q h_1(s) \, ds \, d\theta$ but also the items like $\int_{t-\tau(t)}^{t-\tau_1} \int_{t-\tau_1}^{t} x^T(s) Q x(s) \, ds \, d\theta$ and $\int_{t-\tau(t)}^{t-\tau_1} \int_{t-\tau_1}^{t} x^T(s) Q x(s) \, ds$ involved in the adjustable parameter $\alpha$, which can be viewed as a generalized delay-partition approach due to the adjustable parameter $\alpha$. Generally, an appropriate delay-partition approach can bring less conservative stability criteria (see [19] and the references therein).

Note that free-weighting matrices (e.g., matrices $M$, $S$, and $N$ above) have been employed in both [17, 18] to obtain less conservative stability criteria. However, it is emphasized in [18, Remark 2] that free-weighting matrices may produce a super-high amount of computation for the feasible solutions of LMIs. In order to overcome the disadvantage in this paper, we will propose a delay-range partition (DRP) approach to estimate accurately the item $\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s) P h_1(s) \, ds$ (see (27) and (30)), where no free-weighting matrix is involved. By employing an appropriate Lyapunov-Krasovskii functional and introducing a DRP approach, a mean square stability criterion for GRNs with time-varying delays and stochastic disturbances is first established. Then it is theoretically shown that the proposed stability criterion is less conservative than [17, Theorem 1]. Finally, a numerical example is given to illustrate the theoretical results proposed here. The main contribution of this paper can be listed as follows: (i) the Lyapunov-Krasovskii functional employed in this paper does not include the items like $\int_{t-\tau(t)}^{t-\tau_1} \int_{t-\tau_1}^{t} h_1^T(s) Q h_1(s) \, ds \, d\theta$ which are required in [17, 18]; (ii) the items like $\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s) P h_1(s) \, ds$ in the stochastic differential of Lyapunov-Krasovskii functional are estimated accurately by proposing a DRP approach; (iii) theoretical comparison of the stability criterion [17, Theorem 1] and the one proposed in this paper is given; and (iv) there is no free-weighting matrix involved, which reduces the computational complexity.

The rest of the paper is organized as follows. In Section 2, the model of GRNs to be studied is described. A DRP-based mean square stability criterion (Theorem 3 below) for GRNs with time-varying delays and stochastic disturbances is established in Section 3. The theoretical comparison of Theorem 3 and [17, Theorem 1] is presented in Section 4. In Section 5, an example is given to show the validity of the obtained results. Finally, in Section 6, the conclusions are drawn.

**Notation.** For a positive integer $n$, set $\langle n \rangle = \{1, 2, \ldots, n\}$. $R^n$ denotes the $n$-dimensional Euclidean space. We denote by $R^{m \times n}$ the set of all $m \times n$ matrices over $R$. $A^T$ and $A^{-1}$ represent the transpose and inverse of a matrix $A$, respectively. For real symmetric matrices $X$ and $Y$, the notation $X \succeq Y$ ($X > Y$) means that the matrix $X - Y$ is positive definite (positive semidefinite). $I_n$ is the $n \times n$ identity matrix. In a symmetric matrix, $*$ denotes the entries implied by symmetry.

### 2. Model Description

The following differential equations have been used recently to describe GRNs [7]:

$$\dot{m}_i(t) = -a_i m_i(t) + b_i \left( p_i(t - \sigma(t)) , p_1(t - \sigma(t)), \ldots, p_n(t - \sigma(t)) \right),$$

$$p_n(t - \sigma(t)),$$  \hspace{1cm} (3a)

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t - \tau(t)), \quad i \in \langle n \rangle,$$  \hspace{1cm} (3b)

where $m_i(t)$ and $p_i(t)$ are the concentrations of the $i$th mRNA and protein at time $t$, respectively; $\sigma(t)$, $d_i > 0$, $c_i > 0$, and $a_i > 0$ are constants, representing the degradation rate of the $i$th mRNA, the degradation rate of the $i$th protein, and the translation rate of the $i$th mRNA to $i$th protein, respectively; both $\sigma(t)$ and $\tau(t)$ are transcriptional and translational delays, respectively; $b_i$ is the regulatory function of the $i$th gene, which is generally a nonlinear function of the variables $p_i(t), p_1(t), \ldots, p_n(t)$, but it is monotonic with each variable.

For convenience, we give the following assumptions throughout the paper.

**Assumption 1.** The delays $\sigma(t)$ and $\tau(t)$ are differentiable functions satisfying

$$0 \leq \sigma_1 \leq \sigma(t) \leq \sigma_2, \quad 0 \leq \tau_1 \leq \tau(t) \leq \tau_2,$$  \hspace{1cm} (4a)

$$\dot{\sigma}(t) \leq \sigma_d < +\infty, \quad \dot{\tau}(t) \leq \tau_d < +\infty,$$  \hspace{1cm} (4b)

where $\sigma_1, \sigma_2, \sigma_d, \tau_1, \tau_2$, and $\tau_d$ are constants.

