Research Article

Stochastic Separated Continuous Conic Programming: Strong Duality and a Solution Method

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We study a new class of optimization problems called stochastic separated continuous conic programming (SSCCP). SSCCP is an extension to the optimization model called separated continuous conic programming (SCCP) which has applications in robust optimization and sign-constrained linear-quadratic control. Based on the relationship among SSCCP, its dual, and their discretization counterparts, we develop a strong duality theory for the SSCCP. We also suggest a polynomial-time approximation algorithm that solves the SSCCP to any predefined accuracy.

1. Introduction

Stochastic programming is one of the branches of optimization which enjoys a fast development in recent years. It tries to find optimal decisions in problems involving uncertain data, so it is also called “optimization under uncertainty” [1]. Since the problems in reality often involve uncertain data, stochastic programming has a lot of applications.

Many deterministic optimization models have their stochastic counterpart; for example, the stochastic counterpart of linear programming is stochastic linear programming. In this paper, we consider the stochastic counterpart of a kind of optimization model called separated continuous conic programming (SCCP) which has the following form:

(SCCP) \[ \max \int_0^T [(y + (T-t)c)'u(t) + d'x(t)] \, dt \]
\[ \text{s.t. } \alpha + ta - \int_0^t Gu(s) \, ds - Fx(t) \in \mathcal{K}_1, \]
\[ b - Hu(t) \in \mathcal{K}_2, \]
\[ u(t) \in \mathcal{K}_3, \quad x(t) \in \mathcal{K}_4, \quad t \in [0,T]. \] (1)

Here the control and state variables (both are decision variables), \( u(t) \) and \( x(t) \), are vectors of bounded measurable functions of time \( t \in [0,T] \). \( \mathcal{K}_i, i = 1, 2, 3, 4, \) are closed convex cones in the Euclidean space with appropriate dimensions, \( y, c, d, \alpha, a, b \) are vectors, \( G, F, H \) are matrices, and the superscript ‘\( \cdot' \) denotes the transpose operation.

SCCP was first studied by Wang et al. [2]. They developed a strong duality theory for SCCP under some mild and verifiable conditions and suggested an approximation algorithm to solve SCCP with predefined precision. SCCP has a variety of applications in robust optimization and sign-constrained linear-quadratic control. However, many applications of SCCP are stochastic in nature in the sense that the values of some parameters in the resulted SCCP models may change over time with some probability distribution.

To incorporate this kind of randomness into the SCCP model, we introduce the following stochastic counterpart of SCCP which we call stochastic separated continuous conic programming (SSCCP) problem:

\[ \max \int_0^{T_1} [(y + (T-t)c)'u(t) + d'x(t)] \, dt \]
\[ + \mathbb{E}_\xi \left( \int_{T_1}^{T_2} [(y(\xi)+(T-t)c(\xi))'u(t)+d(\xi)'x(t)] \, dt \right) \]
\[ \text{s.t. } \alpha + ta - \int_0^t Gu(s) \, ds - Fx(t) \in \mathcal{K}_1, \quad t \in [0,T_1]. \] (2)
where $\xi$ is a random variable.

SSCCP is formulated with the similar idea as that of the stochastic linear programming [1, 3]. There are two stages in this problem; the values of some parameters in the second stage depend on the value of a random variable $\xi$.

Our goal in this paper is developing the strong duality for SSCCP and suggesting a solution method to solve it approximately with predefined precision. Here is a summary of our main results. Through discretization, we connect SSCCP and its dual to two ordinary conic programs, and we show that strong duality holds for SSCCP and its dual under some mild (and verifiable) conditions on these two ordinary conic programs. Furthermore, the optimal values of those two conic programs provide an explicit bound on the duality gap between SSCCP and its dual, based on which we suggest a polynomial-time approximation algorithm that solves SSCCP to any predefined accuracy. According to our knowledge, we are the first to raise the SSCCP model and there have been no other results on SSCCP besides those in this paper.

The paper is organized as follows. In Section 2, we present an overview on the related literature. We also give a concrete example to show the application of SSCCP. In Section 3, we construct a dual for SSCCP. We also discretize SSCCP and its dual into two ordinary conic programs, and bring out their relations. In Section 4, we discuss the strong feasibility for SSCCP, its dual, and their discretizations. We then establish the strong duality result for SSCCP and its dual in Section 5. This leads to a polynomial-time approximation algorithm with an explicit error bound, detailed in Section 6. In Section 7, we summarize what we get for SSCCP and point out some future research directions.

For simpler presentation, in the remainder of this paper, we will concentrate on the following problem, which is the corresponding SSCCP when $\xi$ is a discrete variable and only takes two different values with probability $\theta$ and $1-\theta$, that is, there are only two scenarios in the second stage of SSCCP:

\[
\begin{align*}
\max \ & \int_{T_1}^{T} \left[ (\tilde{y}_2 + (T - t) \tilde{c}_2) w(t) + \tilde{d}_2 z(t) \right] dt \\
\text{s.t.} \ & \alpha + ta - \int_{0}^{T} Gu(s) ds - Fx(t) \in \mathcal{K}_1, \quad t \in [0, T], \\
& b - Hu(t) \in \mathcal{K}_2, \quad t \in [0, T], \\
& u(t) \in \mathcal{K}_3, \quad x(t) \in \mathcal{K}_4, \quad t \in [0, T], \\
& \alpha_1 + t\alpha_1 - \int_{0}^{T_1} Gu(s) ds \\
& - \int_{T_1}^{T} Gu(s) ds - Fx(t) \in \mathcal{K}_1, \quad t \in [T_1, T],
\end{align*}
\]

where the first-stage control and state variables are $u(t)$ and $x(t)$, $t \in [0, T_1]$, and the second-stage control and state variables are $v(t)$, $w(t)$, $y(t)$, and $z(t)$, $t \in (T_1, T)$. Also $\alpha_1 = \alpha + T_1 a - \int_{0}^{T_1} Gu(s) ds$, $\alpha_2 = \alpha + T_1 a - \int_{0}^{T_1} Gu(s) ds$, $\theta = \theta y_2$, $\tilde{c}_1 = \theta c_2$, $\tilde{d}_1 = \theta d_2$, $\tilde{c}_2 = (1-\theta) c_2$, $\tilde{d}_2 = (1-\theta) d_2$.

Note that although (3) is a deterministic optimization problem, it is not an SCCP. To see why this is the case, one can try to formulate (3) into the form of SCCP and it then becomes clear that (3) cannot fit into the SCCP form.

In the rest of this paper, we will use some results on conic programming without explanations. Interested readers can consult the books on conic programming (e.g., [4]) for the related results.

2. Literature Review

Bellman [5, 6] first introduced the so-called continuous linear programming (CLP), which has the following form:

\[
\begin{align*}
\text{(CLP)} \quad & \max \int_{0}^{T} c(t) x(t) dt \\
\text{s.t.} \ & B x(t) - \int_{0}^{T} K x(s) ds \leq b(t) \\
& x(t) \geq 0, \quad t \in [0, T].
\end{align*}
\]

Here $x(t)$ is a decision variable. The model has wide-ranging applications (e.g., the bottleneck problem [5]). But CLP is
very difficult to solve in its general form. Later, Anderson [7] introduced separated continuous linear programming (SCLP) (see (5)), a special case of CLP, to model the job-shop scheduling problems:

\[
\text{(SCLP)} \quad \max \int_{0}^{T} \left[ (y(T-t))u(t) + d'x(t) \right] dt \\
\text{s.t.} \quad \int_{0}^{T} Gu(s) ds + Fx(t) \leq \alpha + ta, \\
Hu(t) \leq b, \\
u(t) \geq 0, \quad x(t) \geq 0, \quad t \in [0, T].
\]

The word “separated” refers to the fact that there are two kinds of constraints in SCLP: the constraints involving integration and the instantaneous constraints [7].

Anderson et al. [8] studied the properties of the extreme solutions of the SCLP, based on which Anderson and Philpott [9] developed a simplex type of algorithm for a network-based SCLP. Refer to Anderson and Philpott [10] and Anderson and Nash [11] for their other results on SCLP. Pullan [12–18] continues studying SCLP in a series of papers. He systematically developed a duality theory and solution algorithms for the SCLP.

There are other researches focused on other forms of SCLP, including Luo and Bertsimas [19], Shapiro [20], Fleischer and Sethuraman [21], Weiss [22], and Nasrabadi et al. [23].

One of the extensions of SCLP is SCCP introduced by Wang et al. [2] in which the constraints involve the convex cone in their right hand side. When all the convex cones are nonnegative orthants, SCCP reduces to SCLP. In [2], based on the relationship among SCCP, its dual, and their discretization counterparts, they develop a strong duality theory for the SCCP. They also suggest a polynomial-time approximation algorithm that solves the SCCP to any predefined accuracy.

Wang [24, 25] extends SCCP to generalized separated continuous conic programming (GSCCP) by allowing the parameters in (1) to be piece-wise constants and extends the results of [2] for SCCP to GSCCP. In this paper, we extend SCCP to SSSCP by allowing the changes of values of some parameters in SCCP in the second stage. We also extend the results of [2] for SCCP to SSSCP.

2.1. A Motivated Example for SSSCP. We consider a problem which appears in [2]; for completeness, we reproduce the problem description and the formulation below.

A network processes a continuous flow of jobs at two machines. The jobs visit machines 1 and 2 in the order 1 → 2 → 1, that is, a total of three processing steps; see Figure 1. Corresponding to each processing step, there is a buffer holding the fluid. At \( t = 0 \), the initial levels of fluid at the three steps are 50, 20, and 120 units. The input rates of fluid from outside to the three buffers are 0.01, 0.01, and 0.01. To process each unit of job (“fluid”), the time requirements at the three steps are 0.4, 0.8, and 0.2 time units.

The problem is to find the processing rates at the three steps, \( u_i(t), \ i = 1, 2, 3 \), which determine the fluid levels in the three buffers, \( x_i(t), \ i = 1, 2, 3 \), during a given time interval \([0, T]\) such that the fluid levels in the three buffers are maintained as close as possible to a prespecified constant level \( d = (30 \ 10 \ 80)' \).

The problem can be formulated as follows:

\[
\min \int_{0}^{T} \left[ (x(T-t) - d)' (x(T-t) - d) \right] dt \\
\text{s.t.} \quad \int_{0}^{T} Gu(s) ds + x(t) = \alpha + ta, \\
b - Hu(t) \geq 0, \\
u(t) \geq 0, \quad x(t) \geq 0, \quad t \in [0, T],
\]

where

\[
G = \begin{bmatrix} 1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.4 & 0 & 0.2 \\
0 & 0.8 & 0 \end{bmatrix}, \\
\alpha = \begin{bmatrix} 50 \\
20 \\
120 \end{bmatrix}, \quad a = \begin{bmatrix} 0.01 \\
0.01 \\
0.01 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\
1 \end{bmatrix}.
\]

We can further express the above problem in the form of SCCP. Please refer to [2] for the details.

In reality, the values of \( a \) and \( b \) could be changed during \([0, T]\) for example, when the machine 1 experiences partial breakdown within \([T_1, T]\); where \( 0 < T_1 < T \), the corresponding value of capacity vector for machine 1, \( h_1 \), will change during \([T_1, T]\). This makes the formulation of the problem an SSSCP. We omit the details here.

