Research Article

Exponential Stability of Neutral Stochastic Functional Differential Equations with Two-Time-Scale Markovian Switching

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Received 22 December 2013; Accepted 16 January 2014; Published 16 March 2014

Academic Editor: Weihai Zhang

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We develop exponential stability of neutral stochastic functional differential equations with two-time-scale Markovian switching modeled by a continuous-time Markov chain which has a large state space. To overcome the computational effort and the complexity, we split the large-scale system into several classes and lump the states in each class into one class by the different states of changes of the subsystems; then, we give a limit system to effectively “replace” the large-scale system. Under suitable conditions, using the stability of the limit system as a bridge, the desired asymptotic properties of the large-scale system with Brownian motion and Poisson jump are obtained by utilizing perturbed Lyapunov function methods and Razumikhin-type criteria. Two examples are provided to demonstrate our results.

1. Introduction

In many practical dynamical systems such as neural networks, computer aided design, population ecology, chemical process simulation, and automatic control, stochastic differential equations represent the class of important dynamics (see [1–4]). During the recent several years, the asymptotic properties of neutral stochastic functional differential equations have been investigated by many authors (see [5–14]). Mao [10, 11] gave the exponential stability of neutral stochastic functional differential equations by using the Razumikhin-type theorems. Zhou and Hu [14] used the same argument to discuss the exponential stability in p th moment of neutral stochastic functional differential equations and neutral stochastic functional differential equations with Markovian switching. Wu et al. [13] examined the almost sure robust stability of nonlinear neutral stochastic functional differential equations with infinite delay, including the exponential stability and the polynomial stability. Song and Shen [12] investigated the asymptotic behavior of neutral stochastic functional differential equations under the more general conditions than the classical linear growth condition. Chen et al. [5] considered the exponential stability in mean square

moment of mild solution for impulsive neutral stochastic partial functional differential equations by employing the inequality technique. The attraction and quasi-invariant sets of neutral stochastic partial functional differential equations were also studied in the recent paper [9].

In this paper, we will consider neutral stochastic functional differential equations with two-time-scale Markovian switching modeled by a continuous-time Markov chain which has a large state space. The computational effort and the complexity become a real concern. To overcome the difficulties, we have devoted much effort to the modeling and analysis of such systems, in which one of the main ideas is to split a large-scale system into several classes and lump the states in each class into one state (see [3, 15–21]). Khasminskii et al. for the first time established the asymptotic properties of the Markov chain $r^\varepsilon(\cdot)$ by introducing a small parameter $\varepsilon > 0$ (see [22]). Yin and Zhang developed the method in their book [4] that a complicated system can be replaced by the corresponding limit system that has a much simpler structure. Motivated by the papers [16, 21], under suitable conditions, using the stability of the limit system as a bridge, we will study the exponential stability of neutral stochastic functional differential equations with Brownian

motion and Poisson jump by utilizing perturbed Lyapunov function methods and Razumikhin-type criteria.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and notions needed in our investigation. In Section 3, we state our main results, that is, exponential stability of neutral stochastic functional differential equations with two-time-scale Markovian switching. The exponential stability for neutral stochastic functional differential equations driven by pure jumps is also discussed in Section 4. Finally, two examples are presented to justify and illustrate applications of the theory in Section 5.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Let $W(t) = (W_1(t), \dots, W_m(t))^T$ be an m -dimensional Brownian motion defined on the probability space. For $\tau > 0$, let $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}^n$ with norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T , while its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Denote by $C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$ measurable and bounded $C([-\tau, 0]; \mathbb{R}^n)$ -valued random variables. For $p > 0$ and $t \geq 0$, denote by $L_{\mathcal{F}_t}^p([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_t$ -measurable $C([-\tau, 0]; \mathbb{R}^n)$ -valued random variables $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau \leq \theta \leq 0} E|\phi(\theta)|^p < \infty$. We will denote the indicator function of a set G by I_G .

Consider an n -dimensional neutral stochastic functional differential equation with Markovian switching as follows:

$$\begin{aligned} d[x(t) - D(x_t, r(t))] \\ = f(x_t, t, r(t)) dt + g(x_t, t, r(t)) dw(t), \end{aligned} \quad (1)$$

on $t \geq 0$ with initial data $x_0 = \xi \in C([-\tau, 0]; \mathbb{R}^n)$ and $x_t = x(t + \theta) : -\tau \leq \theta \leq 0$, which is regarded as a $C([-\tau, 0]; \mathbb{R}^n)$ -valued stochastic process. Moreover, $f : C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathbb{S} \rightarrow \mathbb{R}^n$, $g : C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathbb{S} \rightarrow \mathbb{R}^{n \times m}$, $D : C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{S} \rightarrow \mathbb{R}^n$.

Let $r(t)$ ($t \geq 0$) be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, M\}$ with generator $\Gamma = (\gamma_{ij})_{M \times M}$ given by

$$\mathbb{P}\{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases} \quad (2)$$

where $\Delta > 0$. Here, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$.

We assume the Markov $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is well known that almost every sample path $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$. As a standing hypothesis, we assume that the Markov

chain is irreducible. This is equivalent to the condition that, for any $i, j \in \mathbb{S}$, we can find $i_1, i_2, \dots, i_k \in \mathbb{S}$, such that

$$\gamma_{i,i_1} \gamma_{i_1,i_2} \cdots \gamma_{i_k,j} > 0. \quad (3)$$

Then, Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\text{rank}(\Gamma) = M - 1$. Under this condition, the Markov chain has a unique stationary distribution $\pi\Gamma = 0$, subject to $\sum_{j=1}^M \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$. For a real-valued function $\sigma(\cdot)$ defined on $\mathbb{S}$, we define

$$\Gamma\sigma(\cdot)(i) := \sum_{j \in \mathbb{S}} \gamma_{ij} \sigma(j) = \sum_{j \neq i} \gamma_{ij} (\sigma(j) - \sigma(i)), \quad (4)$$

for each $i \in \mathbb{S}$.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, which are continuously twice differentiable in x and once differentiable in t . If $V(x, t, i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(\varphi, t, i) = & V_t(\varphi(0) - D(\varphi, i), t, i) \\ & + V_x(\varphi(0) - D(\varphi, i), t, i) f(\varphi, t, i) \\ & + \frac{1}{2} \text{trace} \left[g^T(\varphi, t, i) \right. \\ & \quad \times V_{xx}(\varphi(0) - D(\varphi, i), t, i) \\ & \quad \left. \times g(\varphi, t, i) \right] \\ & + \sum_{j=1}^l \gamma_{ij} V(\varphi(0) - D(\varphi, j), t, j), \end{aligned} \quad (5)$$

where

$$\begin{aligned} \varphi \in C([-\tau, 0]; \mathbb{R}^n), \quad V_t = \frac{\partial V(x, t, i)}{\partial t}, \\ V_x = \left(\frac{\partial V(x, t, i)}{\partial x_1}, \frac{\partial V(x, t, i)}{\partial x_2}, \dots, \frac{\partial V(x, t, i)}{\partial x_n} \right), \quad (6) \\ V_{xx} = \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j} \right)_{n \times n}. \end{aligned}$$

For a parameter $\varepsilon > 0$, we rewrite the Markov chain $r(t)$ as $r^\varepsilon(t)$ and the generator Γ as Γ^ε . Γ^ε is given by

$$\Gamma^\varepsilon = \frac{1}{\varepsilon} \bar{\Gamma} + \hat{\Gamma}, \quad (7)$$

where $\bar{\Gamma}/\varepsilon$ represents the fast varying motions and $\hat{\Gamma}$ represents the slowly changing dynamics. Set $\Gamma^\varepsilon = (\gamma_{ij}^\varepsilon)_{M \times M}$, $\bar{\Gamma} = (\bar{\gamma}_{ij})_{M \times M}$, and $\hat{\Gamma} = (\hat{\gamma}_{ij})_{M \times M}$. For the sake of simplicity, suppose that

$$\mathbb{S} = \mathbb{S}^1 \cup \mathbb{S}^2 \cup \cdots \cup \mathbb{S}^l, \quad (8)$$

with $\mathbb{S}^k = \{s_{k1}, \dots, s_{kM_k}\}$, $M = M_1 + M_2 + \cdots + M_l$, and

$$\bar{\Gamma} = \text{diag}(\bar{\Gamma}^1, \dots, \bar{\Gamma}^l), \quad (9)$$

where $\bar{\Gamma}^k$ is a generator of a Markov chain taking values in $\mathbb{S}^k$, for every $k \in \{1, \dots, l\}$.

We give the first assumption as follows.

Assumption 1. For each $k \in \{1, \dots, l\}$, $\bar{\Gamma}^k$ is irreducible.

In order to emphasize the effect of the fast switching, (1) can be given by

$$\begin{aligned} d[x^\varepsilon - D(x^\varepsilon, r^\varepsilon(t))] \\ = f(x^\varepsilon, t, r^\varepsilon(t)) dt + g(x^\varepsilon, t, r^\varepsilon(t)) dw(t), \quad (10) \\ x_0^\varepsilon = \xi \in C([- \tau, 0]; \mathbb{R}^n), \quad r^\varepsilon = r_0. \end{aligned}$$

To assure the existence and uniqueness of the solution, we give the following standard assumptions.

Assumption 2 (local Lipschitz condition). For each integer $\alpha \geq 1$, there exists a constant $L_\alpha > 0$ such that

$$\begin{aligned} |f(\varphi, t, i) - f(\phi, t, i)| \vee |g(\varphi, t, i) - g(\phi, t, i)| \\ \leq L_\alpha \|\varphi - \phi\|^2, \quad (11) \end{aligned}$$

for all $i \in \mathbb{S}$, $t \geq 0$ and those $\varphi, \phi \in C([- \tau, 0]; \mathbb{R}^n)$ with $\|\varphi\| \vee \|\phi\| \leq \alpha$, and $f(0, t, i) \equiv 0$, $g(0, t, i) \equiv 0$.