**Assumption 2.** The function $b_i$ is taken as

$$b_i \left( p_1(t), p_2(t), \ldots, p_n(t) \right) = \sum_{j=1}^{n} b_{ij} \left( p_j(t) \right),$$  \hspace{1cm} (5)
which is called SUM logic. Here, \( b_{ij} \) is a monotonic function of the Hill form; that is,

\[
b_{ij}(x) = \begin{cases} 
\frac{\alpha_{ij} (x/\beta_i)^{H_i}}{1 + (x/\beta_i)^{H_i}} & \text{if transcription factor } j \text{ is an activator of gene } i, \\
\frac{1}{1 + (x/\beta_i)^{H_i}} & \text{if transcription factor } j \text{ is a repressor of gene } i,
\end{cases}
\]

where \( H_i \) is the Hill coefficient, \( \beta_i \) is a scalar, and \( \alpha_{ij} \) is a bounded constant, which denotes the dimensionless transcriptional rate of transcription factor \( j \) to gene \( i \).

Clearly, GRN ((3a), (3b)) can be rewritten as

\[
m_i(t) = - \alpha_i m_i(t) + \sum_{j=1}^{n} w_{ij} h_j \left( p_j(t - \sigma(t)) \right) + \nu_i, \tag{7a}
\]

\[
\dot{p}_i(t) = - \epsilon_i p_i(t) + d_i m_i(t - \tau(t)), \quad i \in \{n\} \tag{7b}
\]

where

\[
w_{ij} = \begin{cases} 
\alpha_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i, \\
0 & \text{if there is no connection between } j \text{ and } i, \\
-\alpha_{ij} & \text{if transcription factor } j \text{ is a repressor of gene } i,
\end{cases}
\]

\[
h_j(x) = \frac{(x/\beta_i)^{H_i}}{1 + (x/\beta_i)^{H_i}}, \quad \nu_i = \sum_{j \in \mathcal{Y}_i} \alpha_{ij},
\]

and \( \mathcal{Y}_i \) is the set of all the transcription factors \( j \) which is a repressor of gene \( i \).

Rewriting GRN ((7a), (7b)) into compact matrix form, we obtain

\[
m(t) = -Am(t) + Wh(p(t - \sigma(t))) + \nu, \tag{9a}
\]

\[
\dot{p}(t) = -Cp(t) + Dm(t - \tau(t)), \tag{9b}
\]

where

\[
A = \text{diag}(a_1, a_2, \ldots, a_n),
\]

\[
W = [w_{ij}]_{n \times n},
\]

\[
C = \text{diag}(c_1, c_2, \ldots, c_n),
\]

\[
D = \text{diag}(d_1, d_2, \ldots, d_n),
\]

\[
\nu = \text{col}(\nu_1, \nu_2, \ldots, \nu_n),
\]

\[
m(t) = \text{col}(m_1(t), m_2(t), \ldots, m_n(t)),
\]

\[
p(t) = \text{col}(p_1(t), p_2(t), \ldots, p_n(t)),
\]

\[
h(p(t)) = \text{col}(h_1(p_1(t)), h_2(p_2(t)), \ldots, h_n(p_n(t))).
\]

Let \((m^*, p^*)\) be an equilibrium point of ((9a), (9b)); that is, \((m^*, p^*)\) is a solution of the following equation:

\[
-Am^* + Wh(p^*) + \nu = 0, \quad -Cp^* + Dm^* = 0. \tag{11}
\]

For convenience, we shift the equilibrium point \((m^*, p^*)\) to the origin by using the transformations \(x(t) = m(t) - m^*\) and \(y(t) = p(t) - p^*\); then we have

\[
x(t) = -Ax(t) + Wf(y(t - \sigma(t))), \tag{12a}
\]

\[
y(t) = -Cy(t) + Dx(t - \tau(t)), \tag{12b}
\]

where \(f(y(t)) = h(y(t) + \nu)\).

From the relationship between \(h\) and \(f\), one can easily find that \(f\) satisfies the following sector condition:

\[
f_i(0) = 0, \quad l^-_i \leq \frac{f_i(s)}{s} \leq l^+_i, \quad i \in \{n\}, \quad \forall 0 \neq s \in \mathbb{R}, \tag{13}
\]

where \(l^-_i\) and \(l^+_i\) are a pair of nonnegative scalars and \(f_i(s)\) is the \(i\)th entry of \(f(s)\). Since \(h_i\) is a monotonically increasing and differentiable function with saturation, we have to choose \(l^-_i\) as zero or a small positive number. Let \(L^- = \text{diag}(l^-_1, l^-_2, \ldots, l^-_n)\) and \(L^+ = \text{diag}(l^+_1, l^+_2, \ldots, l^+_n)\).