3. The Dual and Discretizations

3.1. The Dual. The dual of SSSCP that we will focus on is the following problem:

\[
\begin{align*}
\text{(SSCP*)} & \quad \min \int_{0}^{T-T_1} \left[ (\alpha_1 + (T-t) a_1)' h(t) + b_1' l(t) \right] dt \\
& + \int_{0}^{T-T_1} \left[ (\alpha_2 + (T-t) a_2)' p(t) + b_2' q(t) \right] dt \\
& + \int_{T-T_1}^{T} \left[ (\alpha + (T-t) a)' f(t) + b' g(t) \right] dt
\end{align*}
\]
where the decision variables \( h(t), l(t), p(t), q(t), f(t), g(t) \) and \( g(t) \) are bounded measurable functions. \( \mathcal{K}_i^* \) are the dual cones of \( \mathcal{K}_j^* \) and \( \mathcal{K}_{ij}^* \), \( i = 1, 2, 3, 4 \), \( j = 1, 2 \), respectively.

The derivation of the above dual problem is similar to the derivation of the dual problem for LP (see, e.g., [26]) and we omit the details here. Because SSCCP involves time, to achieve some degree of symmetry in the dual (to facilitate the analysis), we choose to write the dual in the reversed time; that is, \( t \) in the dual is \( T - t \) in the primal.

The following weak duality is readily shown from the derivation of (SSCCP*).

**Proposition 1.** The weak duality holds between SSCCP and SSCCP*; that is, if \( (u(t), x(t), v(t), y(t), w(t), z(t)) \), \( t \in [0, T] \), is a feasible solution for SSCCP and \( h(t), l(t), p(t), q(t), f(t), g(t) \), \( t \in [0, T - T_1] \), is a feasible solution for SSCCP*, then

\[
\begin{align*}
\int_0^T \left[ (y + (T - t) c) u(t) + d' x(t) \right] dt \\
+ \int_{T-T_1}^T \left[ (\tilde{y}_1 + (T - t) \tilde{c}_1) v(t) + d'_1 y(t) \right] dt \\
+ \int_{T-T_1}^T \left[ (\tilde{y}_2 + (T - t) \tilde{c}_2) w(t) + d'_2 z(t) \right] dt
\end{align*}
\]

\[
\leq \int_0^{T-T_1} \left[ (\alpha_1 + (T - t) a_1) h(t) + b'_1 l(t) \right] dt \\
+ \int_0^{T-T_1} \left[ (\alpha_2 + (T - t) a_2) p(t) + b'_2 q(t) \right] dt \\
+ \int_{T-T_1}^T \left[ (\alpha + (T - t) a) f(t) + b' g(t) \right] dt.
\]

(9)

Next we will introduce the discretizations for SSCCP and SSCCP*, respectively, and discuss the relationships among SSCCP, SSCCP*, and their discretizations. But first, we need the following notation and conventions which mostly follow that of [2].

**Notation and Conventions**

(i) When we say \((u(t), x(t), v(t), y(t), w(t), z(t))\) is a feasible solution to SSCCP, we mean \((u(t), x(t), v(t), y(t), w(t), z(t), t \in (T_1, T))\) is a feasible solution to SSCCP.

(ii) By default, all vectors are column vectors. One exception is when we denote the solutions to SSCCP and its dual (or their variations) as \((u, x, v, y, w, z)\) and \((h, l, p, q, f, g)\), we mean \((u', x', v', y', w', z')^T\) and \((h', l', p', q', f', g')^T\).

(iii) \( \pi = \{t_0, t_1, \ldots, t_{m_1+m_2}\} \) denotes a partition of \([0, T]\) into \( m_1 + m_2 \) segments:

\[
0 = t_0 < t_1 < \cdots < t_{m_1} = T_1 < t_{m_1 + 1} < \cdots < t_{m_1+m_2} = T,
\]

where \( m_1 \) and \( m_2 \) are positive integer numbers.

(iv) Given a partition \( \pi = \{t_0, \ldots, t_m\} \) and a vector \( \bar{r} := (\bar{r}(t_0), \bar{r}(t_1), \ldots, \bar{r}(T_m)) \), where \( \bar{r}(\cdot) \) is a right continuous function, the following (continuous) function

\[
r(t) = \left\{ \begin{array}{ll}
\bar{r}(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}} \bar{r}(t_i) & \text{for } t \in [t_{i-1}, t_i], \\
\bar{r}(t_{i-1}) & \text{for } i = 1, \ldots, m,
\end{array} \right.
\]

(11)

is called a piecewise linear extension of \( \bar{r} \), whereas the following (right-continuous) function

\[
r(t) = \left\{ \begin{array}{ll}
\bar{r}(t_{i-1}) & \text{for } i = 1, \ldots, m, \\
\bar{r}(T_{m-1}) & \text{for } i = T,
\end{array} \right.
\]

(12)

is called a piecewise constant extension of \( \bar{r} \).

(v) When \((u(t), x(t), v(t), y(t), w(t), z(t))\) is a feasible solution to SSCCP, with \((u(t), v(t), w(t))\) being piecewise constant and \((x(t), y(t), z(t))\) piecewise linear, we assume \(u(t), v(t), w(t)\) is right continuous, and \(x(t), y(t), z(t)\) is continuous, with \(y(T_1+) = x(T_1)\), \(z(T_1+) = x(T_1)\), and the pieces of both \( u \) and \( x \) correspond to a common partition for \([0, T_1]\), and the pieces of both \( v, w \) and \( y, z \) correspond to a common partition for \((T_1, T)\).
When \((h(t), l(t), p(t), q(t), f(t), g(t))\) is a feasible solution to \(SSCCP^*\), with \(h(t), p(t), f(t)\) being piecewise constant and \(l(t), q(t)\) and \(g(t)\) are continuous, with \((g(T-T_1^+) = l(T-T_1^+) + q(T-T_1))\), and the pieces of \(h, l, p, q\), and \(g\) correspond to a common partition for \([0, T-T_1]\), and the pieces of both \(f\) and \(g\) correspond to a common partition for \((T-T_1, T)\).

(vi) For \(i = 1, 2, 3, 4\), denote \(\mathcal{K}_{jm} := \mathcal{K}_j \times \cdots \times \mathcal{K}_j\), and similarly denote \(\mathcal{K}^{*}_{jm} := \mathcal{K}^*_j \times \cdots \times \mathcal{K}^*_j\).

3.2. The Discretizations. We start with introducing the following discretization of \(SSCCP\) based on the partition \(\pi\) of \([0, T-T_1, T]\), where \(0 = t_0 < t_1 < \cdots < t_{m_1} = T_1 < t_{m_1 + 1} < \cdots < t_{m_1 + m_2} = T\):

\[
\text{(SCP}_1(\pi)\text{)}
\]

\[
\begin{align*}
\max_{\pi} & \sum_{i=1}^{m_1} \left( y + \left( T - \frac{t_i + t_{i-1}}{2} \right) c \right) u_i \\
& + \sum_{i=1}^{m_2} \left( y_i + \left( T - \frac{t_{m_1+i} + t_{m_1+i-1}}{2} \right) c \right) v_i \\
& + \sum_{i=1}^{m_2} \left( y_i + \left( T - \frac{t_{m_1+i} - t_{m_1+i-1}}{2} \right) c \right) w_i \\
& + \sum_{i=1}^{m_2} \left( y_i + \left( T - \frac{t_{m_1+i} - t_{m_1+i-1}}{2} \right) c \right) z_i
\end{align*}
\]

s.t. \(\alpha + t_1 a_1 - \left[ G \bar{u}_1 + \cdots + G \bar{u}_m \right] \in \mathcal{K}_1, i = 1, 2, \ldots, m_1;\)
\(\left( t_i - t_{i-1} \right) b - H \bar{u}_i \in \mathcal{K}_2, i = 1, \ldots, m_1;\)
\(\bar{u}_i \in \mathcal{K}_3, i = 1, \ldots, m_1;\)
\(\alpha_2 + t_{m_1+i} a_2 - \left[ G \bar{u}_1 + \cdots + G \bar{u}_m \right] \]
\(- \left[ G \bar{u}_1 + \cdots + G \bar{u}_m \right] - F \tilde{\bar{z}} \in \mathcal{K}_1, i = 1, 2, \ldots, m_2;\)
\(\left( t_{m_1+i} - t_{m_1+i-1} \right) b_1 - H \bar{v}_i \in \mathcal{K}_2, i = 1, \ldots, m_2;\)
\(\bar{v}_i \in \mathcal{K}_3, i = 1, \ldots, m_2;\)
\(\tilde{\bar{z}} \in \mathcal{K}_4, i = 1, \ldots, m_2;\)
\(\tilde{\bar{z}} \in \mathcal{K}_4, i = 1, \ldots, m_2.\)

Note that here we require that \(\bar{v}_0 = \tilde{\bar{z}}_0 = \bar{z}_m\) and \(\alpha - F \tilde{\bar{z}}_0 \in \mathcal{K}_1, \bar{z}_0 \in \mathcal{K}_4.\)

Clearly, \((SCP_1(\pi))\) is a conic program.

**Lemma 2.** From a feasible solution for \(SCP(\pi)\), one can get a feasible solution for \(SSCCP\) with the same objective values, if \(\mathcal{K}_1 \subseteq \mathcal{K}_1, \mathcal{K}_1 \subseteq \mathcal{K}_1, \mathcal{K}_4 \subseteq \mathcal{K}_4, \mathcal{K}_4 \subseteq \mathcal{K}_4.\)

**Proof.** Suppose \((\bar{u}, \bar{v}, \tilde{\bar{z}}, \bar{z}, \tilde{\bar{z}})\) is a feasible solution for \(SCP(\pi)\). Let

\[
\begin{cases}
\bar{u}_i, & t \in [t_{i-1}, t_i), i = 1, \ldots, m_1, \\
\bar{v}_i, & t = t_{m_1}, \\
x, & (0) = \bar{z}_0, \\
x(t) = \frac{t_{i-1} - t}{t_{i-1} - t_{i-1}} \tilde{\bar{y}}_{i-1} + \frac{t - t_{i-1}}{t_{i-1} - t_{i-1}} \tilde{\bar{y}}_i, i = 1, \ldots, m_1, \\
v(t) = \frac{\tilde{\bar{y}}_i}{t_{m_1+i} - t_{m_1+i-1}}, t \in (t_{m_1+i-1}, t_{m_1+i}], i = 1, \ldots, m_2, \\
y(T_1^+) = \bar{z}_m, \\
y(t) = \frac{t_{m_1+i} - t}{t_{m_1+i} - t_{m_1+i-1}} \tilde{\bar{y}}_{i-1} + \frac{t - t_{m_1+i-1}}{t_{m_1+i} - t_{m_1+i-1}} \tilde{\bar{y}}_i, i = 1, \ldots, m_2, \\
\omega (t) = \frac{\tilde{\bar{z}}_i}{t_{m_1+i} - t_{m_1+i-1}}, t \in (t_{m_1+i-1}, t_{m_1+i}], i = 1, \ldots, m_2, \\
\omega (t) = \frac{\tilde{\bar{z}}_i}{t_{m_1+i} - t_{m_1+i-1}} \tilde{\bar{z}}_{i-1} + \frac{t - t_{m_1+i-1}}{t_{m_1+i} - t_{m_1+i-1}} \tilde{\bar{z}}_i, i = 1, \ldots, m_2.
\end{cases}
\]

then we have \(u(t) \in \mathcal{K}_3, x(t) \in \mathcal{K}_2, t \in [0, T_1], v(t) \in \mathcal{K}_1, \omega(t) \in \mathcal{K}_3, t \in (T_1, T], u(t) \in \mathcal{K}_3, \omega(t) \in \mathcal{K}_4, t \in (T_1, T].\)
Because $\hat{x}_{m_1} \in \mathcal{K}_4$ and $\hat{y}_0 = \hat{z}_0 = \hat{x}_{m_1}$, $\hat{y}_0 \in K_4$ and $\hat{z}_0 \in K_4$. When $K_4 \subseteq K_{41}$, $K_4 \subseteq K_{42}$, we have $y(t) \in K_{41}$, and $z(t) \in K_{42}$, and $t \in (T_1, T)$.