Assumption 3 (linear growth condition). There is an $L > 0$, for any $\varphi \in C([- \tau, 0]; \mathbb{R}^n)$, $t \geq 0$, $i \in \mathbb{S}$ such that

$$|f(\varphi, t, i)|^2 \vee |g(\varphi, t, i)|^2 \leq L(1 + \|\varphi\|^2). \quad (12)$$

Assumption 4. For all $i \in \mathbb{S}$ and those $\varphi, \phi \in C([- \tau, 0]; \mathbb{R}^n)$, there is a constant $0 < \kappa < 1$ such that

$$\begin{aligned} |D(\varphi, i) - D(\phi, i)| &\leq \kappa \|\varphi - \phi\|^2, \\ D(0, i) &\equiv 0. \end{aligned} \quad (13)$$

Under Assumptions 2, 3, and 4, (10) has a unique solution denoted by $x^{\varepsilon, \xi, i}(t)$ on $t \geq 0$, where $x^{\varepsilon, \xi, i}$ is dependent on the initial value (ξ, i) (see [8]). Moreover, for every $p > 0$ and any compact subset G of $C([- \tau, 0]; \mathbb{R}^n)$, there is a positive constant H which is independent of ε such that

$$\sup_{(\xi, i) \in G \times \mathbb{S}} E \left[\sup_{-\tau \leq s \leq t} |x^{\varepsilon, \xi, i}(s)|^p \right] \leq H, \quad t \geq 0. \quad (14)$$

Since the state space of the Markov chain is large, it is too complicated to deal with directly. We need to analyse the limit equation of (10). To continue, make all the states in each $\mathbb{S}^k$ into a single state and define an aggregated process $\tilde{r}^\varepsilon(\cdot)$ as

$$\tilde{r}^\varepsilon(t) = k, \quad \text{if } r^\varepsilon(t) \in \mathbb{S}^k. \quad (15)$$

Denote the state space of $\tilde{r}^\varepsilon(t)$ by $\tilde{\mathbb{S}} = \{1, \dots, l\}$, the stationary distribution $\tilde{\Gamma}^k$ by $\mu^k = (\mu_1^k, \dots, \mu_{M_k}^k) \in \mathbb{R}^{1 \times M_k}$, and $\bar{\mu} = \text{diag}(\mu^1, \dots, \mu^l) \in \mathbb{R}^{l \times M}$. Define

$$\tilde{\Gamma} = (\tilde{\gamma}_{ij})_{l \times l} = \bar{\mu} \tilde{\Gamma} \mathbf{1} \quad (16)$$

with $\mathbf{1} = \text{diag}(\mathbf{1}_{M_1}, \dots, \mathbf{1}_{M_l})$ and $\mathbf{1}_{M_k} = (1, \dots, 1)^T \in \mathbb{R}^{M_k \times 1}$, $k = 1, \dots, l$. It has been known that $\tilde{r}^\varepsilon(\cdot)$ converges weakly to $\tilde{r}(\cdot)$ as $\varepsilon \rightarrow 0$, where $\tilde{r}(\cdot)$ is a continuous-time Markov chain with generator $\tilde{\Gamma}$ and state space $\tilde{\mathbb{S}}$ (see [4]). Define

$$\begin{aligned} \bar{D}(\varphi, k) &= \sum_{j=1}^{M_k} \mu_j^k D(\varphi, s_{kj}), \\ \tilde{f}(\varphi, t, k) &= \sum_{j=1}^{M_k} \mu_j^k f(\varphi, t, s_{kj}), \end{aligned} \quad (17)$$

$$\tilde{g}(\varphi, t, k) \tilde{g}^T(\varphi, t, k) = \sum_{j=1}^{M_k} \mu_j^k g(\varphi, t, s_{kj}) g^T(\varphi, t, s_{kj}),$$

for each $s_{kj} \in \mathbb{S}^k$ with $k \in \{1, \dots, l\}$ and $j \in \{1, \dots, M_k\}$. It is easy to know that $\bar{D}(\varphi, k)$, $\tilde{f}(\varphi, t, k)$, and $\tilde{g}(\varphi, t, k)$ are the limits with respect to the stationary distribution of the Markov chain. Consider that, for any $\varphi \neq 0$, $g(\varphi, t, s_{kj}) g^T(\varphi, t, s_{kj})$ are nonnegative definite matrices, so we denote its “square root” of $g(\varphi, t, s_{kj}) g^T(\varphi, t, s_{kj})$ by $\tilde{g}(\varphi, t, k)$. For degenerate diffusions, we can see the argument in [23].

The limit equation of (10) is defined as follows:

$$\begin{aligned} d[\tilde{\varphi}(0) - \bar{D}(\tilde{\varphi}, \tilde{r}(t))] \\ = \tilde{f}(\tilde{\varphi}, t, \tilde{r}(t)) dt + \tilde{g}(\tilde{\varphi}, t, \tilde{r}(t)) dw(t), \quad (18) \\ \tilde{x}_0 = \xi, \quad \tilde{r} = \tilde{r}_0. \end{aligned}$$

3. Exponential Stability of NSFDE with Two-Time-Scale Markovian Switching

In this section, we establish the Razumikhin-type theorem on the exponential stability for (10). Denote by $C^p(\mathbb{R}^n \times \mathbb{R}_+ \times \tilde{\mathbb{S}}; \mathbb{R}_+)$ the family of nonnegative real-valued functions defined on $\mathbb{R}^n \times \mathbb{R}_+ \times \tilde{\mathbb{S}}$ that are p -times continuously differentiable with respect to x . At the same time, we need another assumption and a lemma with respect to $V(x, t, i) \in C^p(\mathbb{R}^n \times \mathbb{R}_+ \times \tilde{\mathbb{S}}; \mathbb{R}_+)$ for some $p \geq 4$.

Assumption 5. For each $k \in \tilde{\mathbb{S}}$, $V(x, t, i) \rightarrow \infty$ as $|x| \rightarrow \infty$. Moreover, $\partial^p V(x, t, i) = O(1)$, $\partial^i V(x, t, i) (|x|^i + |y|^i) \leq K(|x|^p + |y|^p + 1)$ for $1 \leq i \leq p-1$, where $\partial^i V(x, t, i)$ denotes the i th derivative of $V(x, t, i)$ with respect to x and $O(y)$ denotes the function of y satisfying $\sup_y |O(y)|/|y| < \infty$.

Lemma 6. Suppose that $p \geq 1$; there is a positive constant $\kappa \in (0, 1)$ such that

$$\begin{aligned} \mathbb{E}|D(\varphi, k)|^p &\leq \kappa^p \sup_{-\tau \leq \theta \leq 0} e^{\gamma \theta} \|\varphi\|^p, \\ (\varphi, i) &\in L_{\mathcal{F}_t}^p([- \tau, 0]; \mathbb{R}^n) \times \mathbb{S}. \end{aligned} \quad (19)$$

Then, for any $\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, the solution for (10) satisfies

$$\begin{aligned} & \sup_{-\infty < s \leq t} e^{\gamma s} \mathbb{E} |x(s)|^p \\ & \leq \frac{\|\xi\|^p}{1-\kappa} \vee \frac{\sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E} |x(s) - D(x_s, r(s))|^p}{(1-\kappa)^p}, \quad (20) \\ & t \geq 0. \end{aligned}$$

Proof. Note the following elementary inequality:

$$\begin{aligned} (x+y)^p &= (1-\kappa_1)^{1-p} (x^p + \kappa_1^{1-p} y^p), \\ & \forall x, y \geq 0, \quad \kappa_1 > 0. \end{aligned} \quad (21)$$

We have from condition (20) that, for any $t \geq 0$,

$$\begin{aligned} e^{\gamma t} \mathbb{E} |x(t)|^p &\leq e^{\gamma t} \left[(1-\kappa)^{1-p} \mathbb{E} |x(t) - D(x_t, r(t))|^p \right. \\ & \quad \left. + \kappa^{1-p} \mathbb{E} |D(x_t, r(t))|^p \right] \\ &\leq (1-\kappa)^{1-p} e^{\gamma t} \mathbb{E} |x(t) - D(x_t, r(t))|^p \\ & \quad + \kappa e^{\gamma t} \sup_{-\tau \leq \theta \leq 0} e^{\gamma \theta} \mathbb{E} |x(t+\theta)|^p \\ &\leq (1-\kappa)^{1-p} \sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E} |x(s) - D(x_s, r(s))|^p \\ & \quad + \kappa \sup_{-\tau \leq \theta \leq 0} e^{\gamma(s+\theta)} \mathbb{E} |x(s+\theta)|^p \\ &\leq (1-\kappa)^{1-p} \sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E} |x(s) - D(x_s, r(s))|^p \\ & \quad + \kappa \sup_{-\infty < s \leq t} e^{\gamma s} \mathbb{E} |x(s)|^p. \end{aligned} \quad (22)$$

Then,

$$\begin{aligned} & \sup_{-\infty < s \leq t} e^{\gamma s} \mathbb{E} |x(s)|^p \\ & \leq \left[\sup_{-\tau \leq \theta \leq 0} \mathbb{E} |x(\theta)|^2 \right] \\ & \vee \left[(1-\kappa)^{1-p} \sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E} |x(s) - D(x_s, r(s))|^p \right. \\ & \quad \left. + \kappa \sup_{-\infty < s \leq t} e^{\gamma s} \mathbb{E} |x(s)|^p \right]. \end{aligned} \quad (23)$$