As shown in [15–17] the gene regulation is an intrinsically noisy process. For this reason, in this paper, we consider a class of GRNs with both time delays and noise disturbances by the following model:

\[
dx(t) = \left[-Ax(t) + Wf(y(t - \sigma(t)))\right] dt
\]

\[
+ H(x(t), x(t - \tau(t)), y(t), y(t - \sigma(t))) d\omega(t), \tag{14a}
\]

\[
dy(t) = \left[-Cy(t) + Dx(t - \tau(t))\right] dt
\]

\[
+ \nu^T(t - \sigma(t)) H_4 y(t - \sigma(t)), \tag{14b}
\]

where \(\omega(t)\) is an \(m\)-dimensional Brown motion, \(m \geq 1\), and \(H(x(t), x(t - \tau(t)), y(t), y(t - \sigma(t)))\) is the noise intensity matrix at time \(t\) such that

\[
\text{trace}(HH^T) \leq x^T(t) H_1 x(t) + y^T(t) H_2 y(t)
\]

\[
+ x^T(t - \tau(t)) H_3 x(t - \tau(t))
\]

\[
+ y^T(t - \sigma(t)) H_4 y(t - \sigma(t)),
\]

where \(H_i\) \((i = 1, 2, 3, 4)\) are real symmetric positive semidefinite matrices.

For simplicity, set

\[
h_1(t) = -Ax(t) + Wf(y(t - \sigma(t))),
\]

\[
h_2(t) = -Cy(t) + Dx(t - \tau(t)),
\]

\[
h_3(t) = H(x(t), x(t - \tau(t)), y(t), y(t - \sigma(t))).
\]

Then, GRN ((14a), (14b)) can be represented as

\[
dx(t) = h_1(t) dt + h_3(t) d\omega(t), \tag{17a}
\]

\[
dy(t) = h_2(t) dt. \tag{17b}
\]
3. Stability Criterion

In the following theorem, we will propose a DRP approach to present an asymptotical stability criterion in the mean square sense for GRNs with time-varying delays and stochastic disturbance.

**Theorem 3.** For given scalars \( \alpha \in (0, 1) \), \( \tau_2 > \tau_1 > 0 \), \( \sigma_2 > \sigma_1 > 0 \), \( \tau_d \) and \( \sigma_d \), and positive integers \( s_1 \) and \( s_2 \), under the conditions (15) and ((4a), (4b)), we can conclude that GRN ((14a), (14b)) is asymptotically stable in the sense of mean square, if there exist a scalar \( \rho > 0 \) and matrices \( U_i^T = U_i > 0 \) and \( W_j^T = W_j > 0 \) for given \( i = 1, 2, \ldots, 7 \), and \( \Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0 \) such that the following LMI holds:

\[
U_1 \leq \rho I, \quad \Psi_k : = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6
\]

where

\[
\Psi_1 = \tilde{\Psi}_1 + \tilde{\Psi}_1^T + \rho \left( e_t^T H_1 e_t + e_2^T H_2 e_2 + e_3^T H_3 e_3 + e_4^T H_4 e_4 \right),
\]

\[
\tilde{\Psi}_1 = e_t^T U_1 (-Ae_1 + We_1) + e_2^T U_2 (-Ce_2 + De_3),
\]

\[
\Psi_2 = e_1^T (V_1 + V_5) e_1 + e_4^T (V_2 - V_1) e_4 + e_5^T (V_3 - V_2) e_5
\]

\[
+ e_2^T \left[ (1 + \alpha \tau_1) V_4 - (1 - \alpha \tau_d) V_5 \right] e_2
\]

\[
- (1 - \tau_d) e_5^T V_4 e_3 - e_6^T V_5 e_6,
\]

\[
\Psi_3 = e_7^T (W_1 + W_2) e_7 + e_8^T (W_2 - W_1) e_8
\]

\[
+ e_9^T (W_3 - W_2) e_9
\]

\[
+ e_10^T \left[ (1 + \alpha \sigma_1) W_4 - (1 - \alpha \sigma_d) W_3 \right] e_8
\]

\[
- (1 - \sigma_d) e_7^T W_4 e_6 - e_11^T W_5 e_12,
\]

\[
\Psi_4 = (-Ae_1 + We_1)^T (\tau_1 V_6 + \tau_2 V_7) (-Ae_1 + We_1)
\]

\[
+ (-Ce_2 + De_3)^T (\sigma_2 W_6 + \sigma_1 W_2) (-Ce_2 + De_3),
\]

\[
\Psi_5 = \tilde{\Psi}_5 + \tilde{\Psi}_5^T,
\]

\[
\Psi_6 = -\frac{1}{\alpha \tau_1} (e_4 - e_1)^T V_6 (e_4 - e_1)
\]

\[
- \frac{1}{(1 - \alpha) \tau_1} (e_5 - e_2)^T V_6 (e_5 - e_4)
\]

\[
- \frac{1}{\alpha \sigma_1} (e_10 - e_7) W_6 (e_10 - e_7)
\]

\[
- \frac{1}{(1 - \alpha) \sigma_1} (e_{11} - e_{10})^T W_6 (e_{11} - e_{10}),
\]

\[
\Psi_k = -\frac{s_1}{\kappa \alpha \tau_1} (e_2 - e_5)^T V_7 (e_2 - e_5)
\]

\[
- \frac{s_1}{(1 - \alpha) \kappa \tau_1} (e_3 - e_2)^T V_7 (e_3 - e_2)
\]

\[
- \frac{s_1}{(s_1 - k + 1) \tau_1} (e_6 - e_3)^T V_7 (e_6 - e_3),
\]

\[
\Psi_i = -\frac{s_2}{\kappa \alpha \sigma_1} (e_8 - e_{11})^T W_7 (e_8 - e_{11})
\]

\[
- \frac{s_2}{(1 - \alpha) \kappa \sigma_1} (e_9 - e_8)^T W_7 (e_9 - e_8)
\]

\[
- \frac{s_2}{(s_2 - l + 1) \sigma_1} (e_{12} - e_9)^T W_7 (e_{12} - e_9),
\]

\[
e_i = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, n,
\]

\[
\sigma_{i+1} = \sigma_i - 1, \quad \tau_{i+1} = \tau_i - 1.
\]

**Proof.** Let \( \beta(t) = \tau_1 + \alpha (\tau(t) - \tau_1) \) and \( \gamma(t) = \sigma_1 + \alpha (\sigma(t) - \sigma_1) \).