For $t = 0$, 
\[
\alpha + ta - \int_0^t G u(s) ds - F x(t) = \alpha - F \hat{x}_0 \in \mathcal{K}_1.
\]

For $t \in (t_0, t_1)$, 
\[
\alpha + ta - \int_0^t G u(s) ds - F x(t) = \alpha + ta - G \frac{\hat{u}_1}{t_1} t - F \left( \frac{t_1 - t}{t_1} \hat{x}_0 + \frac{t}{t_1} \hat{x}_1 \right)
\]
\[
= \frac{t}{t_1} \left( \alpha + ta - G \hat{u}_1 \right) + \frac{t_{i-1} - t}{t_1} \left( \alpha - F \hat{x}_0 \right)
\]
\[
\in \mathcal{K}_1.
\]

For $t \in [T_{i-1}, T_i)$, $i = 1, \ldots, m_1$,
\[
\alpha + ta - \int_0^t G u(s) ds - F x(t) = \alpha + ta - \left( \int_0^{T_{i-1}} G u(s) ds + \cdots + \int_0^t G u(s) ds \right) - F x(t)
\]
\[
= \alpha + ta - \left( G \frac{\hat{u}_1}{t_1} (t_1 - t_0) + \cdots + G \frac{\hat{u}_1}{t_i - t_{i-1}} (t - t_{i-1}) \right)
\]
\[
- F \left( \frac{t_{i-1} - t}{t_i - t_{i-1}} \hat{x}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \hat{x}_i \right)
\]
\[
= \frac{t_{i-1} - t}{t_i - t_{i-1}} \left( \alpha + ta - \left( \hat{u}_1 + \cdots + \hat{u}_{i-1} \right) \right) - F \hat{x}_{i-1}
\]
\[
\in \mathcal{K}_1.
\]

For $t = t_{m_1}$,
\[
\alpha + ta - \int_0^t G u(s) ds - F x(t) = \alpha + t_{m_1} a - \left( \hat{u}_1 + \cdots + \hat{u}_{m_1-1} + \hat{u}_{m_1} \right) - F \hat{x}_{m_1}
\]
\[
\in \mathcal{K}_1.
\]

For $t \in [T_{m_1-1}, T_{m_1})$, $i = 1, \ldots, m_2$,
\[
b - Hu(t) = b - H \frac{\hat{u}_i}{t_i - t_{i-1}}
\]
\[
= \frac{1}{t_i - t_{i-1}} ((t_i - t_{i-1}) b - H \hat{u}_i)
\]
\[
\in \mathcal{K}_2.
\]

For $t = t_{m_1}$,
\[
b - Hu(t) = b - H \frac{\hat{u}_{m_1}}{t_{m_1} - t_{m_1-1}}
\]
\[
= \frac{1}{t_{m_1} - t_{m_1-1}} \left( (t_{m_1} - t_{m_1-1}) b - H \hat{u}_{m_1} \right)
\]
\[
\in \mathcal{K}_2.
\]

For $t \in (t_{m_1}, t_{m_1+1})$,
\[
\alpha_1 + ta_1 - \int_0^{T_{i-1}} G v(s) ds - F y(t) = \alpha_1 + ta_1 - \left( \hat{v}_1 + \cdots + \hat{v}_{m_1} \right)
\]
\[
- \frac{G \hat{v}_1}{t_{m_1+1} - t_{m_1}} (t - t_{m_1})
\]
\[
- F \left( \frac{t_{m_1+1} - t_{m_1+1}}{t_{m_1+1} - t_{m_1}} \hat{v}_{m_1} \right)
\]
\[
\in \mathcal{K}_1, \text{ when } \mathcal{K}_1 \subseteq \mathcal{K}_{11}.
\]

For $t \in (t_{m_1+i-1}, t_{m_1+i})$, $i = 1, \ldots, m_2$,
\[
\alpha_1 + ta_1 - \left( \hat{v}_1 + \cdots + \hat{v}_{m_1} \right)
\]
\[
- \left( \hat{v}_1 + \cdots + \hat{v}_{m_1+i-1} \right) - F \hat{v}_{m_1+i-1}
\]
\[
\in \mathcal{K}_1.
\]
\[
\begin{align*}
\mathcal{H}_1 \subseteq \mathcal{H}_2 \iff & \mathcal{H}_4 \subseteq \mathcal{H}_4, \\
\mathcal{H}_4 \subseteq \mathcal{K}_2 \iff & \mathcal{K}_4 \subseteq \mathcal{K}_4.
\end{align*}
\]

Note here we require that \( \tilde{g}_0 = \tilde{h}_m + \tilde{a}_m \), and
\[
\begin{align*}
H^T \tilde{l}_0 - \tilde{y}_1 & \in \mathcal{K}_3^+, \\
\tilde{l}_0 & \in \mathcal{K}_4^+, \\
H^T \tilde{q}_0 - \tilde{y}_2 & \in \mathcal{K}_3^+, \\
\tilde{q}_0 & \in \mathcal{K}_4^+.
\end{align*}
\]

Clearly SCP_{2}(\pi') is also a conic program.

We now show the following.

**Lemma 3.** For any two convex cones \( \mathcal{H}_1, \mathcal{H}_2 \),
\[
\mathcal{H}_1 \subseteq \mathcal{H}_2 \implies \mathcal{H}_2^* \subseteq \mathcal{H}_1^*.
\]

**Proof.** \( \implies \): Because \( \mathcal{H}_1 \subseteq \mathcal{H}_2 \), for any \( x_1 \in \mathcal{H}_1 \), \( x_1 \in \mathcal{H}_2 \), so for any \( y_2 \in \mathcal{H}_2^*, x_1^T y_2 \geq 0 \). So \( y_2 \in \mathcal{H}_1^* \). So \( \mathcal{H}_2^* \subseteq \mathcal{H}_1^* \).
\( \Leftarrow \): Because \( \mathcal{H}_2^* \subseteq \mathcal{H}_1^* \), for any \( y_2 \in \mathcal{H}_2^*, y_2 \in \mathcal{H}_1^* \), so for any \( x_1 \in \mathcal{H}_1 \), \( x_1^T y_2 \geq 0 \). So \( x_1 \in \mathcal{H}_2 \), so \( \mathcal{H}_1 \subseteq \mathcal{H}_2 \).
if \( \mathcal{K}_2 \subseteq \mathcal{K}_{21}, \mathcal{K}_2 \subseteq \mathcal{K}_{22}, \mathcal{K}_3 \subseteq \mathcal{K}_{31}, \mathcal{K}_3 \subseteq \mathcal{K}_{32}, \gamma_1 + \gamma_2 = \gamma, \gamma_1 + \gamma_2 = \gamma, \tilde{c}_1 + \tilde{c}_2 = c \).

**Proof.** Suppose \( (\tilde{h}, \tilde{l}, \tilde{p}, \tilde{q}, \tilde{f}, \tilde{g}) \) is a feasible solution for \( \text{SCP}_2(\pi') \). Let

\[
\begin{align*}
h(t) &= \begin{cases} \frac{\tilde{h}_i}{t_i - t_{i-1}}, & t \in [t_{i-1}, t_i), \ i = 1, \ldots, m_2, \\ \tilde{h}_{m_2}, & t = t_{m_2}, \\ \frac{t_{m_2} - t_{m_2-1}}{t_{m_2} - t_{m_2-1}}, & t = t_{m_2}, \end{cases} \\
l(0) &= \tilde{l}_0, \\
l(t) &= \frac{t - t_{i-1}}{t_i - t_{i-1}} \tilde{l}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \tilde{l}_i, \\
t \in [t_{i-1}, t_i), \ i = 1, \ldots, m_2, \\
p(t) &= \begin{cases} \frac{\tilde{h}_i}{t_i - t_{i-1}}, & t \in [t_{i-1}, t_i), \ i = 1, \ldots, m_2, \\ \tilde{p}_i, & t = t_{m_2}, \\ \frac{t_{m_2} - t_{m_2-1}}{t_{m_2} - t_{m_2-1}}, & t = t_{m_2}, \end{cases} \\
qu(0) &= \tilde{q}_0, \\
qu(t) &= \frac{t - t_{i-1}}{t_i - t_{i-1}} \tilde{q}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \tilde{q}_i, \\
t \in [t_{i-1}, t_i), \ i = 1, \ldots, m_2, \\
f(t) &= \frac{\tilde{f}_i}{t_{m_2+i} - t_{m_2+i-1}}, \ t \in (t_{m_2+i}, t_{m_2+i}], \\
g(T - T_i) &= \tilde{g}_0, \\
g(t) &= \frac{t_{m_2+i} - t}{t_{m_2+i} - t_{m_2+i-1}} \tilde{g}_{i-1} + \frac{t - t_{m_2+i-1}}{t_{m_2+i} - t_{m_2+i-1}} \tilde{g}_i, \\
t \in (t_{m_2+i}, t_{m_2+i}], \ i = 1, \ldots, m_1, \\
&= 1, 2, 3, 4, j = 1, 2, \) with nonempty interiors, the following holds:

\[
\begin{align*}
\alpha + ta &= \int_0^T Gu(s) \, ds - Fx(t) \in \mathcal{H}_1, \ t \in (0, T_1], \\
\alpha - Fx(0) &\in \mathcal{H}_1, \\
b - Hu(t) &\in \mathcal{H}_2, \ t \in [0, T_1], \\
u(t) &\in \mathcal{H}_3, \ x(t) \in \mathcal{H}_4, \\
t \in (0, T_1], \ x(0) &\in \mathcal{H}_4, \\
\alpha_1 + ta_1 &= \int_0^{T_1} Gu(s) \, ds \\
- \int_{T_1}^T Gv(s) \, ds - Fy(t) &\in \mathcal{H}_{11}, \ t \in (T_1, T], \\
b_1 - Hv(t) &\in \mathcal{H}_{21}, \ t \in (T_1, T], \\
v(t) &\in \mathcal{H}_{31}, \ y(t) \in \mathcal{H}_{41}, \ t \in (T_1, T], \\
\alpha_2 + ta_2 &= \int_0^{T_1} Gu(s) \, ds \\
- \int_{T_1}^T Gw(s) \, ds - Fz(t) &\in \mathcal{H}_{12}, \ t \in (T_1, T], \\
b_2 - Hw(t) &\in \mathcal{H}_{22}, \ t \in (T_1, T], \\
w(t) &\in \mathcal{H}_{32}, \ z(t) \in \mathcal{H}_{42}, \ t \in (T_1, T].
\end{align*}
\]

We say that \( \text{SSCCP} \) is **strongly feasible** if there exists a strongly feasible solution. The similar notions apply to the dual problem \( \text{SSCCP}^* \).