Therefore, the desired result holds. $\square$

Theorem 7. Let Assumptions 1–4 hold and let c_1, c_2, λ, p be all positive numbers and $q > 1$. Assume that there exists a function $V(x, t, k) \in C^p(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ satisfying Assumption 5, such that

$$c_1 |x|^p \leq V(x, t, k) \leq c_2 |x|^p, \quad (24)$$

for all $(x, t, k) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, $t \geq 0$, $k \in \mathbb{S}$. Consider the following:

$$\mathbb{E} |D(\varphi, k)|^p \leq \kappa^p \sup_{-\tau \leq \theta \leq 0} e^{\gamma \theta} \|\varphi\|^p, \quad (25)$$

$$\kappa = \max \{\kappa_1, \dots, \kappa_k\}, \quad \varphi \in L^p_{\mathcal{F}_t},$$

for all $t \geq 0$, $0 < \kappa_\sigma < 1$, $\sigma = \{1, \dots, k\}$, and

$$\begin{aligned} & \mathbb{E} \left[\max_{k \in \mathbb{S}} \mathcal{L}V(\varphi, t, k) \right] \\ & \leq -\lambda \mathbb{E} \left[\max_{k \in \mathbb{S}} V(\varphi(0) - D(\varphi, k), t, k) \right], \end{aligned} \quad (26)$$

provided $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\} \in L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$, satisfying

$$\begin{aligned} & \mathbb{E} \left[\min_{k \in \mathbb{S}} V(\varphi(\theta), t + \theta, k) \right] \\ & < q \mathbb{E} \left[\max_{k \in \mathbb{S}} V(\varphi(0) - D(\varphi, k), t, k) \right], \end{aligned} \quad (27)$$

for all $-\tau \leq \theta \leq 0$. Then, for all $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $t \geq 0$,

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} |x^\varepsilon(t)|^p \leq \frac{c_2(1+\kappa)^p}{c_1(1-\kappa)^p} \|\xi\|^p e^{-\gamma t}, \quad (28)$$

where

$$\gamma = \min \left\{ \bar{\gamma}, \frac{1}{\tau} \log \frac{q}{(c_2/c_1)(1-\kappa)^p} \right\}, \quad (29)$$

$\bar{\gamma}$ being the root of the following equation:

$$\frac{c_2}{c_1} (1-\kappa)^p e^{\gamma \tau} = \lambda. \quad (30)$$

In other words, the trivial solution of (10) is p th moment exponentially stable and the p th moment Lyapunov exponent is not greater than $-\gamma$.

Proof. Let

$$\widetilde{V}(\varphi, t, j) = \sum_{k=1}^l V(\varphi, t, k) I_{\{j \in \mathbb{S}^k\}} = V(\varphi, t, k), \quad \text{if } j \in \mathbb{S}^k. \quad (31)$$

By the definition of $\widetilde{V}$, we know that

$$\begin{aligned} & \widetilde{V}(\varphi^\varepsilon, t, r^\varepsilon(t)) = V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)), \\ & \sum_{i=1}^M \tilde{\gamma}_{li} \widetilde{V}(\varphi, t, i) = \sum_{i=1}^M \tilde{\gamma}_{li} \sum_{k=1}^l V(\varphi, t, k) I_{\{i \in \mathbb{S}^k\}} = 0. \end{aligned} \quad (32)$$

Extend $r(t)$ to $[-\tau, 0]$ by setting $r(t) = r(0)$. Recalling the facts that $x(t)$ is continuous for all $-\tau \leq \theta \leq 0$ and $r(t)$ is

right continuous, it is easy to see that $\mathbb{E}V(x(t), t, r(t))$ is right continuous on $t \geq -\tau$. Let $\gamma \in (0, \nu)$ be arbitrary and define

$$\begin{aligned} U(t) &:= \sup_{-\tau \leq \theta \leq 0} \left[e^{\gamma(t+\theta)} \mathbb{E}V(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, \tilde{r}^\varepsilon(t+\theta)), \right. \\ &\quad \left. t+\theta, \tilde{r}^\varepsilon(t+\theta)) \right] \\ &= \sup_{-\tau \leq \theta \leq 0} \left[e^{\gamma(t+\theta)} \mathbb{E}\tilde{V}(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, r^\varepsilon(t+\theta)), \right. \\ &\quad \left. t+\theta, r^\varepsilon(t+\theta)) \right], \end{aligned} \quad (33)$$

for all $t \geq 0$. We claim that

$$D^+U(t) = \limsup_{h \rightarrow 0^+} \frac{U(t+h) - U(t)}{h} \leq 0, \quad \forall t \geq 0. \quad (34)$$

Note that, for each $t \geq 0$, either $U(t) > e^{\gamma t} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))$ or $U(t) = e^{\gamma t} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))$.

If $U(t) > e^{\gamma t} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))$, because $\mathbb{E}V(x(t), t, r(t))$ is right continuous on $t \geq -\tau$, it is easy to obtain that, for all $h > 0$ sufficiently small, $U(t) > e^{\gamma(t+h)} \mathbb{E}V(x^\varepsilon(t+h) - D(x_{t+h}^\varepsilon, \tilde{r}^\varepsilon(t+h)), t+h, \tilde{r}^\varepsilon(t+h))$; hence, $U(t+h) \leq U(t)$ and $D^+U(t) \leq 0$.

If $U(t) = e^{\gamma t} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))$, we have

$$\begin{aligned} &e^{\gamma(t+\theta)} \mathbb{E}V(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, \tilde{r}^\varepsilon(t+\theta)), t+\theta, \tilde{r}^\varepsilon(t+\theta)) \\ &\leq e^{\gamma t} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)), \end{aligned} \quad (35)$$

for all $-\tau \leq \theta \leq 0$.

Then,

$$\begin{aligned} &\mathbb{E}V(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, \tilde{r}^\varepsilon(t+\theta)), t+\theta, \tilde{r}^\varepsilon(t+\theta)) \\ &\leq e^{-\gamma\theta} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\ &\leq e^{\gamma\tau} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)), \end{aligned} \quad (36)$$

for all $-\tau \leq \theta \leq 0$.

On the other hand, by Lemma 6, we derive

$$\begin{aligned} &e^{\gamma(t+\theta)} \mathbb{E}V(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, \tilde{r}^\varepsilon(t+\theta)), t+\theta, \tilde{r}^\varepsilon(t+\theta)) \\ &\leq c_2 e^{\gamma(t+\theta)} \mathbb{E}|x^\varepsilon(t+\theta)|^p \\ &\leq c_2 (1-\kappa)^p \sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E}|x^\varepsilon(s) - D(x_s^\varepsilon, \tilde{r}(s))|^p \\ &\leq \frac{c_2}{c_1} (1-\kappa)^p \sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E}V(x^\varepsilon(s) - D(x_s^\varepsilon, \tilde{r}(s)), s, \tilde{r}(s)) \\ &\leq \frac{c_2}{c_1} (1-\kappa)^p e^{\gamma t} \mathbb{E}V(x^\varepsilon(t) - D(x_t^\varepsilon, \tilde{r}(t)), t, \tilde{r}(t)). \end{aligned} \quad (37)$$

Then,

$$\begin{aligned} &\mathbb{E}V(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, \tilde{r}^\varepsilon(t+\theta)), t+\theta, \tilde{r}^\varepsilon(t+\theta)) \\ &< q \mathbb{E}V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)), \end{aligned} \quad (38)$$

where $q > (c_2/c_1)(1-\kappa)^p e^{\gamma\tau}$; that is, $\gamma < (1/\tau)(\log(q/(c_2/c_1)(1-\kappa)^p))$.

Consequently, there exists a sufficiently small $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} &\mathbb{E} \left[\min_{k \in \mathbb{S}} V(\varphi^\varepsilon(\theta), t+\theta, k) \right] \\ &< q \mathbb{E} \left[\max_{k \in \mathbb{S}} V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, k), t, k) \right], \end{aligned} \quad (39)$$

for all $-\tau \leq \theta \leq 0$. Thus,

$$\begin{aligned} &\mathbb{E} \left[\max_{k \in \mathbb{S}} \mathcal{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \right] \\ &\leq -\lambda \mathbb{E} \left[\max_{k \in \mathbb{S}} V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \right], \end{aligned} \quad (40)$$

which implies that

$$\begin{aligned} &\mathbb{E}[\mathcal{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\ &\leq -\lambda \mathbb{E}[V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))]. \end{aligned} \quad (41)$$

By the condition of $\gamma < \nu \leq \lambda$, we get

$$\begin{aligned} &\mathbb{E}[\mathcal{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\ &\leq -\gamma \mathbb{E}[V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))]. \end{aligned} \quad (42)$$

Next, we consider

$$\begin{aligned} &U(t+h) - U(t) \\ &= \limsup_{\varepsilon \rightarrow 0} \left[e^{\gamma(t+\theta+h)} \mathbb{E}[V(x^\varepsilon(t+\theta+h) - D(x_{t+\theta+h}^\varepsilon, \tilde{r}^\varepsilon(t+\theta+h)), \right. \\ &\quad \left. t+\theta+h, \tilde{r}^\varepsilon(t+\theta+h)) \right] \\ &\quad - e^{\gamma(t+\theta)} \mathbb{E}[V(x^\varepsilon(t+\theta) - D(x_{t+\theta}^\varepsilon, \tilde{r}^\varepsilon(t+\theta)), \\ &\quad \left. t+\theta, \tilde{r}^\varepsilon(t+\theta)) \right] \\ &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \int_{t+\theta}^{t+\theta+h} e^{\gamma s} [\mathcal{L}V(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\ &\quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), \\ &\quad \left. s, \tilde{r}^\varepsilon(s))] ds \\ &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \int_{t+\theta}^{t+\theta+h} e^{\gamma s} [\mathcal{L}\tilde{V}(\varphi^\varepsilon, s, r^\varepsilon(s)) \\ &\quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), \\ &\quad \left. s, \tilde{r}^\varepsilon(s))] ds. \end{aligned} \quad (43)$$