Choose a Lyapunov-Krasovskii functional candidate as

\[
V(t) = \sum_{i=1}^{4} V_i(t),
\]

where

\[
V_1(t) = x^T(t) U_1 x(t) + y^T(t) U_2 y(t),
\]

\[
V_2(t) = \int_{t-\alpha \tau_1}^{t} x^T(s) V_1 x(s) \, ds + \int_{t-\tau_1}^{t} x^T(s) V_2 x(s) \, ds + \int_{t-\beta(\tau(t))}^{t} x^T(s) V_3 x(s) \, ds + \int_{t-\tau(t)}^{t} x^T(s) V_4 x(s) \, ds,
\]

\[
V_3(t) = \int_{t-\alpha \sigma_1}^{t} y^T(s) W_1 y(s) \, ds + \int_{t-\sigma_1}^{t} y^T(s) W_2 y(s) \, ds + \int_{t-\gamma(t)}^{t} y^T(s) W_3 y(s) \, ds + \int_{t-\sigma(t)}^{t} y^T(s) W_4 y(s) \, ds,
\]

\[
V_4(t) = \int_{t-\alpha \tau_2}^{t} y^T(s) W_5 y(s) \, ds + \int_{t-\tau_2}^{t} y^T(s) W_6 y(s) \, ds.
\]
\[ V_4(t) = \int_{t_{\tau_1}}^{t} h_1^T(s) V_2 h_1(s) \, ds \, d\theta + \int_{t_{\sigma_1}}^{t} h_2^T(s) V_2 h_2(s) \, ds \, d\theta + \int_{t_{\tau_1}}^{t} h_1^T(s) V_3 h_1(s) \, ds \, d\theta + \int_{t_{\sigma_1}}^{t} h_2^T(s) V_3 h_2(s) \, ds \, d\theta, \]

and the matrices \( U_j^T = U_j > 0 \) \( (i = 1, 2) \), \( V_j^T = V_j > 0 \), and \( W_j^T = W_j > 0 \) \( (j = 1, 2, \ldots, 7) \) are taken from a feasible solution to (18) and (19). By Itô's formula, we can obtain the following stochastic differential:

\[ dV(t) = \sum_{i=1}^{4} \mathcal{L} V_i(t) \, dt + 2x^T(t) U_i h_3(t) \, d\omega(t), \quad \mathcal{L} \]

where \( \mathcal{L} \) is the weak infinitesimal operator and

\[ \mathcal{L} V_1(t) = 2x^T(t) U_1 h_1(t) + 2y^T(t) U_2 h_2(t) + h_2^T(t) U_1 h_1(t) \leq \eta^T(t) \Psi_1 \eta(t), \]
\[ \mathcal{L} V_2(t) = x^T(t) (V_1 + V_3) x(t) + x^T(t - \alpha\tau_1) (V_2 - V_1) x(t - \alpha\tau_1) + x^T(t - \tau_1) (V_3 - V_2) x(t - \tau_1) + (1 - \beta(t)) x^T(t - \beta(t)) (V_4 - V_3) x(t - \beta(t)) - (1 - \gamma(t)) x^T(t - \tau(t)) (V_4 - V_3) x(t - \tau(t)) - x^T(t - \tau_2) V_3 x(t - \tau_2) \leq \eta^T(t) \Psi_2 \eta(t), \]
\[ \mathcal{L} V_3(t) = y^T(t) (W_1 + W_3) y(t) + y^T(t - \alpha\sigma_1) (W_2 - W_1) y(t - \alpha\sigma_1) + y^T(t - \sigma_1) (W_3 - W_2) y(t - \sigma_1) + (1 - \phi(t)) y^T(t - \phi(t)) (W_4 - W_3) y(t - \phi(t)) - (1 - \sigma(t)) y^T(t - \tau(t)) W_4 y(t - \sigma(t)) - y^T(t - \sigma_2) W_3 y(t - \sigma_2) \leq \eta^T(t) \Psi_3 \eta(t), \]
\[ \mathcal{L} V_4(t) = h_1^T(t) (\tau_1 V_6 + \tau_2 V_7) h_1(t) - \int_{t_{\tau_1}}^{t} h_1^T(s) V_2 h_1(s) \, ds - \int_{t_{\sigma_1}}^{t} h_2^T(s) V_2 h_2(s) \, ds + h_2^T(t) (\sigma_1 W_6 + \sigma_2 W_7) h_2(t) - \int_{t_{\tau_1}}^{t} h_2^T(s) W_2 h_2(s) \, ds - \int_{t_{\sigma_1}}^{t} h_2^T(s) W_2 h_2(s) \, ds \leq \eta^T(t) \Psi_4 \eta(t), \]