Next we will show that the strong feasibility of \( \text{SSCCP} \) and \( \text{SSCCP}^* \) can be determined by the strong feasibility of the following two conic programs:

\[
(\text{CP}_1) \quad \max \ T_j c^T \hat{u} - (T_j d - (T_j) (\tilde{d}_1 + \tilde{d}_2))^T \hat{x} \\
+ (T - T_j) \tilde{c}_1 \tilde{v} + (T - T_j) \tilde{d}_1 \tilde{y} \\
+ (T - T_j) \tilde{c}_2 \tilde{w} + (T - T_j) \tilde{d}_2 \tilde{z} \\
s.t. \quad \alpha + T_j a -GU - F \tilde{x} \in \mathcal{H}_1, \\
T_j b - Hu \in \mathcal{H}_2, \\
\hat{u} \in \mathcal{H}_3, \ \hat{x} \in \mathcal{H}_4, \\
\alpha_1 + T_a \hat{u} - GU \hat{v} - F \tilde{y} \in \mathcal{H}_{11}, \\
(T - T_j) b_1 - Hv \in \mathcal{H}_{21}, \\
\hat{v} \in \mathcal{H}_{31}, \ \hat{y} \in \mathcal{H}_{41}, \\
\alpha_2 + T_a \hat{u} - GU \hat{w} - F \tilde{z} \in \mathcal{H}_{12}, \\
(T - T_j) b_2 - Hw \in \mathcal{H}_{22}, \\
\hat{w} \in \mathcal{H}_{32}, \ \hat{z} \in \mathcal{H}_{42},
\]

(31)

4. Strong Feasibility

We say that \((u, x, v, y, w, z)\) is a **strongly feasible** solution to \( \text{SSCCP} \), if for the closed and convex cones \( \mathcal{H}_i, \mathcal{H}_{ij} \), then \( h(t) \in \mathcal{K}_1^*, l(t) \in \mathcal{K}_2^*, p(t) \in \mathcal{K}_3^*, q(t) \in \mathcal{K}_4^*, t \in [0, T - T_j], f(t) \in \mathcal{K}_5^*, \) and \( t \in (T - T_j, T] \). When \( \mathcal{K}_2 \subseteq \mathcal{K}_{21}, \mathcal{K}_2 \subseteq \mathcal{K}_{22}, \mathcal{K}_3 \subseteq \mathcal{K}_{31}, \mathcal{K}_3 \subseteq \mathcal{K}_{32}, \), then; because \( \hat{g}_0 = \hat{q}_0 + \hat{f}_0 = \hat{q}_m + \hat{f}_m \in \mathcal{K}_2^* \) and \( \hat{g}_m \in \mathcal{K}_3^* \), we have \( \hat{g}_0 \in \mathcal{K}_3^* \), so \( g(t) \in \mathcal{K}_3^* \) and \( t \in (T - T_j, T] \).

The remaining proof is similar to that in proving Lemma 2 and we omit the details here. \( \square \)
\[(CP_2) \quad \min \ (T - T_1) a_1^\top \hat{h} + ((T - T_1) b_1 - T_1 b)^\top \tilde{q} + T_1 a_1 \tilde{f} + T_1 b^\top \tilde{g} \\
\text{s.t.} \quad G^\top \hat{h} + H^\top \tilde{q} - (\gamma + (T - T_1) c_1) \in \mathcal{K}_{31}^*, \\
F^\top \hat{h} - (T - T_1) d_1 \in \mathcal{K}_{41}^*, \\
\hat{h} \in \mathcal{K}_{11}^*, \quad \tilde{q} \in \mathcal{K}_{21}^*, \\
G^\top \tilde{p} + H^\top \tilde{q} - (\gamma_2 + (T - T_1) c_2) \in \mathcal{K}_{32}^*, \\
F^\top \tilde{p} - (T - T_1) d_2 \in \mathcal{K}_{42}^*, \\
\tilde{p} \in \mathcal{K}_{12}^*, \quad \tilde{q} \in \mathcal{K}_{22}^*, \\
G^\top (\hat{h} + \tilde{p}) + G^\top \tilde{f} + H^\top \tilde{g} - (\gamma + Tc) \in \mathcal{K}_3^*, \\
F^\top \tilde{f} - T_1 d \in \mathcal{K}_4^*, \\
\tilde{f} \in \mathcal{K}_1^*, \quad \tilde{g} \in \mathcal{K}_2^*. \tag{32} \]

Note that the constraints of CP\(_1\) and CP\(_2\) above are the same as the constraints of SCP\(_1(\pi)\) and SCP\(_2(\pi')\), respectively, when \(m_1 = m_2 = 1\). The objectives of CP\(_1\) and CP\(_2\), however, are different from those of SCP\(_1(\pi)\) and SCP\(_2(\pi')\). The choice of these objectives is to facilitate the explicit derivation of a bound on the duality gap; see the proof of Theorem II.

**Lemma 5.** If the conic programs CP\(_1\) are strongly feasible and \(\mathcal{K}_1 \subseteq \mathcal{K}_{11}\), \(\mathcal{K}_1 \subseteq \mathcal{K}_{12}\), \(\mathcal{K}_4 \subseteq \mathcal{K}_{41}\), and \(\mathcal{K}_4 \subseteq \mathcal{K}_{42}\), then SSCCP is strongly feasible, and so is SCP\(_1(\pi)\).

**Proof.** (1) We first show that when CP\(_1\) is strongly feasible, SSCCP is strongly feasible.

Suppose \((\bar{u}, \bar{x}, \bar{v}, \bar{y}, \bar{z})\) is a strongly feasible solution to CP\(_1\). We have

\[
\alpha + T_1 a - G\bar{u} - F\bar{x} \in \mathcal{K}_1, \\
T_1 b - H\bar{u} \in \mathcal{K}_2, \\
\bar{u} \in \mathcal{K}_3, \quad \bar{x} \in \mathcal{K}_4, \\
\alpha_1 + Ta_1 - G\bar{u} - G\bar{v} - F\bar{y} \in \mathcal{K}_{11}, \\
(T - T_1) b_1 - Hv \in \mathcal{K}_{21}, \\
\bar{v} \in \mathcal{K}_{31}, \quad \bar{y} \in \mathcal{K}_{41}, \\
\alpha_2 + Ta_2 - G\bar{u} - G\bar{w} - F\bar{z} \in \mathcal{K}_{12}, \\
(T - T_1) b_2 - H\bar{w} \in \mathcal{K}_{22}, \\
\bar{w} \in \mathcal{K}_{32}, \quad \bar{z} \in \mathcal{K}_{42}, \\
\text{and } \bar{x}_0 \text{ is such a constant that} \\
\alpha - F\bar{x}_0 \in \mathcal{K}_1, \quad \bar{x}_0 \in \mathcal{K}_4. \tag{33} \]

Let

\[
u(t) = \frac{\bar{u}}{T_1}, \quad t \in [0, T_1], \\
x(0) = \bar{x}_0, \quad x(T_1) = \bar{x}, \\
x(t) = \frac{T_1 - t}{T_1} \bar{x}_0 + \frac{t}{T_1} \bar{x}, \quad t \in (0, T_1), \\
v(t) = \frac{\bar{v}}{T - T_1}, \quad t \in (T_1, T], \\
y(t) = \frac{T - t}{T - T_1} \bar{y} + \frac{T_1 - t}{T - T_1} \bar{y}, \quad t \in (T_1, T), \\
w(t) = \frac{\bar{w}}{T - T_1}, \quad t \in (T_1, T], \\
z(t) = \bar{z}, \quad z(t) = \frac{T - t}{T - T_1} \bar{z} + \frac{T_1 - t}{T - T_1} \bar{z}, \quad t \in (T_1, T). \tag{35} \]

We have \(u(t) \in \text{int } \mathcal{K}_3, x(t) \in \text{int } \mathcal{K}_4, \ t \in (0, T_1], x(0) \in \mathcal{K}_4, v(t) \in \text{int } \mathcal{K}_{31}, w(t) \in \text{int } \mathcal{K}_{32}, \text{ and } t \in (T_1, T]. \)

Because \(\mathcal{K}_4 \subseteq \mathcal{K}_{41}\) and \(\mathcal{K}_4 \subseteq \mathcal{K}_{42}\), then \(y(t) \in \text{int } \mathcal{K}_{41}\), \(t \in (T_1, T]\), \(z(t) \in \text{int } \mathcal{K}_{42}\) and \(t \in (T_1, T]\).

For \(t = 0\),

\[
\alpha - F\bar{x}_0 = \alpha - F\bar{x}_0 \in \mathcal{K}_1. \tag{36} \]

For \(t \in (0, T_1]\),

\[
\alpha + ta - \int_0^t Gu(s) ds - Fx(t) = \frac{t}{T_1} (\alpha + Ta a - G\bar{u} - F\bar{x}) + \frac{T_1 - t}{T_1} (\alpha - F\bar{x}_0) \tag{37} \]

For \(t \in [0, T_1]\),

\[
b - Hu(t) = b - H\bar{u}_1 = \frac{1}{T_1} (T_1 b - H\bar{u}) \in \text{int } \mathcal{K}_2. \tag{38} \]

For \(t \in (T_1, T]\),

\[
\alpha_1 + ta_1 - \int_0^t Gu(s) ds - \int_0^t Gv(s) ds - Fy(t) = \frac{t - T_1}{T - T_1} (\alpha + Ta a - G\bar{u} - F\bar{y}) \tag{39} \]

\[\epsilon \in \text{int } \mathcal{K}_1. \]

Note that when \(\mathcal{K}_1 \subseteq \mathcal{K}_{11}\), \(\alpha + T_1 a - G\bar{u} - F\bar{x} \in \text{int } \mathcal{K}_{11}\).
For \( t \in (T_1, T]\),
\[
b_1 - H v(t) = b_1 - \frac{\bar{v}}{T - T_1}
= \frac{1}{T - T_1} ( (T - T_1) b_1 - H \bar{v}) \in \text{int } \mathcal{K}_{21}.
\]

Similarly, we can get (by noting that \( \mathcal{K}_1 \subseteq \mathcal{K}_{12} )
\[
\alpha_2 + t \alpha_2 - \int_{T_1}^{T} G(u(s)) ds - \int_{T_1}^{T} Gw(s) ds - Fz(t) \in \text{int } \mathcal{K}_{12},
\]
\[
t \in (T_1, T],
\]
\[
b_2 - H w(t) \in \text{int } \mathcal{K}_{22}, \quad t \in (T_1, T] .
\]

We can see that \((u(t), x(t), v(t), y(t), w(t), z(t))\) is a (two-piece) strongly feasible solution for SSSCP. So SSSCP is strongly feasible.

(2) Now we will show that from this strongly feasible solution for SSSCP, we can get a strongly feasible solution for SCP\(1(\pi)\).