By the definition of operator $\mathcal{L}$, we have

$$\begin{aligned}
& \mathcal{L}\tilde{V}(\varphi^\varepsilon, t, r^\varepsilon(t)) \\
&= \tilde{V}_t(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad + \tilde{V}_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad \times f(\varphi^\varepsilon, t, r^\varepsilon(t)) \\
&\quad + \frac{1}{2} \text{trace} \left[g^T(\varphi^\varepsilon, t, r^\varepsilon(t)) \right. \\
&\quad \quad \times \tilde{V}_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad \quad \times g(\varphi^\varepsilon, t, r^\varepsilon(t)) \left. \right] \\
&\quad + \sum_{i=1}^M \gamma_{r^\varepsilon(t)i}^\varepsilon \tilde{V}(\varphi^\varepsilon(0) - D(x_t^\varepsilon, i), t, i) \\
&= \tilde{V}_t(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad + \tilde{V}_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad \times f(\varphi^\varepsilon, t, r^\varepsilon(t)) \\
&\quad + \frac{1}{2} \text{trace} \left[g^T(\varphi^\varepsilon, t, r^\varepsilon(t)) \right. \\
&\quad \quad \times \tilde{V}_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad \quad \times g(\varphi^\varepsilon, t, r^\varepsilon(t)) \left. \right] \\
&\quad + \sum_{i=1}^M \hat{\gamma}_{r^\varepsilon(t)i}^\varepsilon \tilde{V}(\varphi^\varepsilon(0) - D(x_t^\varepsilon, i), t, i) \\
&= V_t(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\quad + V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\quad \times \tilde{f}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \\
&\quad + \frac{1}{2} \text{trace} \left[\tilde{g}^T(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \right. \\
&\quad \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\quad \quad \times \tilde{g}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \left. \right] \\
&\quad + \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(t)k}^\varepsilon V(\varphi^\varepsilon(0) - D(x_t^\varepsilon, k), t, k) \\
&\quad + V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\quad \times [f(\varphi^\varepsilon, t, r^\varepsilon(t)) - \tilde{f}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\
&\quad + \frac{1}{2} \text{trace} \left[g^T(\varphi^\varepsilon, t, r^\varepsilon(t)) \right. \\
&\quad \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\quad \quad \times g(\varphi^\varepsilon, t, r^\varepsilon(t)) - \tilde{g}^T(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \left. \right]
\end{aligned}$$

$$\begin{aligned}
& \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
& \times \tilde{g}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \left. \right] \\
& + \sum_{i=1}^M \hat{\gamma}_{r^\varepsilon(t)i}^\varepsilon \tilde{V}(\varphi^\varepsilon(0) - D(x_t^\varepsilon, i), t, i) \\
& - \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(t)k}^\varepsilon V(\varphi^\varepsilon(0) - D(x_t^\varepsilon, k), t, k) \\
& = \mathcal{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \\
& \quad + V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
& \quad \times [f(\varphi^\varepsilon, t, r^\varepsilon(t)) - \tilde{f}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\
& \quad + \frac{1}{2} \text{trace} \left[g^T(\varphi^\varepsilon, t, r^\varepsilon(t)) \right. \\
& \quad \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
& \quad \quad \times g(\varphi^\varepsilon, t, r^\varepsilon(t)) - \tilde{g}^T(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \\
& \quad \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
& \quad \quad \times \tilde{g}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \left. \right] \\
& \quad + \sum_{i=1}^M \hat{\gamma}_{r^\varepsilon(t)i}^\varepsilon \tilde{V}(\varphi^\varepsilon(0) - D(x_t^\varepsilon, i), t, i) \\
& \quad - \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(t)k}^\varepsilon V(\varphi^\varepsilon(0) - D(x_t^\varepsilon, k), t, k). \tag{44}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& U(t+h) - U(t) \\
&= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \quad \times [\mathcal{L}V(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\
&\quad \quad \quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s))] ds \\
&+ \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma t} V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
&\quad \quad \times [f(\varphi^\varepsilon, s, r^\varepsilon(s)) - \tilde{f}(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s))] ds \\
&+ \frac{1}{2} \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s}
\end{aligned}$$

$$\begin{aligned}
& \times \text{trace} \left[g^T(\varphi^\varepsilon, s, r^\varepsilon(s)) \right. \\
& \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
& \quad \times g(\varphi^\varepsilon, s, r^\varepsilon(s)) - \tilde{g}^T(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\
& \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
& \quad \left. \times \tilde{g}(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \right] ds \\
& + \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
& \quad \times \left(\sum_{i=1}^M \hat{\gamma}_{r^\varepsilon(s)i} \tilde{V}(\varphi^\varepsilon(0) - D(x_s^\varepsilon, i), s, i) \right. \\
& \quad \left. - \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(s)k} V(\varphi^\varepsilon(0) - D(x_s^\varepsilon, k), s, k) \right) ds \\
& =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{45}$$

By the definition of $\tilde{f}$,

$$\begin{aligned}
& f(\varphi^\varepsilon, t, r^\varepsilon(t)) - \tilde{f}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \\
& = \sum_{k=1}^l \sum_{j=1}^{M_k} f(\varphi^\varepsilon, t, s_{kj}) \times \left[I_{\{r^\varepsilon(t)=s_{kj}\}} - \mu_j^k I_{\{\tilde{r}^\varepsilon(t)=k\}} \right].
\end{aligned} \tag{46}$$

This implies that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \mathbb{E} \int_{t+\theta}^{t+\theta+h} e^{\gamma s} V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
& \quad \times [f(\varphi^\varepsilon, s, r^\varepsilon(s)) - \tilde{f}(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s))] ds \\
& \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_{t+\theta}^{t+\theta+h} e^{\gamma s} V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \right. \\
& \quad \left. \times [f(\varphi^\varepsilon, s, r^\varepsilon(s)) - \tilde{f}(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s))] ds \right]^{1/2} \\
& = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_{t+\theta}^{t+\theta+h} e^{\gamma s} V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \right. \\
& \quad \times \sum_{k=1}^l \sum_{j=1}^{M_k} f(\varphi^\varepsilon, s, s_{kj}) \\
& \quad \left. \times [I_{\{r^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\tilde{r}^\varepsilon(s)=k\}}] ds \right]^{1/2}
\end{aligned}$$

$$\begin{aligned}
& \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_{t+\theta}^{t+\theta+h} \sum_{k=1}^l \sum_{j=1}^{M_k} e^{\gamma s} L(1 + \|\varphi\|^p) \right. \\
& \quad \left. \times [I_{\{r^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\tilde{r}^\varepsilon(s)=k\}}] ds \right]^{1/2}.
\end{aligned} \tag{47}$$

By the argument of Lemma 7.14 in [4], the right side of the above inequality is equivalent to 0; that is, $I_2 = 0$. Similarly, we can show that

$$\begin{aligned}
I_3 &= \frac{1}{2} \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
& \quad \times \text{trace} [g^T(\varphi^\varepsilon, s, r^\varepsilon(s)) \\
& \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
& \quad \times g(\varphi^\varepsilon, s, r^\varepsilon(s)) \\
& \quad - \tilde{g}^T(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\
& \quad \times V_{xx}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
& \quad \times \tilde{g}(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s))] ds = 0.
\end{aligned} \tag{48}$$

By the definition of $\hat{\Gamma}$ and $\tilde{\Gamma}$, we have

$$\begin{aligned}
& \sum_{i=1}^M \hat{\gamma}_{r^\varepsilon(t)i} \tilde{V}(\varphi^\varepsilon(0) - D(x_t^\varepsilon, i), t, i) \\
& = \hat{\Gamma} \tilde{V}(\varphi^\varepsilon(0) - D(x_t^\varepsilon, i), t, \cdot)(r^\varepsilon(t)), \\
& \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(t)k} V(\varphi(0) - D(x_t^\varepsilon, k), t, k) \\
& = \tilde{\Gamma} V(\varphi(0) - D(x_t^\varepsilon, k), t, \cdot)(\tilde{r}^\varepsilon(t)).
\end{aligned} \tag{49}$$

Hence,

$$\begin{aligned}
I_4 &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
& \quad \times \left(\sum_{i=1}^M \hat{\gamma}_{r^\varepsilon(s)i} \tilde{V}(\varphi^\varepsilon(0) - D(x_s^\varepsilon, i), s, i) \right. \\
& \quad \left. - \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(s)k} V(\varphi^\varepsilon(0) - D(x_s^\varepsilon, k), s, k) \right) ds
\end{aligned}$$

$$\begin{aligned}
&= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \times \left(\widehat{\Gamma} \widetilde{V}(\varphi^\varepsilon(0) - D(x_s^\varepsilon, i), s, \cdot)(r^\varepsilon(s)) \right. \\
&\quad \left. - \widetilde{\Gamma} V(\varphi^\varepsilon(0) - D(x_s^\varepsilon, k), s, \cdot)(s_{kj}) \right) ds \\
&= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \sum_{k=1}^l \sum_{j=1}^{M_k} \widehat{\Gamma} \widetilde{V}(\varphi^\varepsilon(0) - D(x_s^\varepsilon, i), s, \cdot)(s_{kj}) \\
&\quad \times \left[I_{\{r^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\widetilde{r}^\varepsilon(s)=k\}} \right] ds \\
&\leq \limsup_{\varepsilon \rightarrow 0} \left[\mathbb{E} \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \right. \\
&\quad \times \sum_{k=1}^l \sum_{j=1}^{M_k} \widehat{\Gamma} \widetilde{V}(\varphi^\varepsilon(0) - D(x_s^\varepsilon, i), s, \cdot)(s_{kj}) \\
&\quad \times \left. \left[I_{\{r^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\widetilde{r}^\varepsilon(s)=k\}} \right] \right]^{1/2}.
\end{aligned} \tag{50}$$

By the argument of Lemma 7.14 in [4], the right side of the above inequality is equivalent to 0; that is, $I_4 = 0$. Therefore,

$$\begin{aligned}
&U(t+h) - U(t) \\
&= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \times [\mathcal{L}V(\varphi^\varepsilon, s, \widetilde{r}^\varepsilon(s)) \\
&\quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \widetilde{r}^\varepsilon(s)), s, \widetilde{r}^\varepsilon(s))] ds \\
&\leq 0.
\end{aligned} \tag{51}$$

That is

$$U(t+h) \leq U(t). \tag{52}$$

So, $U(t+h) = U(t)$ for all $h > 0$ sufficiently small, and hence $D^+U(t) = 0$. Inequality (34) holds.