For any scalars \( a, b \) with \( a < b \), it follows from (16) that \( \int_{a}^{b} h_1(t) \, dt = x(b) - x(a) - \int_{a}^{b} h_3(t) \, d\omega(t) \) and \( \int_{a}^{b} h_2(t) \, dt = y(b) - y(a) \), and hence

\[ \mathbb{E} \left( \int_{a}^{b} h_1^T(t) v V_1 V_j h_1(t) \, dt \right) \geq [x(b) - x(a)]^T V_j [x(b) - x(a)], \quad j = 6, 7, \]
\[ \int_{a}^{b} h_2^T(t) v W_j h_2(t) \, dt = [y(b) - y(a)]^T W_j [y(b) - y(a)], \quad j = 6, 7. \]
where \( \mathcal{B} \) represents the mathematical expectation operator. Next, from the sector condition (13), we can obtain that

\[
0 \leq -2 \left[ (L^*)^{-1} f \left( y(t - \sigma(t)) \right) - y(t - \sigma(t)) \right]^T \\
\times A \left[ f \left( y(t - \sigma(t)) \right) - L^* y(t - \sigma(t)) \right] \\
= \eta(t)^T \Psi_{444}(t) \eta(t).
\]

When \( \tau(t) \in [\tau_1 + (k-1)/s_1, \tau_1 + k/s_1] \) for some positive integer \( k \in \{s_1\} \) and \( \sigma(t) \in [\sigma_1 + (l-1)/s_2, \sigma_1 + l/s_2] \) for some positive integer \( l \in \{s_2\} \), it is easy to see that

\[
\frac{1}{\beta(t) - \tau_1} \geq \frac{s_1}{k\alpha_1}, \quad \frac{1}{\tau(t) - \beta(t)} \geq \frac{s_1}{(1-\alpha)k\alpha_1}, \quad \frac{1}{\tau_2 - \tau(t)} \geq \frac{s_1}{(s_1 - k + 1)\alpha_1},
\]

\[
\frac{1}{\gamma(t) - \sigma_1} \geq \frac{s_2}{l\alpha_2}, \quad \frac{1}{\sigma(t) - \gamma(t)} \geq \frac{s_2}{(1-\alpha)l\alpha_2}, \quad \frac{1}{\sigma_2 - \sigma(t)} \geq \frac{s_2}{(s_2 - l + 1)\alpha_2}.
\]

Then, the combination of (21)–(31) gives

\[
\mathcal{V}(t) \leq \eta(t)^T \Psi_{444}(t) \eta(t),
\]

where

\[
\eta(t) = \begin{bmatrix} x(t), x(t - \beta(t)), x(t - \tau(t)), \\
x(t - \sigma_1), x(t - \tau_1), x(t - \tau_2), \\
y(t), y(t - \gamma(t)), y(t - \sigma(t)), y(t - \alpha_1), \\
y(t - \sigma_1), y(t - \sigma_2), f(y(t - \sigma(t))) \end{bmatrix}.
\]

Due to (19), we have \( \mathcal{V}(t) < 0 \), and hence GRN ((14a), (14b)) is asymptotically stable in the mean square sense.

Remark 4. In the above theorem a DRP approach has been proposed to establish an asymptotic mean square stability criterion for GRN ((14a), (14b)). Both DRP approach and the so-called piecewise analysis method (see, e.g., [20]) divide the delay-varying intervals into some parts with equal length. Then DRP approach enlarges the expectations of weak infinitesimal operator of the same Lyapunov-Krasovskii functional in every subinterval, while the piecewise analysis method constructs different Lyapunov-Krasovskii functional in every subinterval.

Remark 5. Comparing with the Lyapunov-Krasovskii functionals employed in Theorem 3 and [17, Theorem 1], we remove the items \( \int_{-\tau_1}^{\tau_1} \int_{0}^{\sigma_1} \text{trace}(f_1^T(s)Z_2f_1^T(s))dsd\theta \) and \( \int_{-\sigma_2}^{\sigma_2} \int_{0}^{\tau_2} \text{trace}(f_2^T(s)Z_4f_2^T(s))dsd\theta \), which is required in [17].

This will reduce the number of LMI variables to be solved, and hence Theorem 3 requires less computer time than [17, Theorem 1]. Furthermore, it will be shown in the next section that Theorem 3 is certainly less conservative than [17, Theorem 1].

4. Theoretical Comparisons

In this part we will offer a theoretical comparison on conservativeness of Theorem 3 and [17, Theorem 1]. For this reason, we introduce [17, Theorem 1] as follows.