Let
\[
\bar{u}_i = u(t)(t_i - t_{i-1}), \quad \bar{x}_i = x(t_i), \quad i = 1, \ldots, m_1 ;
\]
\[
\bar{v}_i = v(t)(t_i - t_{i-1}), \quad \bar{y}_i = y(t), \quad i = 1, \ldots, m_2 ;
\]
\[
\bar{w}_i = w(t)(t_i - t_{i-1}), \quad \bar{z}_i = z(t), \quad i = 1, \ldots, m_2, .
\]

Then \( \bar{u}_i \in \text{int } \mathcal{K}_3, \bar{x}_i \in \text{int } \mathcal{K}_4, i = 1, \ldots, m_1, \bar{v}_i \in \text{int } \mathcal{K}_{31}, \bar{y}_i \in \text{int } \mathcal{K}_{41}, \bar{w}_i \in \text{int } \mathcal{K}_{32}, \bar{z}_i \in \text{int } \mathcal{K}_{42}, i = 1, \ldots, m_2.

For \( i = 1, \ldots, m_1 \),
\[
\alpha + t_1 \alpha - \left[ G\bar{u}_1 + \cdots + G\bar{u}_m \right] - \left[ G\bar{v}_1 + \cdots + G\bar{v}_m \right]
- F\bar{z}_i \in \text{int } \mathcal{K}_{12},
\]
\[
(t_{m_1} - t_{m_1 - 1}) b - H\bar{u}_i \in \text{int } \mathcal{K}_{22}.
\]

Similarly, we can get.

For \( i = 1, \ldots, m_2 \),
\[
\alpha_2 + t_{m_2} \alpha_2 - \left[ G\bar{u}_1 + \cdots + G\bar{u}_m \right] - \left[ G\bar{v}_1 + \cdots + G\bar{v}_m \right]
- F\bar{z}_i \in \text{int } \mathcal{K}_{12},
\]
\[
(t_{m_2} - t_{m_2 - 1}) b - H\bar{u}_i \in \text{int } \mathcal{K}_{22}.
\]

So \((\bar{u}, \bar{x}, \bar{v}, \bar{y}, \bar{w}, \bar{z})\) is a strongly feasible solution for SCP\(1(\pi)\), and SCP\(1(\pi)\) is strongly feasible.

\(\square\)

**Lemma 6.** If the conic programs CP\(2_1\) are strongly feasible, \( \mathcal{K}_2 \subseteq \mathcal{K}_{21}, \mathcal{K}_3 \subseteq \mathcal{K}_{22}, \mathcal{K}_5 \subseteq \mathcal{K}_{31}, \mathcal{K}_3 \subseteq \mathcal{K}_{32}, \gamma_1 + \gamma_2 = \gamma, \ c_1 + c_2 = c, \) then SSSCP\(^\pi\) is strongly feasible and so is SCP\(2(\pi')\).

**Proof.** The proof is similar to that of Lemma 5; the details are omitted here.

\(\square\)

We shall focus on one partition for \([0, T_1, T]\), denoted by \(\pi(\epsilon_1, \epsilon_2)\) and one partition for \([0, T - T_1, T]\), denoted by \(\pi'(\epsilon_1, \epsilon_2)\). \(\pi(\epsilon_1, \epsilon_2)\) divides the interval \([0, T_1]\) into \(m_1\) equal segments, each of length \(\epsilon_1\), and divides the interval \([T_1, T]\) into \(m_2\) equal segments, each of length \(\epsilon_2\). \(\pi'(\epsilon_1, \epsilon_2)\) divides the interval \([0, T - T_1]\) into \(m_3\) equal segments, each of length \(\epsilon_3\), and divides the interval \([T - T_1, T]\) into \(m_4\) equal segments, each of length \(\epsilon_4\).

If we reverse the inner order of \(u, x, v, y, w, z\) in SCP\(1(\pi(\epsilon_1, \epsilon_2))\), that is, for example, change \(u = (u_1, \ldots, u_{m_1})\) to \((u_{m_1}, \ldots, u_1)\), we get the following problem:

\[
(\text{SCP}_1(\pi(\epsilon_1, \epsilon_2)))
\]
\[
\max h^T_{11}\bar{u} + h^T_{12}\bar{x} + h^T_{13}\bar{y} + \frac{c_1}{2} d_1^T \bar{x}_0
\]
\[ \begin{align*}
&\text{s.t. } g_{11} - \tilde{C}_{m_1} \tilde{u} - \tilde{f}_{m_1} \tilde{x} \in \mathcal{K}_{1,m_1}, \\
f_{11} - \tilde{H}_{m_1} \tilde{u} \in \mathcal{K}_{2,m_1}, \\
g_{12} - \tilde{C}_{m_2,m_1} \tilde{u} - \tilde{c}_{m_2} \tilde{y} - \tilde{F}_{m_2} \tilde{y} \in \mathcal{K}_{11,m_2}, \\
f_{12} - \tilde{H}_{m_2} \tilde{v} \in \mathcal{K}_{12,m_2}, \\
g_{13} - \tilde{C}_{m_3,m_1} \tilde{u} - \tilde{c}_{m_3} \tilde{w} - \tilde{F}_{m_3} \tilde{w} \in \mathcal{K}_{12,m_2}, \\
f_{13} - \tilde{H}_{m_3} \tilde{w} \in \mathcal{K}_{22,m_2}, \\
\tilde{u} \in \mathcal{K}_{3,m_1}, \quad \tilde{x} \in \mathcal{K}_{4,m_1}, \\
\tilde{v} \in \mathcal{K}_{31,m_2}, \quad \tilde{y} \in \mathcal{K}_{41,m_2}, \\
\tilde{w} \in \mathcal{K}_{32,m_3}, \quad \tilde{z} \in \mathcal{K}_{42,m_3}, \\
g_{l1} = m_1 \left\{ \begin{array}{c}
\epsilon_1 b \\
\epsilon_1 b \\
\vdots \\
\epsilon_1 b
\end{array} \right\}, \\
g_{l2} = m_2 \left\{ \begin{array}{c}
\epsilon_2 b_1 \\
\epsilon_2 b_1 \\
\vdots \\
\epsilon_2 b_1
\end{array} \right\}, \\
g_{l3} = m_2 \left\{ \begin{array}{c}
\alpha + T_1 a \\
\alpha + T_1 a \\
\vdots \\
\alpha + T_1 a
\end{array} \right\}, \\
h_{l1} = m_1 \left\{ \begin{array}{c}
y + \left( T - T_1 + \frac{\epsilon_1}{2} c \right) \\
y + \left( T - T_1 + \frac{\epsilon_1}{2} c \right) \\
\vdots \\
y + \left( T - T_1 + \frac{\epsilon_1}{2} c \right)
\end{array} \right\}, \\
h_{l2} = m_2 \left\{ \begin{array}{c}
\hat{y}_1 + \frac{\epsilon_2}{2} \hat{c}_1 \\
\hat{y}_1 + \frac{\epsilon_2}{2} \hat{c}_1 \\
\vdots \\
\hat{y}_1 + \frac{\epsilon_2}{2} \hat{c}_1
\end{array} \right\}, \\
h_{l3} = m_2 \left\{ \begin{array}{c}
\hat{y}_2 + \frac{\epsilon_2}{2} \hat{c}_2 \\
\hat{y}_2 + \frac{\epsilon_2}{2} \hat{c}_2 \\
\vdots \\
\hat{y}_2 + \frac{\epsilon_2}{2} \hat{c}_2
\end{array} \right\}, \\
d_{11} = m_1 \left\{ \begin{array}{c}
\frac{\epsilon_1}{2} d + \frac{\epsilon_2}{2} \tilde{d}_1 + \frac{\epsilon_2}{2} \tilde{d}_2 \\
\frac{\epsilon_1}{2} d + \frac{\epsilon_2}{2} \tilde{d}_1 + \frac{\epsilon_2}{2} \tilde{d}_2 \\
\vdots \\
\frac{\epsilon_1}{2} d + \frac{\epsilon_2}{2} \tilde{d}_1 + \frac{\epsilon_2}{2} \tilde{d}_2
\end{array} \right\}.
\end{align*} \]
Similarly, we have

\[
\begin{align*}
\text{(SCP}_{2}(\pi'(e_1, e_2))) & : \\
& \min \ g_{u1}' \bar{h} + f_{u1}' \bar{f} + \frac{e_2}{2} b_1 \bar{h}_0 + g_{u1}' \bar{p} \\
& + f_{u1}' \bar{g} + \frac{e_2}{2} b_2 \bar{g}_0 + g_{u1}' \bar{f} + f_{u1}' \bar{g} \\
\text{s.t.} & \quad \bar{G}_{m_1} \bar{h} + \bar{H}_{m_1} \bar{f} - h_{u1} \in \mathcal{X}_{31,m_1}, \\
& \quad \bar{F}_{m_1} \bar{h} - d_{u1} \in \mathcal{X}_{41,m_1}, \\
& \quad \bar{f} \in \mathcal{X}_{1,m_1}, \quad \bar{g} \in \mathcal{X}_{2,m_1}, \\
& \quad \bar{G}_{m_1} \bar{p} + \bar{H}_{m_1} \bar{q} - h_{u3} \in \mathcal{X}_{32,m_1}, \\
& \quad \bar{F}_{m_1} \bar{p} - d_{u3} \in \mathcal{X}_{42,m_1}, \\
& \quad \bar{p} \in \mathcal{X}_{12,m_1}, \quad \bar{q} \in \mathcal{X}_{22,m_1}; \\
\end{align*}
\]
\[ g_{u1} = m_1 \begin{cases} \left( \alpha + \left( T_1 - \frac{e_1}{2} \right) a \right) \\ \left( \alpha + \left( T_1 - \frac{3e_1}{2} \right) a \right) \\ \vdots \\ \left( \alpha + \frac{e_1}{2} a \right) \end{cases}, \]

\[ g_{u2} = m_2 \begin{cases} \left( \alpha_1 + \left( T - \frac{e_2}{2} \right) d_1 \right) \\ \left( \alpha_1 + \left( T - \frac{3e_2}{2} \right) d_1 \right) \\ \vdots \\ \left( \alpha_1 + \frac{e_2}{2} d_1 \right) \end{cases}, \]

\[ g_{u3} = m_2 \begin{cases} \left( \alpha_2 + \left( T - \frac{e_2}{2} \right) d_2 \right) \\ \left( \alpha_2 + \left( T - \frac{3e_2}{2} \right) d_2 \right) \\ \vdots \\ \left( \alpha_2 + \frac{e_2}{2} d_2 \right) \end{cases}, \]

\[ h_{u1} = m_1 \begin{cases} \left( y + \left( T - T_1 + e_1 \right) c \right) \\ \left( y + \left( T - T_1 + 2e_1 \right) c \right) \\ \vdots \\ \left( y + Tc \right) \end{cases}, \]

\[ h_{u2} = m_2 \begin{cases} \left( \tilde{y}_1 + e_2 \tilde{c}_1 \right) \\ \left( \tilde{y}_1 + 2e_2 \tilde{c}_1 \right) \\ \vdots \\ \left( \tilde{y}_1 + \left( -T \right) \tilde{c}_1 \right) \end{cases}, \]

\[ h_{u3} = m_2 \begin{cases} \left( \tilde{y}_2 + e_2 \tilde{c}_2 \right) \\ \left( \tilde{y}_2 + 2e_2 \tilde{c}_2 \right) \\ \vdots \\ \left( \tilde{y}_2 + \left( -T \right) \tilde{c}_2 \right) \end{cases}, \]

\[ d_{u1} = m_1 \begin{cases} \left( e_1 d_1 \right) \\ \left( e_1 d_1 \right) \\ \vdots \\ \left( e_1 d_1 \right) \end{cases}, \quad d_{u2} = m_2 \begin{cases} \left( e_2 d_1 \right) \\ \left( e_2 d_1 \right) \\ \vdots \\ \left( e_2 d_1 \right) \end{cases}, \]

\[ d_{u3} = m_2 \begin{cases} \left( e_2 d_2 \right) \\ \left( e_2 d_2 \right) \\ \vdots \\ \left( e_2 d_2 \right) \end{cases}. \]

Note that in \( g_{u1} \), the difference between two adjacent items is \( e_1 a \); in \( g_{u2} \), the difference between two adjacent items is \( e_2 a \); in \( g_{u3} \), the difference between two adjacent items is \( e_2 a \). In \( h_{u1} \), the difference between two adjacent items is \( e_1 c \); in \( h_{u2} \), the difference between two adjacent items is \( e_2 \tilde{c}_1 \); in \( h_{u3} \), the difference between two adjacent items is \( e_2 \tilde{c}_1 \). All the items in \( d_{u1} \) are the same, and this is also true for \( d_{u2} \) and \( d_{u3} \).