It follows from (34) that $U(t) \leq U(0)$ for all $t \geq 0$. By the definition of $U(t)$,

$$\begin{aligned}
&\limsup_{\varepsilon \rightarrow 0} e^{\gamma t} \mathbb{E} |x^\varepsilon(t) - D(x_t^\varepsilon, \widetilde{r}^\varepsilon(t))|^p \\
&\leq c_2 \limsup_{\varepsilon \rightarrow 0} \sup_{\tau \leq \theta \leq 0} e^{\gamma \theta} \mathbb{E} |x^\varepsilon(\theta) - D(x_\theta^\varepsilon, \widetilde{r}^\varepsilon(\theta))|^p \\
&\leq c_2 \limsup_{\varepsilon \rightarrow 0} \sup_{-\tau \leq \theta \leq 0} (1 + \kappa)^{p-1} \\
&\quad \times [\mathbb{E} |x^\varepsilon(\theta)|^p + \kappa^{1-p} \mathbb{E} |D(x_\theta^\varepsilon, \widetilde{r}^\varepsilon(\theta))|^p] \\
&\leq c_2 (1 + \kappa)^p \|\xi\|^p, \quad t \geq 0.
\end{aligned} \tag{53}$$

By Lemma 6, we derive

$$\limsup_{\varepsilon \rightarrow 0} e^{\gamma t} \mathbb{E} |x^\varepsilon(t)|^p \leq \frac{c_2(1 + \kappa)^p}{c_1(1 - \kappa)^p} \|\xi\|^p. \tag{54}$$

That is,

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} |x^\varepsilon(t)|^p \leq \frac{c_2(1 + \kappa)^p}{c_1(1 - \kappa)^p} \|\xi\|^p e^{-\gamma t}, \quad \forall t \geq 0. \tag{55}$$

□

4. Neutral Stochastic Functional System with Pure Jump

In this section, we discuss the stability of the following neutral stochastic functional system with pure jump:

$$\begin{aligned}
&d[\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t))] \\
&= f(\varphi^\varepsilon, t, r^\varepsilon(t)) dt + \int_{\mathbb{R}^m} b(x_{t-}^\varepsilon, t, r^\varepsilon(t), z) \widetilde{N}(dt, dz) \\
&\quad x_0 = \xi \in C([- \tau, 0]; \mathbb{R}^n), \quad r(0) \in \mathbb{S},
\end{aligned} \tag{56}$$

where $x_{t-}^\varepsilon = \lim_{s \uparrow t} x_s^\varepsilon$, $D : C([- \tau, 0]; \mathbb{R}^n) \times \mathbb{S} \rightarrow \mathbb{R}^n$, $b : C([- \tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathbb{S} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$. We assume that each column $b^{(\delta)}$ of the $n \times m$ matrix $b = [b_{ij}]$ depends on z only through the δ th coordinate z_δ ; that is,

$$\begin{aligned}
&b^{(\delta)}(\varphi, t, i, z) = b^{(\delta)}(\varphi, t, i, z_\delta); \\
&z = (z_1, \dots, z_m) \in \mathbb{R}^m, \quad i \in \mathbb{S}.
\end{aligned} \tag{57}$$

$N(t, z)$ is an m -dimensional Poisson process and the compensated Poisson process is defined by

$$\begin{aligned}
&\widetilde{N}(dt, dz) = (\widetilde{N}_1(dt, dz_1), \dots, \widetilde{N}_m(dt, dz_m)) \\
&= (N_1(dt, dz_1) - \lambda_1(dz_1) dt, \dots, N_m(dt, dz_m) \\
&\quad - \lambda_m(dz_m) dt),
\end{aligned} \tag{58}$$

where $\{N_\delta, \delta = 1, \dots, m\}$ are independent one-dimensional Poisson random measures with characteristic measure

$\{\lambda_\delta, \delta = 1, \dots, m\}$ coming from m independent one-dimensional Poisson point processes. The limit system of (56) is defined as follows:

$$\begin{aligned} & d[\varphi^\varepsilon(0) - D(\tilde{x}_t^\varepsilon, \tilde{r}^\varepsilon(t))] \\ &= \tilde{f}(\tilde{x}_t^\varepsilon, t, \tilde{r}^\varepsilon(t)) dt \\ &+ \int_{\mathbb{R}^n} \tilde{b}(\tilde{x}_t^\varepsilon, t, \tilde{r}^\varepsilon(t), z) \tilde{N}(dt, dz) \\ &\tilde{x}_0 = \xi \in C([- \tau, 0]; \mathbb{R}^n), \quad \tilde{r}(0) \in \mathbb{S}, \end{aligned} \quad (59)$$

where $\tilde{x}_{t-}^\varepsilon = \lim_{s \uparrow t} \tilde{x}_s^\varepsilon$ and $\tilde{b}: C([- \tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \tilde{\mathbb{S}} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$. Similar to the definition of $\tilde{f}$, we define

$$\begin{aligned} \bar{D}(\varphi, k) &= \sum_{j=1}^{N_m} \mu_j^k D(\varphi, s_{kj}), \\ \bar{b}(\varphi, t, k, z) &= \sum_{j=1}^{N_m} \mu_j^k b(\varphi, t, s_{kj}, z), \end{aligned} \quad (60)$$

for each $s_{kj} \in \mathbb{S}^k$ with $k \in \{1, \dots, l\}$ and $j \in \{1, \dots, N_m\}$.

To assure the existence and uniqueness of the solution of (59), we also give the following standard assumptions.

Assumption 8. For any integer ζ , there is a constant $L_\zeta > 0$, such that

$$\begin{aligned} & |f(\varphi, t, i) - f(\phi, t, i)| \\ & \vee \sum_{\delta=1}^m \int_{\mathbb{R}} |b^{(\delta)}(\varphi, t, i, z_\delta) - b^{(\delta)}(\phi, t, i, z_\delta)| \lambda_\delta(dz_\delta) \\ & \leq L_\zeta \|\varphi - \phi\|^2, \end{aligned} \quad (61)$$

for all $i \in \mathbb{S}$ and those $\varphi, \phi \in C([- \tau, 0]; \mathbb{R}^n)$ with $\|\varphi\| \vee \|\phi\| \leq \zeta$, and $f(0, t, i) \equiv 0, b(0, t, i, z) \equiv 0$.

Assumption 9. There is an $\bar{L} > 0$, such that, for any $\varphi, \phi \in C([- \tau, 0]; \mathbb{R}^n), i \in \mathbb{S}$,

$$\begin{aligned} & |f(\varphi, t, i)| \vee \sum_{\delta=1}^m \int_{\mathbb{R}} |b^{(\delta)}(\varphi, t, i, z_\delta)| \lambda_\delta(dz_\delta) \\ & \leq \bar{L} (1 + \|\varphi\|^2). \end{aligned} \quad (62)$$

Assumption 10. For all $i \in \mathbb{S}$ and those $\varphi, \phi \in C([- \tau, 0]; \mathbb{R}^n)$, there is a constant $0 < \kappa < 1$ such that

$$\begin{aligned} & |D(\varphi, i) - D(\phi, i)| \leq \kappa \|\varphi - \phi\|^2, \\ & D(0, i) \equiv 0. \end{aligned} \quad (63)$$

Given that $V \in C^p(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, define an operator $\mathbb{L}V$ by

$$\begin{aligned} & \mathbb{L}V(\varphi, t, i) \\ &= V_t(\varphi(0) - D(\varphi, i), t, i) \\ &+ V_x(\varphi(0) - D(\varphi, i), t, i) f(\varphi, t, i) \\ &+ \sum_{j=1}^N \gamma_{ij} V(\varphi(0) - D(\varphi, i), t, j) \\ &+ \int_{\mathbb{R}} \sum_{\delta=1}^m \{V(\varphi(0) - D(\varphi, i) + b^{(\delta)}(\varphi, t, i, z_\delta), t, i) \\ &- V(\varphi(0) - D(\varphi, i), t, i) \\ &- V_x(\varphi(0) - D(\varphi, i), t, i) \\ &\times b^{(\delta)}(\varphi, t, i, z_\delta)\} \lambda_\delta(dz_\delta), \end{aligned} \quad (64)$$

where

$$\begin{aligned} & V_x(\varphi(0) - D(\varphi, i), t, i) \\ &= \left(\frac{\partial V(\varphi(0) - D(\varphi, i), t, i)}{\partial x_1}, \dots, \right. \\ & \left. \frac{\partial V(\varphi(0) - D(\varphi, i), t, i)}{\partial x_m} \right). \end{aligned} \quad (65)$$

Lemma 11 (see [20]). Let Assumptions 1, 8, and 9 hold, as $\varepsilon \rightarrow 0$; then, $(x^\varepsilon(\cdot), \tilde{r}^\varepsilon(\cdot))$ converges weakly to $(\tilde{x}(\cdot), \tilde{r}(\cdot))$ in $D([0, \infty), \mathbb{R}^n \times \tilde{\mathbb{S}})$, where $D([0, \infty), \mathbb{R}^n \times \tilde{\mathbb{S}})$ is the space of functions defined on $[0, \infty)$ that are right continuous and have left limits taking values in $\mathbb{R}^n \times \tilde{\mathbb{S}}$ and are endowed with the Skorohod topology.