**Lemma 6** (see [17, Theorem 1]). When \( L^* = 0 \) and \( L^* = K \), GRN ((14a), (14b)) subject to ((4a), (4b)) is asymptotically stable in the mean square sense, if there exist positive definite matrices \( P_i, R_i \) \((i = 1, 2)\), \( Q_j \) \((j = 1, 2, \ldots, 8)\), and \( Z_k \) \((k = 1, 2, \ldots, 6)\), a diagonal positive matrix \( A \), matrices \( S_i, J_i, N_i, N_i, M_i, U_i, V_i, L_i, T_i \) \((i = 1, 2)\) of appropriate sizes, and positive scalars \( \rho_1, \rho_2, \rho_3 \) such that the following LMIs hold:

\[
\begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{12} & \Omega_{22} & 0 & \Omega_{24} \\
\Omega_{13} & 0 & \Psi_{14} & 0 \\
\Omega_{14} & \Omega_{24} & 0 & \Psi_{14} \\
\end{bmatrix} < 0, \quad i = 1, 2, 3, 4,
\]

where

\[
\Psi_{144} = \begin{bmatrix}
\tau_{12} (1 - \alpha) \Sigma_{12} \Sigma_{12} T L \Sigma_{12} U \Sigma_{12} T \Sigma_{12} T & 0 & 0 \Sigma_{12} U \Sigma_{12} U \Sigma_{12} U \Sigma_{12} T \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]
\[ \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix}, \]

\[ \Xi_{11} = -P_1C - C^TP_2 + Q_8 + (\rho_1 + \tau_1\rho_2 + \tau_{12}\rho_3)H_1 \\
+ V_1 + V_1^T + R_2, \]

\[ \Xi_{22} = (1 + \alpha\sigma_d)Q_5 - (1 - \alpha\sigma_d)Q_6 + L_1 + L_1^T - U_2 - U_2^T, \]

\[ \Xi_{33} = -(1 - \sigma_d)Q_5 + (\rho_1 + \tau_1\rho_2 + \tau_{12}\rho_3)H_4 - L_2 \\
- I_2^T + T_2 + T_2^T, \]

\[ \Xi_{44} = Q_7 - Q_8, \]

\[ \Xi_{55} = Q_9 - Q_7. \]

In order to show that Theorem 3 is less conservative than [17, Theorem 1], the following propositions are required.

**Proposition 7.** Let \( \Sigma^T = \Sigma, J_j, X_j \) \( (j = 1, 2, 3) \), and \( W^T = W > 0 \) be given real matrices of appropriate sizes. For given scalars \( \alpha \in (0, 1) \) and \( c > 0 \), set

\[ S_1 = [(1 - \alpha)J_1 \alpha J_2], \quad T_1 = \text{diag}(((1 - \alpha)W, \alpha W)), \]

\[ S_2 = J_3, \quad T_2 = W, \]

\[ S_0 = J_1X_1 + J_2X_2 + J_3X_3. \]

(36)

If

\[ \left[ \begin{array}{c} S + S_0 + S_0^T \\ \sqrt{\gamma S_0} \\ -T_1 \end{array} \right] < 0, \quad i = 1, 2, \]

(37)

then there exists a (sufficiently large) positive integer \( s \) such that

\[ \Sigma - \frac{s}{k(1 - \alpha) - c^{-1}X_1TWX_1} - \frac{s}{k\alpha}c^{-1}X_2^TWX_2 \\
- \frac{s}{s - k + 1}c^{-1}X_3^TWX_3 < 0, \quad \forall k \in \langle s \rangle. \]

(38)

**Proof.** It follows from (37) and the Schur complementary lemma [21] that

\[ \Sigma + S_0 + S_0^T + cS_1T_1^{-1}sI_1 < 0, \quad i = 1, 2, \]

(39)

and hence there exists a (sufficiently large) positive integer \( s \) such that

\[ \Sigma + S_0 + S_0^T + \frac{s + 1}{s}cS_1T_1^{-1}sI_1 < 0, \quad i = 1, 2. \]

(40)

For an arbitrary but fixed \( k \in \langle s \rangle \), one can derive from (36) and (40) that

\[ \Sigma + J_1X_1 + X_1^TJ_1 + \frac{k}{s}c(1 - \alpha)J_1W^TJ_1^T \\
+ J_2X_2 + X_2^TJ_2 + \frac{k}{s}c\alpha J_2W^TJ_2^T \\
+ J_3X_3 + X_3^TJ_3 + \frac{s - k + 1}{s}cJ_3W^TJ_3^T < 0. \]

(41)
Since
\[ J_1 X_1 + X_1^T J_1 + \frac{k(1-\alpha)}{s} c_1 W^{-1} J_1^T + \frac{s}{k(1-\alpha)} c_1^{-1} X_1^T W X_1 \geq 0, \]
\[ J_2 X_2 + X_2^T J_2 + \frac{ka}{s} c_2 W^{-1} J_2^T + \frac{s}{ka} c_2^{-1} X_2^T W X_2 \geq 0, \]
\[ J_3 X_3 + X_3^T J_3 + \frac{s-k+1}{s} c_3 W^{-1} J_3^T + \frac{s}{s-k+1} c_3^{-1} X_3^T W X_3 \geq 0, \]
we obtain from (41) that (38) holds. The proof is completed. 