Now we write down the relationships between the input parameters in SCP(\( n(e_1, e_2) \)) and SCP(\( n(e_1, e_2) \)), and these following relationships will be used in proving Theorem 11 in Section 5:

\[ g_{u1} - g_{u1} = m_1, \quad \text{same} \begin{cases} \left( -\frac{e_1}{2} a \right) \\ \vdots \\ \left( -\frac{e_1}{2} a \right) \end{cases}, \]

\[ g_{u2} - g_{u2} = m_2, \quad \text{same} \begin{cases} \left( -\frac{e_2}{2} d_1 \right) \\ \vdots \\ \left( -\frac{e_2}{2} d_1 \right) \end{cases}, \]

\[ g_{u3} - g_{u3} = m_2, \quad \text{same} \begin{cases} \left( -\frac{e_2}{2} d_2 \right) \\ \vdots \\ \left( -\frac{e_2}{2} d_2 \right) \end{cases}, \]

\[ f_{u1} - f_{u1} = m_1, \quad \text{same} \begin{cases} \left( 0 \right) \\ \vdots \\ \left( 0 \right) \end{cases}, \]

\[ f_{u2} - f_{u2} = m_2, \quad \text{same} \begin{cases} \left( 0 \right) \\ \vdots \\ \left( 0 \right) \end{cases}, \]

\[ f_{u3} - f_{u3} = m_2, \quad \text{same} \begin{cases} \left( 0 \right) \\ \vdots \\ \left( 0 \right) \end{cases}, \]

\[ h_{u1} - h_{u1} = m_1, \quad \text{same} \begin{cases} \left( \frac{e_1}{2} \right) \\ \vdots \\ \left( \frac{e_1}{2} \right) \end{cases}, \]

\[ h_{u2} - h_{u2} = m_2, \quad \text{same} \begin{cases} \left( \frac{e_2}{2} \right) \\ \vdots \\ \left( \frac{e_2}{2} \right) \end{cases}, \]

\[ h_{u3} - h_{u3} = m_2, \quad \text{same} \begin{cases} \left( \frac{e_2}{2} \right) \\ \vdots \\ \left( \frac{e_2}{2} \right) \end{cases}. \]
5. Strong Duality

In this section, we will prove that under some mild and verifiable conditions, strong duality holds between SSCP and its dual.

Let $\pi_1$ denote a partition of $[0,T - T_1, T]$, $\pi_1 = \{t_0, t_1, \ldots, t_m, t_{m+1}, \ldots, t_{m+m+2}\}$, with $t_0 = 0, t_{m+1} = T - T_1$, $t_{m+m+2} = T$, and $t_1 - t_0 = t_{m+1} - t_m = (t_0, t_1)$, $t_{m+2} - t_{m+1} = (t_1/2)$, while $t_j - t_{j-1} = \epsilon_2$, $i = 2, \ldots, m, t_j - t_{j-1} = \epsilon_1$, $j = m_2 + 3, \ldots, m_2 + m_1 + 1$.

**Lemma 7.** If the conic programs $SCP_2(\pi_1)$ are strongly feasible and $\mathcal{K}_1 \subseteq \mathcal{K}_3, \mathcal{K}_2 \subseteq \mathcal{K}_4$, $\mathcal{K}_4 \subseteq \mathcal{K}_2$, then $SCP_3^* (\pi(\epsilon_1, \epsilon_2))$ is strongly feasible.

**Proof.** The constraints of $(SCP_2(\pi_1))$ are:

\[
\begin{array}{l}
\left( \begin{array}{c}
\mathcal{G}_{m_1}^* \\
G' \ldots G' \\
\end{array} \right) \mathbf{h} + \left( \begin{array}{c}
\mathcal{H}_{m_2}^* \\
H' \\
\end{array} \right) \mathbf{i}
\end{array}
\]

is a strongly feasible solution of $SCP_2(\pi_1)$, then

\[
\begin{array}{l}
\left( \begin{array}{c}
\widehat{\mathbf{h}}_1, \ldots, \widehat{\mathbf{h}}_{m_1+1} \\
\widehat{\mathbf{i}}_1, \ldots, \widehat{\mathbf{i}}_{m_2+1} \\
\end{array} \right), \left( \begin{array}{c}
\widehat{\mathbf{p}}_1, \ldots, \widehat{\mathbf{p}}_{m_3+1} \\
\widehat{\mathbf{q}}_1, \ldots, \widehat{\mathbf{q}}_{m_4+1} \\
\end{array} \right)
\end{array}
\]

is a strongly feasible solution to $SCP_3^* (\pi(\epsilon_1, \epsilon_2))$.

From Lemma 5, we know if the conic programs CP are strongly feasible and $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and $\mathcal{K}_3 \subseteq \mathcal{K}_4$, then $SCP_3(\pi(\epsilon_1, \epsilon_2))$ is strongly feasible.

From Lemma 6, we know if the conic programs CP are strongly feasible and $\mathcal{K}_2 \subseteq \mathcal{K}_3$, $\mathcal{K}_3 \subseteq \mathcal{K}_4$, then $SCP_3(\pi(\epsilon_1, \epsilon_2))$ is strongly feasible.

So under the condition that the conic programs (CP) and (CP) are strongly feasible, $\mathcal{K}_1 \subseteq \mathcal{K}_2, \mathcal{K}_2 \subseteq \mathcal{K}_3$, $\mathcal{K}_3 \subseteq \mathcal{K}_4$, $\mathcal{K}_4 \subseteq \mathcal{K}_2$, $\gamma_1 + \gamma_2 = \gamma$ and $\epsilon_1 + \epsilon_2 = \epsilon$, $SCP_3(\pi^*)$ is strongly feasible. Now, from Lemma 7, if additionally $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $\mathcal{K}_2 \subseteq \mathcal{K}_3$, $\mathcal{K}_3 \subseteq \mathcal{K}_4$, $\mathcal{K}_4 \subseteq \mathcal{K}_2$, $\gamma_1 + \gamma_2 = \gamma$, $\epsilon_1 + \epsilon_2 = \epsilon$, $SCP_3(\pi(\epsilon_1, \epsilon_2))$ is solvable.
The dual of SCP₂(\(\pi'(e₁, e₂)\)) without the constant terms \((e₂/2)b₁^{t₁₁} + (e₂/2)b₂^{t₁₂}\) in the objective is the following:

\[
SCP₂^*(\pi'(e₁, e₂)) \quad \max \quad h₁' + d₁' + h₂' + \nu' \quad \text{s.t.} \quad g₁₁ + \nu₁₁ \leq 0, \quad g₁₂ + \nu₁₂ \leq 0, \quad (\nu₁₁, \nu₁₂) \in \mathcal{K}_{1,m₁},
\]

\[
SCP₂(\pi'(e₁, e₂)) \quad \max \quad h₁' + d₁' + h₂' + \nu' \quad \text{s.t.} \quad g₁₁ + \nu₁₁ \leq 0, \quad g₁₂ + \nu₁₂ \leq 0, \quad (\nu₁₁, \nu₁₂) \in \mathcal{K}_{1,m₁},
\]

We use \(\pi₂\) to denote the partition \(\{t₀, t₁, t₂, \ldots, tₘ₁, tₘ₁+1, tₘ₁+2\}\), with \(t₀ = 0, t₁ = t₂ = tₘ₁, tₘ₁+1 = tₘ₁+2 = T\), and \(t₁ - t₀ = tₘ₁ + 1 - t₁ = e₁/2, tₘ₁ + 2 - tₘ₁ + 1 = e₁/2\), while \(t₁ - t₁₋₁ = e₁\), \(i = 2, \ldots, m₁; t_j - t_j₋₁ = e₂\), \(j = m₁ + 1, \ldots, m₁ + m₂ + 1\).

**Lemma 8.** If the conic programs SCP₂(\(\pi₂\)) are strongly feasible and \(\mathcal{H}_2 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \mathcal{H}_4 \subseteq \mathcal{H}_{12}\), \(\mathcal{H}_3 \subseteq \mathcal{H}_4\), then \(SCP₂^*(\pi'(e₁, e₂))\) is strongly feasible.

**Proof.** If we reverse the inner order of \(\bar{u}, \bar{x}, \bar{v}, \bar{w}, \bar{z}\), the constraints of SCP₂(\(\pi₂\)) are:

\[
\begin{align*}
\left(\alpha₂ + Ta₂\right) \quad & - \left(\begin{array}{c}
G \\
G \vdots G \\
G \\
\end{array}\right) \quad \bar{u} \quad - \left(\begin{array}{c}
\bar{u} \\
\bar{u} \\
\bar{u} \\
\end{array}\right) \quad \bar{v} \quad \in \mathcal{H}_{1,m₁+1}, \\
\left(\frac{e₁}{b} \right) \quad & - \left(\begin{array}{c}
H \\
H \\
H \\
\end{array}\right) \quad \bar{u} \quad \in \mathcal{H}_{2,m₁+1}, \\
\left(\alpha₂ + Ta₂\right) \quad & - \left(\begin{array}{c}
G \\
G \vdots G \\
G \\
\end{array}\right) \quad \bar{u} \quad - \left(\begin{array}{c}
\bar{u} \\
\bar{u} \\
\bar{u} \\
\end{array}\right) \quad \bar{v} \quad \in \mathcal{H}_{1,m₁+1}, \\
\left(\frac{e₁}{b} \right) \quad & - \left(\begin{array}{c}
H \\
H \\
H \\
\end{array}\right) \quad \bar{u} \quad \in \mathcal{H}_{2,m₁+1}, \\
\end{align*}
\]

When \(\mathcal{H}_2 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \mathcal{H}_4 \subseteq \mathcal{H}_{12}\), comparing the above with the constraints of SCP₂(\(\pi'(e₁, e₂)\)), we observe that if

\[
\begin{align*}
\left(\bar{u}_1, \ldots, \bar{u}_{m₁+1}\right), \left(\bar{x}_1, \ldots, \bar{x}_{m₁+1}\right), \left(\bar{v}_1, \ldots, \bar{v}_{m₁+1}\right), \\
\left(\bar{y}_1, \ldots, \bar{y}_{m₁+1}\right), \left(\bar{w}_1, \ldots, \bar{w}_{m₁+1}\right), \left(\bar{z}_1, \ldots, \bar{z}_{m₁+1}\right)
\end{align*}
\]

is a strongly feasible solution to SCP₁(\(\pi₂\)), then

\[
\begin{align*}
\left(\bar{u}_1, \ldots, \bar{u}_{m₁+1}\right), \left(\bar{x}_1, \ldots, \bar{x}_{m₁+1}\right), \left(\bar{v}_1, \ldots, \bar{v}_{m₁+1}\right), \\
\left(\bar{y}_1, \ldots, \bar{y}_{m₁+1}\right), \left(\bar{w}_1, \ldots, \bar{w}_{m₁+1}\right), \left(\bar{z}_1, \ldots, \bar{z}_{m₁+1}\right)
\end{align*}
\]

is a strongly feasible solution to SCP₂*(\(\pi'(e₁, e₂)\)).