Theorem 12. Let Assumptions 1 and 8–10 hold and let c_1, c_2, λ, p be all positive numbers and $q > 1$. Assume that there exists a function $V(x, t, k) \in C^p(\mathbb{R}^n \times \mathbb{R}_+ \times \tilde{\mathbb{S}}; \mathbb{R}_+)$ satisfying Assumption 5, such that

$$c_1 |x|^p \leq V(x, t, k) \leq c_2 |x|^p, \quad k \in \tilde{\mathbb{S}}, \quad (66)$$

for all $(x, t, k) \in \mathbb{R}^n \times \mathbb{R}_+ \times \tilde{\mathbb{S}}$ and $t \geq 0, k \in \tilde{\mathbb{S}}$. Consider the following:

$$\mathbb{E}|D(\varphi, k)|^p \leq \kappa^p \sup_{-\tau \leq \theta \leq 0} e^{\bar{\gamma}\theta} \|\varphi\|^p, \quad (67)$$

$$\kappa = \max\{\kappa_1, \dots, \kappa_k\}, \quad \varphi \in L_{\mathcal{F}_t}^p,$$

for all $t \geq 0, 0 < \kappa_\sigma < 1, \sigma = \{1, \dots, k\}$, and

$$\mathbb{E} \left[\max_{k \in \tilde{\mathbb{S}}} \mathbb{L}V(\varphi, t, k) \right] \leq -\gamma \mathbb{E} \left[\max_{k \in \tilde{\mathbb{S}}} V(\varphi(0) - D(\varphi, k)) \right], \quad (68)$$

provided $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\} \in L^p_{\mathcal{F}_t}([- \tau, 0]; \mathbb{R}^n)$, satisfying

$$\begin{aligned} & \mathbb{E} \left[\min_{k \in \mathbb{S}} V(x(t+\theta), t+\theta, k) \right] \\ & < q \mathbb{E} \left[\max_{k \in \mathbb{S}} V(\varphi(0) - D(\varphi, k), t, i) \right], \quad -\tau \leq \theta \leq 0. \end{aligned} \quad (69)$$

Then, for all $\xi \in C([- \tau, 0]; \mathbb{R}^n)$, $t \geq 0$,

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} |x^\varepsilon(t)|^p \leq \frac{c_2(1+\kappa)^p}{c_1(1-\kappa)^p} \|\xi\|^p e^{-\bar{\gamma}t}, \quad (70)$$

where

$$\bar{\gamma} = \min \left\{ \bar{\gamma}, \frac{1}{\tau} \log \frac{q}{(c_2/c_1)(1-\kappa)^p} \right\}, \quad (71)$$

$\bar{\gamma}$ being the root of the following equation:

$$\frac{c_2}{c_1} (1-\kappa)^p e^{\bar{\gamma}\tau} = \lambda. \quad (72)$$

Proof. Define

$$\tilde{V}(\varphi, t, \rho) = \sum_{k=1}^l V(\varphi, t, k) I_{\{\rho \in \mathbb{S}^k\}} = V(\varphi, t, k), \quad \text{if } \rho \in \mathbb{S}^k. \quad (73)$$

Extend $r(t)$ to $[-\tau, 0]$ by setting $r(t) = r(0)$; then, $\mathbb{E}V(x(t), t, r(t))$ is right continuous on $t \geq -\tau$. Let $\gamma \in (0, \bar{\gamma})$ be arbitrary and define

$$\begin{aligned} U(t) &:= \sup_{-\tau \leq \theta \leq 0} \left[e^{\gamma(t+\theta)} \mathbb{E}V(x^\varepsilon(t+\theta) - D(x^\varepsilon_{t+\theta}, \tilde{r}^\varepsilon(t+\theta)), \right. \\ &\quad \left. t+\theta, \tilde{r}^\varepsilon(t+\theta)) \right] \\ &= \sup_{-\tau \leq \theta \leq 0} \left[e^{\gamma(t+\theta)} \mathbb{E}\tilde{V}(x^\varepsilon(t+\theta) - D(x^\varepsilon_{t+\theta}, r^\varepsilon(t+\theta)), \right. \\ &\quad \left. t+\theta, r^\varepsilon(t+\theta)) \right], \end{aligned} \quad (74)$$

for all $t \geq 0$. We claim that

$$D^+U(t) = \limsup_{h \rightarrow 0^+} \frac{U(t+h) - U(t)}{h} \leq 0, \quad \forall t \geq 0. \quad (75)$$

Similar to the proof of Theorem 7, we derive

$$\begin{aligned} & \mathbb{E}V(x^\varepsilon(t+\theta), t+\theta, \tilde{r}^\varepsilon(t+\theta)) \\ & < q \mathbb{E}V(\varphi(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)), \end{aligned} \quad (76)$$

for all $-\tau \leq \theta \leq 0$, where $q > (c_2/c_1)(1-\kappa)^p e^{\gamma\tau}$; that is, $\gamma < (1/\tau)(\log(q/(c_2/c_1)(1-\kappa)^p))$.

Thus,

$$\begin{aligned} & \mathbb{E} \left[\max_{k \in \mathbb{S}} \mathbb{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \right] \\ & \leq -\lambda \mathbb{E} \left[\max_{k \in \mathbb{S}} V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \right], \end{aligned} \quad (77)$$

which implies that

$$\begin{aligned} & \mathbb{E} [\mathbb{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\ & \leq -\lambda \mathbb{E} [V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))]. \end{aligned} \quad (78)$$

By the condition of $\gamma < \bar{\gamma} \leq \lambda$, we get

$$\begin{aligned} & \mathbb{E} [\mathbb{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\ & \leq -\gamma \mathbb{E} [V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t))]. \end{aligned} \quad (79)$$

We now consider

$$\begin{aligned} & U(t+h) - U(t) \\ &= \limsup_{\varepsilon \rightarrow 0} \left[e^{\gamma(t+\theta+h)} \mathbb{E} \right. \\ &\quad \times [V(x^\varepsilon(t+\theta+h) - D(x^\varepsilon_{t+\theta+h}, \tilde{r}^\varepsilon(t+\theta+h)), \\ &\quad \left. t+\theta+h, \tilde{r}^\varepsilon(t+\theta+h)) \right] \\ &\quad - e^{\gamma(t+\theta)} \mathbb{E} \\ &\quad \times [V(x^\varepsilon(t+\theta) - D(x^\varepsilon_{t+\theta}, \tilde{r}^\varepsilon(t+\theta)), \\ &\quad \left. t+\theta, \tilde{r}^\varepsilon(t+\theta)) \right] \\ &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\ &\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} [\mathbb{L}V(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\ &\quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s))] ds \\ &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\ &\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\ &\quad \times [\mathbb{L}\tilde{V}(\varphi^\varepsilon, s, r^\varepsilon(s)) \\ &\quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), s, \tilde{r}^\varepsilon(s))] dt. \end{aligned} \quad (80)$$

By the definition of the operator $\mathbb{L}$, we have

$$\begin{aligned}
& \mathbb{L}\tilde{V}(\varphi^\varepsilon, t, r^\varepsilon(t)) \\
&= \tilde{V}_t(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&+ \tilde{V}_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\times f(\varphi^\varepsilon, t, r^\varepsilon(t)) \\
&+ \sum_{\delta=1}^m \int_{\mathbb{R}} \left\{ \tilde{V}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)) \right. \\
&\quad + b^{(\delta)}(x_{t-}^\varepsilon, t, r^\varepsilon(t), z_\delta), t, r^\varepsilon(t)) \\
&\quad - \tilde{V}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad - \tilde{V}_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t)), t, r^\varepsilon(t)) \\
&\quad \left. \times b^{(\delta)}(x_{t-}^\varepsilon, t, r^\varepsilon(t), z_\delta) \right\} \lambda_\delta(dz_\delta) \\
&+ \sum_{j=1}^N \gamma_{r^\varepsilon(t)j} \tilde{V}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, j), t, j) \\
&= \mathbb{L}V(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t)) \\
&+ V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\times [f(\varphi^\varepsilon, t, r^\varepsilon(t)) - \tilde{f}(\varphi^\varepsilon, t, \tilde{r}^\varepsilon(t))] \\
&+ \sum_{\delta=1}^m \int_{\mathbb{R}} \left\{ V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)) \right. \\
&\quad + b^{(\delta)}(x_{t-}^\varepsilon, t, \tilde{r}^\varepsilon(t), z_\delta), t, \tilde{r}^\varepsilon(t)) \\
&\quad - V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)) \\
&\quad \left. + \tilde{b}^{(\delta)}(x_{t-}^\varepsilon, t, \tilde{r}^\varepsilon(t), z_\delta), t, \tilde{r}^\varepsilon(t)) \right\} \\
&\times \lambda_\delta(dz_\delta) \\
&- \sum_{\delta=1}^m \int_{\mathbb{R}} \left\{ V_x(\varphi^\varepsilon(0) \right. \\
&\quad - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)), t, \tilde{r}^\varepsilon(t)) \\
&\quad \times (b^{(\delta)}(x_{t-}^\varepsilon, t, r^\varepsilon(t), z_\delta) \\
&\quad \left. - \tilde{b}^{(\delta)}(x_{t-}^\varepsilon, t, \tilde{r}^\varepsilon(t), z_\delta)) \right\} \lambda_\delta(dz_\delta) \\
&+ \sum_{j=1}^N \tilde{\gamma}_{r^\varepsilon(t)j} \tilde{V}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, j), t, j) \\
&- \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(t)k} (\varphi^\varepsilon(0) - D(\varphi^\varepsilon, k), t, k).
\end{aligned} \tag{81}$$