Proposition 8. Let \( \Omega^T = \Omega, J_1, J_2, X_1, Y_1 (i = 1, 2, 3), W^T = W > 0, \) and \( Z^T = Z > 0 \) be given real matrices of appropriate sizes and \( c_i > 0 (i = 1, 2) \) and \( \alpha \in (0, 1) \) given scalars. Set
\[ S_{11} = [c_1 (1-\alpha) J_1 c_1 a J_2 c_2 (1-\alpha) L_1 c_2 a L_2], \]
\[ S_{12} = [c_1 (1-\alpha) J_1 c_1 a J_2 c_2 (1-\alpha) Z_1 c_2 a Z_2], \]
\[ S_{21} = [c_1 J_3 c_2 (1-\alpha) J_1 c_2 a L_1], \]
\[ S_{22} = [c_1 J_3 c_2 J_2 L_1], \]
\[ T_{11} = \text{diag} (c_1 (1-\alpha) W_1 c_1 a W_2 c_2 (1-\alpha) Z_2 c_2 a Z_2), \]
\[ T_{12} = \text{diag} (c_1 (1-\alpha) W_1 c_1 a W_2 c_2 Z_2), \]
\[ T_{21} = \text{diag} (c_1 W_1 c_1 a Z_2), \]
\[ T_{22} = \text{diag} (c_1 W_2 c_2 a Z_2), \]
\[ \Omega_1 = J_1 X_1 + J_2 X_2 + J_3 X_3, \]
\[ \Omega_2 = L_1 Y_1 + L_2 Y_2 + L_3 Y_3. \]
If
\[ \Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + S_{ij} T_{ij}^{-1} S_{ij}^T \preceq 0, \quad i, j = 1, 2, \] (44)
then there exists a pair of (sufficiently large) positive integers \( s_1 \) and \( s_2 \) such that
\[ \Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + S_{ij} T_{ij}^{-1} S_{ij}^T \preceq 0, \quad i, j = 1, 2, \]
and hence
\[ \Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + S_{ij} T_{ij}^{-1} S_{ij}^T + (1-\delta) S_{ij}^T T_{ij}^{-1} S_{ij} < 0, \quad i = 1, 2, \forall \delta \in [0, 1], \]

(47)

that is,
\[ \Omega_0 + \Omega_1 + \Omega_1^T \sqrt{S_{ij}^T T_{ij}^{-1} S_{ij}} \preceq 0, \quad i = 1, 2, \forall \delta \in [0, 1], \]

(48)

where
\[ \tilde{S}_1 = [(1-\alpha) J_1 a J_2], \quad \tilde{T}_1 = \text{diag} ((1-\alpha) W, a W), \]
\[ \tilde{S}_2 = J_3, \quad \tilde{T}_2 = W, \]
\[ \Omega_0 = \Omega + \Omega_2 + \Omega_2^T + \delta c_2 J_2 \tilde{T}_1^{-1} \tilde{S}_1 + (1-\delta) c_2 \tilde{S}_2 \tilde{T}_2^{-1} \tilde{S}_2^T, \]
\[ \tilde{S}_1 = [(1-\alpha) L_1 a L_2], \quad \tilde{T}_1 = \text{diag} ((1-\alpha) Z, a Z), \]
\[ \tilde{S}_2 = L_3, \quad \tilde{T}_2 = Z. \]

Applying Proposition 7 to \( \Sigma = \Omega_0, \Sigma_0 = \Omega_1, S_1 = \tilde{S}_1, T_1 = \tilde{T}_1, \) and \( c = c_1, \) we obtain that
\[ \Omega_0 - \frac{s_1}{k(1-\alpha)} c_1^{-1} X_1^T W X_1 - \frac{s_1}{ka} c_1^{-1} X_2^T W X_2 - \frac{s_1}{s_1 - k + 1} c_1^{-1} X_3^T W X_3 < 0, \forall \delta \in [0, 1], \forall k \in \langle s_1 \rangle, \delta \in [0, 1] \]

(50)

for some (sufficiently large) positive integer \( s_1. \)

By the Schur complementary lemma, one can easily derive from (50) that
\[ \left[ \Omega_k + \Omega_2 + \Omega_2^T \sqrt{S_{ij} T_{ij}^{-1} S_{ij}} \right] \preceq 0, \quad i = 1, 2, \forall k \in \langle s_1 \rangle, \]

(51)

where
\[ \Omega_k = \Omega - \frac{s_1}{k(1-\alpha)} c_1^{-1} X_1^T W X_1 - \frac{s_1}{ka} c_1^{-1} X_2^T W X_2 - \frac{s_1}{s_1 - k + 1} c_1^{-1} X_3^T W X_3. \]

Again applying Proposition 7 to \( \Sigma = \Omega_k, \Sigma_0 = \Omega_2, S_1 = \tilde{S}_1, T_1 = \tilde{T}_1, \) and \( c = c_2, \) one can complete the proof. 