From Lemma 5, we know that if the conic program (CP₁) is strongly feasible, and \(\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \mathcal{H}_4 \subseteq \mathcal{H}_{12}\), then SCP₂(\(\pi₂\)) is strongly feasible. From Lemma 8, if additionally, \(\mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \mathcal{H}_4 \subseteq \mathcal{H}_{12}\), \(\mathcal{H}_3 \subseteq \mathcal{H}_{12}\), we have SCP₂*(\(\pi'(e₁, e₂)\)) is strongly feasible.

From Lemma 6, we know that if the conic program (CP₁) is strongly feasible and \(\mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \mathcal{H}_4 \subseteq \mathcal{H}_{12}\), \(\mathcal{H}_3 \subseteq \mathcal{H}_{12}\), then SCP₂(\(\pi'(e₁, e₂)\)) is strongly feasible. So under the condition that the conic programs CP₁ and CP₂ are strongly feasible and \(\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \mathcal{H}_4 \subseteq \mathcal{H}_{12}\), SCP₂*(\(\pi'(e₁, e₂)\)) is solvable.

Now we have the following.
Lemma 9. If the conic programs $\mathcal{CP}_1$ and $\mathcal{CP}_2$ are strongly feasible and $\mathcal{K}_1 \subseteq \mathcal{K}_{11}, \mathcal{K}_2 \subseteq \mathcal{K}_{12}, \mathcal{K}_2 \subseteq \mathcal{K}_{22}, \mathcal{K}_3 \subseteq \mathcal{K}_{31}, \mathcal{K}_3 \subseteq \mathcal{K}_{32}, \mathcal{K}_4 \subseteq \mathcal{K}_{41}, \mathcal{K}_4 \subseteq \mathcal{K}_{42}, \gamma_1 + \gamma_2 = \gamma, \tilde{c}_1 + \tilde{c}_2 = c$, both $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ and $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$ are solvable, then, one has

$$v(\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))) \leq v(\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2)))$$

Proposition 10. Suppose that $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ and $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$ are solvable. Then, one has

$$v(\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))) \leq v(\mathcal{SSCPP}) \leq v(\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2)))$$

Proof. Because $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ and $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$ are solvable, they have finite optimal objective values $v(\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2)))$ and $v(\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2)))$.

From Lemma 2, we know that the optimal of $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$, the first inequality (since $\mathcal{SSCPP}$ is a maximization problem). A similar argument justifies the third inequality. The second inequality follows from the weak duality in Proposition I.

Theorem 11. Suppose $\mathcal{CP}_1$ and $\mathcal{CP}_2$ are strongly feasible, with finite optimal values. $\mathcal{K}_1 \subseteq \mathcal{K}_{11}, \mathcal{K}_2 \subseteq \mathcal{K}_{12}, \mathcal{K}_2 \subseteq \mathcal{K}_{22}, \mathcal{K}_3 \subseteq \mathcal{K}_{31}, \mathcal{K}_4 \subseteq \mathcal{K}_{32}, \mathcal{K}_4 \subseteq \mathcal{K}_{41}, \mathcal{K}_4 \subseteq \mathcal{K}_{42}, \gamma_1 + \gamma_2 = \gamma, \tilde{c}_1 + \tilde{c}_2 = c$. If one lets the number of intervals in $\pi, \pi'$, and both $m$ and $m_2$ be the same and both of them equal to $m$, then there exists a constant $\Gamma > 0$, which is independent of $m$, such that

$$v(\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))) - v(\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))) \leq \frac{\Gamma}{m}.$$  (60)

Consequently, one must have $v(\mathcal{SSCPP}) = v(\mathcal{SSCPP}^*)$; that is, strong duality holds.

Proof. First note that strong duality follows immediately from the inequality in (60) by letting $m \to \infty$, taking into account the inequalities in Proposition 10.

To establish the error bound in (60), consider the following primal–dual pair of conic programs:

$$\text{(SCP)} \quad \max \quad h_{11}^* \tilde{u} + d_{11}^* \tilde{x} + h_{12}^* \tilde{y} + d_{12}^* \tilde{w} + d_{13}^* \tilde{z}$$

$$\text{s.t.} \quad g_1 - \bar{G}_{1,1} \tilde{u} - \bar{F}_{1,1} \tilde{x} \in \mathcal{K}_{1,1},$$

$$f_{11} - \bar{H}_{1,1} \tilde{u} \in \mathcal{K}_{2,1},$$

$$g_2 - \bar{G}_{1,2} \tilde{u} - \bar{G}_{2,2} \tilde{v} - \bar{F}_{2,2} \tilde{y} \in \mathcal{K}_{1,2},$$

$$f_{12} - \bar{H}_{1,2} \tilde{v} \in \mathcal{K}_{2,2},$$

$$g_3 - \bar{G}_{1,3} \tilde{u} - \bar{G}_{3,3} \tilde{w} - \bar{F}_{3,3} \tilde{z} \in \mathcal{K}_{1,3},$$

$$f_{13} - \bar{H}_{1,3} \tilde{w} \in \mathcal{K}_{2,3},$$

$$\quad \tilde{u} \in \mathcal{K}_{3,1}, \quad \tilde{x} \in \mathcal{K}_{4,1}, \quad \tilde{v} \in \mathcal{K}_{3,1},$$

$$\quad \tilde{y} \in \mathcal{K}_{4,1}, \quad \tilde{w} \in \mathcal{K}_{3,1}, \quad \tilde{z} \in \mathcal{K}_{4,1}.$$  \hspace{1cm} (61)

Note that the problem in (62) has the same constraints as $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$ but the objective function of $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$, whereas the problem in (8) has the constraints of $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ but the objective function of $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$.

Hence, both primal and dual are strongly feasible, since $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ and $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$ are. Consequently they both have optimal solutions and their respective optimal objective values coincide.

We denote the optimal solutions for SCP as $(\tilde{u}^*, \tilde{x}^*, \tilde{v}^*, \tilde{w}^*, \tilde{z}^*)$ and the optimal solution for $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ as $(\tilde{h}^*, \tilde{p}^*, \tilde{q}^*, \tilde{f}^*, \tilde{g}^*)$. Note that these are feasible solutions to $\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))$ and $\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$, respectively. Hence, we have

$$v(\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))) \geq h_{11}^* \tilde{u}^* + d_{11}^* \tilde{x}^* + h_{12}^* \tilde{y}^*$$

$$\quad + d_{13}^* \tilde{w}^* + d_{12}^* \tilde{z}^* + \frac{\epsilon_1}{2} d \tilde{x}_0.$$  \hspace{1cm} (63)

Hence,

$$v(\mathcal{SCP}_2(\pi'(\epsilon_1, \epsilon_2))) - v(\mathcal{SCP}_2(\pi(\epsilon_1, \epsilon_2))) \leq g_{12}^* \tilde{h}^* + f_{12}^* \tilde{p}^* + \frac{\epsilon_2}{2} b_{12} \tilde{q}_0 + g_{13}^* \tilde{p}^* + f_{13}^* \tilde{g}^*.$$  \hspace{1cm} (63)
\[-\left(h'_{11}\hat{u}^* + d'_{11}\hat{x}^* + h'_{12}\hat{y}^* + d'_{12}\hat{y}^* + h'_{13}\hat{w}^*ight) + d'_{13}\hat{z}^* + \frac{e_1}{2}d'\hat{x}_0\right]

\[= \begin{pmatrix} \frac{e_2}{2}a_1 \\ \frac{e_2}{2}a_1 \\ \vdots \\ \frac{e_2}{2}a_1 \end{pmatrix}' \tilde{h}^*
\]

\[+ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}' \hat{f}_{12}
\]

\[= \begin{pmatrix} \frac{e_2}{2}a_2 \\ \frac{e_2}{2}a_2 \\ \vdots \\ \frac{e_2}{2}a_2 \end{pmatrix}' \tilde{g}^*
\]

\[+ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}' \hat{f}_{13}
\]

\[= \begin{pmatrix} \frac{e_2}{2}a \\ \frac{e_2}{2}a \\ \vdots \\ \frac{e_2}{2}a \end{pmatrix}' \tilde{g}^*
\]

\[+ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}' \hat{f}_n
\]

\[-\left[h_{11} \hat{u} + d_{11} \hat{x} + h_{12} \hat{y} + d_{12} \hat{y} + h_{13} \hat{w}\right]
\]

\[+ \frac{e_1}{2}d \hat{x}_0\]

\[= \begin{pmatrix} \frac{e_2}{2}d_1 \\ \frac{e_2}{2}d_1 \\ \vdots \\ \frac{e_2}{2}d_1 \end{pmatrix}' \hat{x}^*
\]

\[+ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}' \hat{d}_{u1}
\]

\[= \begin{pmatrix} \frac{e_2}{2}d_2 \\ \frac{e_2}{2}d_2 \\ \vdots \\ \frac{e_2}{2}d_2 \end{pmatrix}' \hat{y}^*
\]

\[+ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}' \hat{d}_{u2}
\]

\[= \begin{pmatrix} \frac{e_2}{2}d_3 \\ \frac{e_2}{2}d_3 \\ \vdots \\ \frac{e_2}{2}d_3 \end{pmatrix}' \hat{w}^*
\]

\[+ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}' \hat{d}_{u3}
\]

\[= \frac{e_2}{2}a' \left(\sum_{j=1}^{m_1} h_j^*\right) + \left(\frac{e_1}{2}b - \frac{e_2}{2}b_0\right)' \hat{g}_{m_2}^*
\]

\[-\frac{e_2}{2}a' \left(\sum_{j=1}^{m_1} h_j^*\right) + \left(\frac{e_2}{2}b - \frac{e_2}{2}b_0\right)' \hat{g}_{m_2}^*
\]

\[-\frac{e_1}{2}a' \left(\sum_{j=1}^{m_1} f_j^*\right) - \frac{e_1}{2}b' \hat{g}_{m_1} + \frac{e_2}{2}b' (\hat{b}_0 + \hat{b}_0' \hat{q}_0)
\]

\[-\frac{e_1}{2}a' \left(\sum_{j=1}^{m_1} f_j^*\right)
\]

\[+ \left(\frac{e_2}{2} (\hat{d}_1 + \hat{d}_2) + \frac{e_1}{2} d\right)' \hat{x}_{m_1}
\]

\[-\frac{e_2}{2}a' \left(\sum_{j=1}^{m_2} v_j^*\right) - \frac{e_2}{2}d_1 \hat{y}_{m_2}^*
\]