This implies that

$$\begin{aligned}
& U(t+h) - U(t) \\
&= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \times [\mathbb{L}V(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\
&\quad + \gamma V(\varphi^\varepsilon(0) \\
&\quad - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s))] ds \\
&+ \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \times V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
&\quad \times [f(\varphi^\varepsilon, s, r^\varepsilon(s)) - \tilde{f}(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s))] ds \\
&+ \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \times \left[\sum_{\delta=1}^m \int_{\mathbb{R}} [V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)) \right. \\
&\quad + b^{(\delta)}(x_{s-}^\varepsilon, s, \tilde{r}^\varepsilon(s), z_\delta), s, \tilde{r}^\varepsilon(s)) \\
&\quad - V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)) \\
&\quad + \tilde{b}^{(\delta)}(x_{s-}^\varepsilon, s, \tilde{r}^\varepsilon(s), z_\delta), \\
&\quad \left. s, \tilde{r}^\varepsilon(s))] \right. \\
&\quad \left. \times \lambda_\delta(dz_\delta) \right] ds \\
&- \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
&\quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
&\quad \times \left\{ \sum_{\delta=1}^m \int_{\mathbb{R}} [V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \right. \\
&\quad \times (b^{(\delta)}(x_{s-}^\varepsilon, s, r^\varepsilon(s), z_\delta) \\
&\quad - \tilde{b}^{(\delta)}(x_{s-}^\varepsilon, s, \tilde{r}^\varepsilon(s), z_\delta))] \\
&\quad \left. \times \lambda_\delta(dz_\delta) \right\} ds
\end{aligned}$$

$$\begin{aligned}
& + \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \\
& \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
& \times \left(\sum_{j=1}^N \tilde{\gamma}_{r^\varepsilon(s)j} \tilde{V}(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, j), s, j) \right. \\
& \quad \left. - \sum_{k=1}^l \tilde{\gamma}_{\tilde{r}^\varepsilon(s)k} (\varphi^\varepsilon(0) - D(\varphi^\varepsilon, k), s, k) \right) ds \\
& =: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{82}$$

By the definition of $\tilde{b}$,

$$\begin{aligned}
& b^{(\delta)}(x_{t-}^\varepsilon, t, r^\varepsilon(t), z_\delta) - \tilde{b}^{(\delta)}(x_{t-}^\varepsilon, t, \tilde{r}^\varepsilon(t), z_\delta) \\
& = \sum_{i=1}^l \sum_{j=1}^{N_k} b^{(\delta)}(x_{t-}^\varepsilon, t, s_{kj}, z_\delta) \\
& \quad \times \left[I_{\{r^\varepsilon(t)=s_{kj}\}} - \mu_j^k I_{\{\tilde{r}^\varepsilon(t)=k\}} \right].
\end{aligned} \tag{83}$$

By Assumption 8, we have

$$\begin{aligned}
J_4 & = \limsup_{\varepsilon \rightarrow 0} \sum_{\delta=1}^m \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
& \quad \times V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s)) \\
& \quad \times \int_{\mathbb{R}} \left[b^{(\delta)}(x_{s-}^\varepsilon, s, r^\varepsilon(s), z_\delta) \right. \\
& \quad \quad \left. - \tilde{b}^{(\delta)}(x_{s-}^\varepsilon, s, \tilde{r}^\varepsilon(s), z_\delta) \right] \\
& \quad \times \lambda_\delta(dz_\delta) ds \\
& = \limsup_{\varepsilon \rightarrow 0} \sum_{\delta=1}^m \sum_{k=1}^l \sum_{j=1}^{N_k} \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\
& \quad \times V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), \\
& \quad \quad s, \tilde{r}^\varepsilon(s)) \\
& \quad \times \int_{\mathbb{R}} b^{(\delta)}(x_{s-}^\varepsilon, s, s_{kj}, z_\delta) \\
& \quad \times \left[I_{\{r^\varepsilon(s)=s_{kj}\}} \right. \\
& \quad \quad \left. - \mu_j^k I_{\{\tilde{r}^\varepsilon(s)=k\}} \right] \\
& \quad \times \lambda_\delta(dz_\delta) ds
\end{aligned}$$

$$\begin{aligned}
& \leq \limsup_{\varepsilon \rightarrow 0} \sum_{\delta=1}^m \sum_{k=1}^l \sum_{j=1}^{N_k} \left[\mathbb{E} \left| \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \right. \right. \\
& \quad \times V_x(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), \\
& \quad \quad s, \tilde{r}^\varepsilon(s)) \\
& \quad \times \int_{\mathbb{R}} b^{(\delta)}(x_{s-}^\varepsilon, s, s_{kj}, z_\delta) \\
& \quad \times \left[I_{\{r^\varepsilon(s)=s_{kj}\}} \right. \\
& \quad \quad \left. - \mu_j^k I_{\{\tilde{r}^\varepsilon(s)=k\}} \right] \\
& \quad \left. \times \lambda_\delta(dz_\delta) ds \right|^2 \Bigg]^{1/2}.
\end{aligned} \tag{84}$$

By the argument of Lemma 7.14 in [4], the right side of the above inequality is equivalent to 0; that is, $J_4 = 0$. Similarly, by mean-value theorem, we can show that there exists $\eta_t^{(\delta)}$ which is between $\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)) + b^{(\delta)}(x_{t-}^\varepsilon, t, \tilde{r}^\varepsilon(t), z_\delta)$ and $\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(t)) + \tilde{b}^{(\delta)}(x_{t-}^\varepsilon, t, \tilde{r}^\varepsilon(t), z_\delta)$ such that

$$\begin{aligned}
J_3 & = \lim_{\varepsilon \rightarrow 0} \sum_{\delta=1}^m \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \left\{ \int_{\mathbb{R}} V_x(\eta_s) \right. \\
& \quad \quad \times \left[b^{(\delta)}(x_{s-}^\varepsilon, s, r^\varepsilon(s), z_\delta) \right. \\
& \quad \quad \quad \left. - \tilde{b}^{(\delta)}(x_{s-}^\varepsilon, s, \tilde{r}^\varepsilon(s), z_\delta) \right] \\
& \quad \quad \times \lambda_\delta(dz_\delta) \Big\} ds \\
& = \lim_{\varepsilon \rightarrow 0} \sum_{\delta=1}^m \sum_{k=1}^l \sum_{j=1}^{N_k} \mathbb{E} \\
& \quad \times \int_{t+\theta}^{t+\theta+h} e^{\gamma s} V_x(\eta_s) \\
& \quad \times \int_{\mathbb{R}} b^{(\delta)}(x_{s-}^\varepsilon, s, s_{kj}, z_\delta) \\
& \quad \times \left[I_{\{r^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\tilde{r}^\varepsilon(s)=k\}} \right] \\
& \quad \times \lambda_\delta(dz_\delta) ds \\
& \leq \lim_{\varepsilon \rightarrow 0} \sum_{\delta=1}^m \sum_{k=1}^l \sum_{j=1}^{N_k} \left[\mathbb{E} \left| \int_{t+\theta}^{t+\theta+h} e^{\gamma s} V_x(\eta_t) \right. \right. \\
& \quad \times \int_{\mathbb{R}} b^{(\delta)}(x_{s-}^\varepsilon, s, s_{kj}, z_\delta)
\end{aligned}$$

$$\begin{aligned} & \times \left[I_{\{r^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\tilde{r}^\varepsilon(s)=k\}} \right] \\ & \times \lambda_\delta (dz_\delta) ds \Bigg|^2 \Bigg]^{1/2}. \end{aligned} \quad (85)$$

By the argument of Lemma 7.14 in [4], we have $J_3 = 0$. Similar to the proof of Theorem 7, we derive $J_2 = 0$, $J_5 = 0$. Therefore, we arrive at

$$\begin{aligned} & U(t+h) - U(t) \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \int_{t+\theta}^{t+\theta+h} e^{\gamma s} \\ & \quad \times [\mathbb{L}V(\varphi^\varepsilon, s, \tilde{r}^\varepsilon(s)) \\ & \quad + \gamma V(\varphi^\varepsilon(0) - D(\varphi^\varepsilon, \tilde{r}^\varepsilon(s)), s, \tilde{r}^\varepsilon(s))] ds \\ & \leq 0. \end{aligned} \quad (86)$$

Then,

$$U(t+h) \leq U(t). \quad (87)$$

Similar to the proof of Theorem 7, we get

$$\mathbb{E}|x^\varepsilon(t)|^p \leq \frac{c_2(1+\kappa)^p}{c_1(1-\kappa)^p} \|\xi\|^p e^{-\bar{\gamma}t}. \quad (88)$$

The proof is therefore completed. $\square$

5. Examples

We will give two examples to illustrate our theory.