Proposition 9. Let \( p, q \in R^n \) and \( M \in R^{n\times n} \) satisfying \( M^T = M > 0. \) Then
\[ (p + q)^T M (p + q) \leq \frac{1}{\alpha} p^T M p + \frac{1}{1-\alpha} q^T M q, \quad \forall \alpha \in (0, 1). \]

(53)

Proof. One has
\[ \frac{1}{\alpha} p^T M p + \frac{1}{1-\alpha} q^T M q - (p + q)^T M (p + q) \]

\[ = \frac{1-\alpha}{\alpha} p^T M p + \frac{\alpha}{1-\alpha} q^T M q - p^T M q - q^T M p \]
= \left( \sqrt{\frac{1 - \alpha}{\alpha}} p - \sqrt{\frac{\alpha}{1 - \alpha}} q \right)^T M \left( \sqrt{\frac{1 - \alpha}{\alpha}} p - \sqrt{\frac{\alpha}{1 - \alpha}} q \right) 
\geq 0.

(54)

Now it is time to show that Theorem 3 is less conservative than [17, Theorem 1] in theory.

**Theorem 10.** Set $L^* = 0$ and $L^* = K$. If the LMIs in (34) are feasible, then the LMIs in (18) and (19) are feasible.

**Proof.** Set $\rho = \rho_1$, $U_j = P_j$ ($j = 1, 2$), $V_j = Q_{5-j}$, $W_j = Q_{9-j}$ ($j = 1, 2, 3, 4$), $V_5 = R_1$, $W_5 = R_2$, $V_k = Z_{k-5}$, and $W_k = Z_{k-1}$ ($k = 6, 7$). Then it follows from (34) and the Schur complementary lemma that (18) holds and

$$
\Sigma_{11} + \Psi_4 + \tau_1 MV_6^{-1} M^T + \sigma_1 V W_6^{-1} V^T + \tilde{S}_{ij} \tilde{T}_{ij}^T < 0,
$$

(55)

and Proposition 9, implies that

$$
\Omega + \Psi_k + \Psi_l < 0,
$$

(56)

and hence the LMIs in (19) are feasible. The proof is completed.

**Remark 11.** In Theorem 10, it has been theoretically investigated that Theorem 3 is certainly less conservative than [17, Theorem 1]. On the other hand, the numbers of LMI variables to be solved in Theorem 3 and [17, Theorem 1] are $8n^2 + 9n + 1$ and $25n^2 + 10n + 3$, respectively, which implies that Theorem 3 will require less computer time than [17, Theorem 1].

**5. An Illustrative Example**

In this section, a numerical example is given to illustrate the effectiveness and less conservativeness of our theoretical results.

We consider a delayed GRN with stochastic disturbances, with the parameters described as

$$
A = \text{diag}(3, 3, 3), \quad C = \text{diag}(2.5, 2.5, 2.5),
$$

$$
D = \text{diag}(0.8, 0.8, 0.8),
$$

$$
W = \begin{bmatrix}
0 & 0 & -2.5 \\
-2.5 & 0 & 0 \\
0 & -2.5 & 0
\end{bmatrix}.
$$

(61)

Let $G_1 = G_2 = G_3 = G_4 = 0.4I$ and $f(x) = x^2/(1 + x^2)$, which means that $L^* = 0$ and $L^* = K = 0.65I$. When $s_1 = s_2 = 10$, for $\tau_1 = \tau_2 = 1$, $\tau_3 = \tau_4 = 10$, $\tau_d = \sigma_d = 0.5$, and $\alpha = 0.1$, by using the MATLAB Toolbox, inequalities (18) and (19) have feasible solutions. By the theorem, the system is asymptotically stable in the mean square sense. Here, some of the solution matrices are given as follows:

$$
U_1 = \begin{bmatrix}
3.9297 & 0.0000 & 0.0000 \\
0.0000 & 3.9297 & 0.0000 \\
0.0000 & 0.0000 & 3.9297
\end{bmatrix},
$$

$$
U_2 = \begin{bmatrix}
7.8311 & -0.0000 & -0.0000 \\
-0.0000 & 7.8311 & -0.0000 \\
-0.0000 & -0.0000 & 7.8311
\end{bmatrix},
$$

(62)

and show the trajectories of the $x(t)$ and $y(t)$ in Figure 1.

When $\tau_1 = \sigma_1 = 1$ and $s_1 = s_2 = 10$, the maximal allowable upper bounds of $\tau_2$ and $\tau_3$ for different values of $\tau_d = \sigma_d$ obtained by Theorem 3 and [17, Theorem 1] are shown in Table 1. It can be seen from Table 1 that Theorem 3 is the less conservative than [17, Theorem 1].
6. Conclusions

In this paper, the stability problem for a class of GRNs with time-varying delays and stochastic disturbances has been investigated. By constructing an appropriate Lyapunov-Krasovskii functional and proposing a DRP approach, a mean square stability criterion is given in terms of LMIs, which can be easily tested by the LMI Toolbox of MATLAB. It is theoretically shown that the proposed result is less conservative than \cite[Theorem 1]{17}. Moreover, the number of LMI variables in this paper is more than the one in \cite[Theorem 1]{17}. A numerical example has been provided to illustrate the theoretical results given in this paper.

Extending the idea of this paper to other system models, including singular delayed systems \cite{19, 22–24}, stochastic systems \cite{25}, Markovian jump systems \cite{20, 26}, and genetic regulatory networks \cite{5, 10, 27}, is under consideration.

Conflict of Interests

The authors declare that there is no commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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