\[-\frac{e_2}{2}a' \left(\sum_{j=1}^{m_2} v_j^*\right) - \frac{e_2}{2}d_2 \hat{z}_{m_2}^* + \frac{e_1}{2} d' \hat{x}_0\]

\[\text{(64)}\]
If \( m_1 = m_2 = m \), then \( \epsilon_1 = T_1/m, \epsilon_2 = (T - T_1)/m \). Hence, we have

\[
\nu \left( \text{SCP} \left( \pi'(\epsilon_1, \epsilon_2) \right) \right) - \nu \left( \text{SCP} \left( \pi(\epsilon_1, \epsilon_2) \right) \right) \\
\leq \frac{1}{2m} \left[ -(T - T_1) a'_1 \left( \sum_{j=1}^{m} \bar{h}'_j \right) + (T_1 b - (T - T_1) b_1) \bar{f}'_m - (T - T_1) a'_2 \left( \sum_{j=1}^{m} \bar{p}'_j \right) + (T_1 b - (T - T_1) b_2) \bar{q}'_m \\
- T_1 a'_1 \left( \sum_{j=1}^{m} \bar{h}'_j \right) - T_1 b'_1 \bar{g}'_m + (T - T_1) \left( b_1 \bar{f}'_0 + b_2 \bar{q}'_0 \right) \\
+ T_1 c'_1 \left( \sum_{j=1}^{m} \bar{u}'_j \right) - (T - T_1) \left( \bar{a}_1 + \bar{a}_2 \right) + T_1 d' \bar{x}'_m \\
+ (T - T_1) c'_2 \left( \sum_{j=1}^{m} \bar{v}'_j \right) + (T - T_1) \bar{d}'_1 \bar{y}'_m \\
+ (T - T_1) c'_2 \left( \sum_{j=1}^{m} \bar{w}'_j \right) + (T - T_1) \bar{d}'_2 \bar{z}'_m - T_1 d' \bar{x}_0 \right].
\]

From the primal feasibility of \((\bar{u}'^*, \bar{x}'^*, \bar{v}'^*, \bar{f}'^*, \bar{w}'^*, \bar{z}'^*)\), we have

\[
\alpha + T_1 a - G \left( \sum_{i=1}^{m} \bar{u}'_i \right) - F \bar{y}'_m \in \mathcal{K}_1, \\
T_1 b - H \left( \sum_{i=1}^{m} \bar{u}'_i \right) \in \mathcal{K}_2, \\
\alpha_1 + T \bar{a}_1 - G \left( \sum_{i=1}^{m} \bar{u}'_i \right) - G \left( \sum_{i=1}^{m} \bar{v}'_i \right) - F \bar{y}'_m \in \mathcal{K}_1, \\
(T - T_1) b_1 - H \left( \sum_{i=1}^{m} \bar{v}'_i \right) \in \mathcal{K}_2, \\
\alpha_2 + T \bar{a}_2 - G \left( \sum_{i=1}^{m} \bar{u}'_i \right) - G \left( \sum_{i=1}^{m} \bar{w}'_i \right) - F \bar{z}'_m \in \mathcal{K}_2, \\
(T - T_1) b_2 - H \left( \sum_{i=1}^{m} \bar{w}'_i \right) \in \mathcal{K}_2, \\
\sum_{i=1}^{m} \bar{u}'_i \in \mathcal{X}_3, \quad \bar{x}'_m \in \mathcal{X}_4, \quad \sum_{i=1}^{m} \bar{v}'_i \in \mathcal{X}_3, \quad \bar{y}'_m \in \mathcal{X}_4, \\
\sum_{i=1}^{m} \bar{w}'_i \in \mathcal{X}_3, \quad \bar{z}'_m \in \mathcal{X}_4.
\]

So we know that \((\sum_{i=1}^{m} \bar{u}'_i, \bar{x}'_m, \sum_{i=1}^{m} \bar{v}'_i, \bar{y}'_m, \sum_{i=1}^{m} \bar{w}'_i, \bar{z}'_m)\) is a feasible solution to \(\text{CP}_1\). Thus,

\[
T_1 c \left( \sum_{i=1}^{m} \bar{u}'_i \right) - \left( (T - T_1) \right) (\bar{a}_1 + \bar{a}_2) + T_1 d \bar{x}'_m \\
+ (T - T_1) c \left( \sum_{i=1}^{m} \bar{v}'_i \right) + (T - T_1) \bar{d}'_1 \bar{y}'_m \\
+ (T - T_1) c \left( \sum_{i=1}^{m} \bar{w}'_i \right) + (T - T_1) \bar{d}'_2 \bar{z}'_m \\
\leq \nu \left( \text{CP}_1 \right).
\]

Similarly, from the dual feasibility of \((\bar{h}'^*, \bar{r}'^*, \bar{p}'^*, \bar{q}'^*, \bar{f}'^*, \bar{g}'^*)\), we have

\[
G' \left( \sum_{i=1}^{m} \bar{h}'_i \right) + H' \bar{r}'_m - (\bar{y}_1 + (T - T_1) \bar{c}_1) \in \mathcal{K}^*_1, \\
F' \left( \sum_{i=1}^{m} \bar{h}'_i \right) - (T - T_1) \bar{a}_1 \in \mathcal{K}^*_1, \\
G' \left( \sum_{i=1}^{m} \bar{p}'_i \right) + H' \bar{q}'_m - (\bar{y}_2 + (T - T_1) \bar{c}_2) \in \mathcal{K}^*_2, \\
F' \left( \sum_{i=1}^{m} \bar{p}'_i \right) - (T - T_1) \bar{d}_2 \in \mathcal{K}^*_2, \\
G' \left( \sum_{i=1}^{m} \bar{h}'_i + \sum_{i=1}^{m} \bar{p}'_i \right) + G' \left( \sum_{i=1}^{m} \bar{r}'_i \right) + H' \bar{g}'_m - (\bar{y} + Tc) \in \mathcal{K}^*_3,
\]

\[
\sum_{i=1}^{m} \bar{h}'_i \in \mathcal{K}^*_1, \quad \bar{r}'_m \in \mathcal{K}^*_2, \quad \sum_{i=1}^{m} \bar{p}'_i \in \mathcal{K}^*_1, \quad \bar{g}'_m \in \mathcal{K}^*_2, \\
\bar{a}'_m \in \mathcal{X}^*_1, \quad \sum_{i=1}^{m} \bar{f}'_i \in \mathcal{X}^*_1, \quad \bar{g}'_m \in \mathcal{K}^*_2.
\]

Hence, \((\sum_{i=1}^{m} \bar{h}'_i, \bar{r}'_m, \sum_{i=1}^{m} \bar{p}'_i, \bar{q}'_m, \sum_{i=1}^{m} \bar{f}'_i, \bar{g}'_m)\) is a feasible solution to \(\text{CP}_2\), and

\[
(T - T_1) a'_1 \left( \sum_{j=1}^{m} \bar{h}'_j \right) - (T_1 b - (T - T_1) b_1) \bar{f}'_m \\
+ (T - T_1) a'_2 \left( \sum_{j=1}^{m} \bar{p}'_j \right) - (T_1 b - (T - T_1) b_2) \bar{q}'_m \\
+ T_1 a'_1 \left( \sum_{j=1}^{m} \bar{f}'_j \right) + T_1 b' \bar{g}'_m \\
\geq \nu \left( \text{CP}_2 \right).
\]
Putting the above together, we have
\[
V(SCP_2(\pi'(e_1, e_2))) - V(SCP_1(\pi(e_1, e_2)))
\leq \frac{1}{2m} \left[ -v(CP_2) + v(CP_1) + (T - T_1) \right. \\
\left. \times (b_1^T\hat{x}_0 + b_2^T\hat{q}_0) - T_1d^T\hat{x}_0 \right].
\]

Hence, we can let
\[
\Gamma := \frac{1}{2} \left[ -v(CP_2) + v(CP_1) + (T - T_1) \right. \\
\left. \times (b_1^T\hat{x}_0 + b_2^T\hat{q}_0) - T_1d^T\hat{x}_0 \right],
\]

and \( \Gamma < \infty \), since \( v(CP_1) < \infty \) and \( v(CP_2) < \infty \) as assumed.

\[\square\]

6. The Approximation Algorithm

Proposition 10 and Theorem 11 suggest that we can solve SSCCP and their dual approximately through solving their discretized versions, the ordinary conic program \( SCP_1(\pi(e_1, e_2)) \) and \( SCP_2(\pi'(e_1, e_2)) \). The latter is readily solvable by standard algorithms, for example, SeDuMi [27], and the (discrete) solution can then be extended into the piecewise-constant control and piecewise-linear state variables as a feasible solution to SSCCP. Furthermore, the explicit error bound in (60) means that we can achieve any required accuracy by partitions \( \pi(e_1, e_2) \) and \( \pi'(e_1, e_2) \) with a sufficiently large number \( m \) to construct the discretized conic programs \( SCP_1(\pi(e_1, e_2)) \) and \( SCP_2(\pi'(e_1, e_2)) \). Specifically, if \( \delta \) is the required accuracy, then we can choose
\[
m = \left\lfloor \frac{\Gamma}{\delta} \right\rfloor,
\]
where (refer to the end of the proof of Theorem 11)
\[
\Gamma := \frac{1}{2} \left[ -v(CP_2) + v(CP_1) + (T - T_1) \right. \\
\left. \times (b_1^T\hat{x}_0 + b_2^T\hat{q}_0) - T_1d^T\hat{x}_0 \right].
\]

Then, from (59) and (60), we have
\[
V(SSCCP^*) - V(SSCCP) \leq \frac{T}{m} \leq \delta.
\]

Note that the constraints of the above three problems originate from (14) and (27). Clearly, maximizing \( d^T\hat{x}_0 \) and minimizing both \( b_1^T\hat{q}_0 \) and \( b_2^T\hat{q}_0 \) improve our estimation of the error bound.

In summary, our algorithm amounts to solving six conic programming problems: \( CP_1, CP_2, (75), \) and \( SCP_1(\pi(e_1, e_2)) \). Conic programs are known to be polynomially solvable. Hence, ours is a polynomial-time algorithm.

Of course, with \( m \) increases, the computational burden in terms of solving the discretized problems increases. However, the discretized problems are all ordinary conic programming problems and they are polynomially solvable. There exist softwares (e.g., SeCuMi [27], CVXOPT [28], etc.) which can solve conic programming problems efficiently. So the increased computational burden does not really pose a problem in this algorithm.

7. Conclusion and Future Work

In this paper, we have developed a duality theory for SSCCP, which is an important extension on SCCP. Specifically, we have shown that the strong duality between SSCCP and its dual is implied by two related ordinary conic programs \( CP_1 \) and \( CP_2 \) being strongly feasible with finite optimal values. We have also developed a polynomial-time approximation algorithm that solves SSCCP to any desired accuracy with an easily computable error bound, based on the strong duality result.

All these results can be readily generalized for SSCCP with three or more stages and with finite number of scenarios in each stage, without essential difficulty.

From Theorem 11, we know that as \( m \to \infty \), the duality gap tends to 0, and the optimal objective value of the discretized conic program \( V(SCP_1(\pi'(e_1, e_2))) \) approaches the optimal objective value of the original SSCCP. In the future, we plan to investigate whether the optimal solution to \( SCP_1(\pi'(e_1, e_2)) \) will also approach the optimal solution to SSCCP.
Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References


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