Example 1. Let $r^\varepsilon(\cdot)$ be a Markov chain generated by Γ^ε given in (14) with

$$\bar{\Gamma} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad (89)$$

$$\hat{\Gamma} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (90)$$

The generator $\bar{\Gamma}$ is made up of two irreducible blocks; by

$$(\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} = 0 \quad (91)$$

and $\pi_1 + \pi_2 + \pi_3 = 1$, we get $\mu^1 = (5/8, 1/8, 1/4)$. In the same way, by

$$(\pi_4 \ \pi_5) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = 0 \quad (92)$$

and $\pi_4 + \pi_5 = 1$, we have $\mu^2 = (1/2, 1/2)$. So,

$$\begin{aligned} \bar{\Gamma} &= \bar{\mu} \hat{\Gamma} \mathbf{1} = \begin{pmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ & \times \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{8} & \frac{3}{8} \\ 1 & -1 \end{pmatrix}. \end{aligned} \quad (93)$$

Consider a one-dimensional neutral stochastic functional differential equation as follows:

$$\begin{aligned} & d[\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t))] \\ &= f(\varphi^\varepsilon, r^\varepsilon(t)) dt + g(\varphi^\varepsilon, r^\varepsilon(t)) dw(t), \end{aligned} \quad (94)$$

with

$$\begin{aligned} D(\varphi, s_{11}) &= -0.6 \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, s_{12}) &= -0.2 \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, s_{13}) &= -0.4 \int_{-1}^0 \varphi(\theta) d\theta, \\ f(\varphi, s_{11}) &= -16\varphi(0) - 8 \cos[\varphi(0)], \\ f(\varphi, s_{12}) &= 8\varphi(0) + 4 \cos[\varphi(0)], \\ f(\varphi, s_{13}) &= 16\varphi(0), \\ g(\varphi, s_{11}) &= \frac{\sqrt{10}}{10} \int_{-1}^0 \varphi(\theta) d\theta \cos \left[\int_{-1}^0 \varphi(\theta) d\theta \right], \\ g(\varphi, s_{12}) &= -\frac{\sqrt{2}}{2} \int_{-1}^0 \varphi(\theta) d\theta \sin \left[\int_{-1}^0 \varphi(\theta) d\theta \right], \\ g(\varphi, s_{13}) &= \frac{\sqrt{3}}{2} \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, s_{21}) &= 0.5 \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, s_{22}) &= 0.5 \int_{-1}^0 \varphi(\theta) d\theta, \\ f(\varphi, s_{21}) &= -2\varphi(0), \quad f(\varphi, s_{22}) = -2\varphi(0), \end{aligned}$$

$$g(\varphi, s_{21}) = \frac{\int_{-1}^0 \varphi(\theta) d\theta \sin \left[\int_{-1}^0 \varphi(\theta) d\theta \right]}{4\sqrt{2}},$$

$$g(\varphi, s_{22}) = \frac{\int_{-1}^0 \varphi(\theta) d\theta \cos \left[\int_{-1}^0 \varphi(\theta) d\theta \right]}{4\sqrt{2}}. \quad (95)$$

For any $\varphi \in L^2_{\mathcal{F}_t}([-1, 0]; \mathbb{R})$ and $\kappa = \max\{0.6, 0.2, 0.4\} = 0.6$, applying the Hölder inequality yields

$$\mathbb{E}|D(\varphi, i)|^2 \leq 0.6^2 \sup_{-1 \leq \theta \leq 0} e^{\gamma\theta} \mathbb{E} \left| \int_{-1}^0 \varphi(\theta) d\theta \right|^2 \leq 0.36 \|\varphi\|^2, \quad (96)$$

which implies condition (24). Then, the limit equation is

$$d[\tilde{\varphi}(0) - D(\tilde{\varphi}, \tilde{r}(t))] = f(\tilde{\varphi}, \tilde{r}(t)) dt + g(\tilde{\varphi}, \tilde{r}(t)) dw(t), \quad (97)$$

where $\tilde{r}$ is the Markov chain generated by $\tilde{\Gamma}$ and

$$\tilde{D}(\varphi, 1) = -0.5 \int_{-1}^0 \varphi(\theta) d\theta,$$

$$\tilde{D}(\varphi, 2) = 0.5 \int_{-1}^0 \varphi(\theta) d\theta,$$

$$\tilde{f}(\varphi, 1) = -5\varphi(0), \quad \tilde{f}(\varphi, 2) = -2\varphi(0),$$

$$\tilde{g}(\varphi, 1) = \frac{1}{2} \int_{-1}^0 \varphi(\theta) d\theta, \quad \tilde{g}(\varphi, 2) = \frac{1}{4} \int_{-1}^0 \varphi(\theta) d\theta. \quad (98)$$

We define $V(x, 1) = 2x^2$, $V(x, 2) = x^2$. And by simple calculation, we can get

$$\mathcal{L}V(\varphi, 1) \leq -20\frac{3}{8}\varphi^2(0) + \frac{13}{32} \left| \int_{-1}^0 \varphi(\theta) d\theta \right|^2, \quad (99)$$

$$\mathcal{L}V(\varphi, 2) \leq -\frac{5}{2}\varphi^2(0) + \frac{13}{16} \left| \int_{-1}^0 \varphi(\theta) d\theta \right|^2.$$

Consequently,

$$\begin{aligned} \max_{i=1,2} \mathcal{L}V(\varphi, i) &\leq -\frac{5}{2}\varphi^2(0) + \frac{13}{16} \left| \int_{-1}^0 \varphi(\theta) d\theta \right|^2 \\ &= -\frac{5}{4} \left[\max_{i=1,2} V(x, i) \right] + \frac{13}{16} \left[\min_{i=1,2} V(x, i) \right]. \end{aligned} \quad (100)$$

It is easy to find a $q > 1$ such that $5/4 - 13q/16 > 0$. Therefore, for any $\phi \in L^2_{\mathcal{F}_t}([-1, 0]; \mathbb{R})$ satisfying $\mathbb{E}[\min_{i \in \mathbb{S}} \phi(\theta)] \leq q\mathbb{E}[\max_{i \in \mathbb{S}} \phi(0)]$ on $-1 \leq \theta \leq 0$, (100) yields

$$\mathbb{E} \left[\max_{i \in \mathbb{S}} \mathcal{L}V(\varphi, i) \right] \leq -\left(\frac{5}{4} - \frac{13q}{16} \right) \mathbb{E} \left[\max_{i=1,2} V(x, i) \right]. \quad (101)$$

Hence, by Theorem 7, the solution $x^\varepsilon(t)$ is mean square stable when ε is sufficiently small.

Example 2. Let $r^\varepsilon(\cdot)$ be a Markov chain generated by

$$\Gamma^\varepsilon = \frac{1}{\varepsilon} \tilde{\Gamma} + \hat{\Gamma} = \frac{1}{\varepsilon} \begin{pmatrix} -2 & 0 & 2 & 0 \\ 1 & -2 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & -2 \end{pmatrix}. \quad (102)$$

Here, we set $\hat{\Gamma} = 0$. By a similar way, we get the stationary distribution $\mu = (2/11, 4/11, 3/11, 2/11)$.

Consider the following one-dimensional equation:

$$\begin{aligned} d[\varphi^\varepsilon(0) - D(\varphi^\varepsilon, r^\varepsilon(t))] \\ = f(\varphi^\varepsilon, r^\varepsilon(t)) dt + \int_0^\infty \sigma(r^\varepsilon(t), z) x_{t-}^\varepsilon \tilde{N}(dt, dz) \end{aligned} \quad (103)$$

with

$$\begin{aligned} D(\varphi, 1) &= -0.9 \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, 2) &= -0.4 \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, 3) &= -0.5 \int_{-1}^0 \varphi(\theta) d\theta, \\ D(\varphi, 4) &= -0.3 \int_{-1}^0 \varphi(\theta) d\theta, \end{aligned} \quad (104)$$

$$f(\varphi, 1) = 2 \sin[\varphi(0)], \quad f(\varphi, 2) = -\frac{11}{2} \varphi(0),$$

$$f(\varphi, 3) = -\frac{11}{3} \varphi(0), \quad f(\varphi, 4) = -2 \sin[\varphi(0)].$$

Let

$$\begin{aligned} \alpha(z) &= \frac{2}{11} \sigma(1, z) + \frac{4}{11} \sigma(2, z) + \frac{3}{11} \sigma(3, z) + \frac{2}{11} \sigma(4, z), \\ \int_0^\infty \alpha^2(z) \lambda(dz) &< 2. \end{aligned} \quad (105)$$

For any $\varphi \in L^2_{\mathcal{F}_t}([-1, 0]; \mathbb{R})$ and $\kappa = \max\{0.9, 0.4, 0.5, 0.3\} = 0.9$, applying the Hölder inequality yields

$$\mathbb{E}|D(\varphi, i)|^2 \leq 0.9^2 \sup_{-1 \leq \theta \leq 0} e^{\gamma\theta} \mathbb{E} \left| \int_{-1}^0 \varphi(\theta) d\theta \right|^2 \leq 0.81 \|\varphi\|^2, \quad (106)$$

which implies condition (67). Then, the limit equation is

$$\begin{aligned} d \left[\varphi(0) + 0.5 \int_{-1}^0 \varphi(\theta) d\theta \right] \\ = -3\varphi(0) dt + \int_0^\infty \alpha(z) \tilde{x}_{t-} \tilde{N}(dt, dz). \end{aligned} \quad (107)$$

Let $V(x) = x^2$; then

$$\mathbb{L}V(\varphi, i) \leq -6\varphi^2(0) + \int_0^\infty \alpha^2(z) \lambda(dz) \left| \int_{-1}^0 \varphi(\theta) d\theta \right|^2. \quad (108)$$

We can find a $q > 1$ such that $6 - 2q > 0$. Therefore, for any $\phi \in L^2_{\mathcal{F}_t}([-1, 0]; \mathbb{R})$ satisfying $\mathbb{E}[\min_{i \in \mathbb{S}} \phi(\theta)] \leq q\mathbb{E}[\max_{i \in \mathbb{S}} \phi(0)]$ on $-1 \leq \theta \leq 0$, (108) yields

$$\mathbb{E} \left[\max_{i \in \mathbb{S}} \mathbb{L}V(\varphi, i) \right] \leq -(6 - 2q) \mathbb{E} \left[\max_{i=1,2} V(x, i) \right]. \quad (109)$$

Hence, by Theorem 12, the solution $x^\varepsilon(t)$ is mean square stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was supported by the National Science Foundation of China with Grant no. 61374085.

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