Research Article

Local Fractional Laplace Variational Iteration Method for Nonhomogeneous Heat Equations Arising in Fractal Heat Flow

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The local fractional Laplace variational iteration method is used for solving the nonhomogeneous heat equations arising in the fractal heat flow. The approximate solutions are nondifferentiable functions and their plots are also given to show the accuracy and efficiency to implement the previous method.

1. Introduction

Fractional calculus [1–4] was used to deal with the heat conduction equation in fractal media. Fractional heat conduction equation was studied by many researchers [5–17]. For example, Povstenko considered the thermoelasticity based on the fractional heat conduction equation [7]. Youssef suggested the generalized theory of fractional-order thermoelasticity [8]. Ezzat and El-Karamany presented the fractional-order conduction in thermoelastic medium [9]. Ezzat proposed the fractional-order heat transfer in thermoelectric fluid [10]. Sherief et al. reported the fractional-order generalized thermoelasticity with one relaxation time [11]. Vázquez et al. used the second law of thermodynamics to fractional heat conduction equation [12]. Hristov considered the inverse Stefan problem and nonlinear heat conduction with Jeffrey's fading memory by using the heat balance integral method [13, 14]. Davey and Prosser gave the solutions of the heat transfer on fractal and prefractal domains [15]. Ostoja-Starzewski investigated thermoelasticity of fractal media [16]. Qi and Jiang discussed space-time fractional Cattaneo diffusion equation [17]. Bhrawy and Alghamdi applied the Legendre tau-spectral method to find time fractional

heat equation with nonlocal conditions [18]. Atangana and Kılıçman suggested the Sumudu transform solving certain nonlinear fractional heat-like equations [19].

Recently, the local fractional calculus [20–22] was used to deal with the discontinuous problem for heat transfer in fractal media [23–25]. The nonhomogeneous heat equations arising in fractal heat flow were considered by using the local fractional Fourier series method [26]. The local fractional heat conduction equation was investigated by the local fractional variation iteration method [27]. The nondifferentiable solution of one-dimensional heat equations arising in fractal transient conduction was found by the local fractional Adomian decomposition method [28]. Local fractional Laplace variational iteration method [29, 30] was considered to deal with linear partial differential equations. In this paper, our aim is to investigate the nonhomogeneous heat equations arising in heat flow with local fractional derivative. The paper is organized as follows. Section 2 introduces the nonhomogeneous heat equations arising in heat flow with local fractional derivative. In Section 3, local fractional Laplace variational iteration method is presented. In Section 4, the nondifferentiable solutions for nonhomogeneous heat equations

arising in heat flow with local fractional derivative are investigated. Finally, conclusions are shown in Section 5.

2. The Nonhomogeneous Heat Equations Arising in Heat Flow with Local Fractional Derivatives

In this section we present the one-dimensional nonhomogeneous heat equations arising in heat flow with local fractional derivatives.

Let the local fractional volume integral of the function $\mathbf{u}$ be defined as [19]

$$\iiint \mathbf{u}(r_p) d\Omega^{(\gamma)} = \lim_{N \rightarrow \infty} \sum_{p=1}^N \mathbf{u}(r_p) \Delta\Omega_p^{(\gamma)}, \quad (1)$$

where the elements of the volume $\Delta\Omega_p^{(\gamma)} \rightarrow 0$ as $N \rightarrow \infty$ and the fractal dimension of the volume γ . The equality $u(x, y, z, t)$ is the temperature at the point $(x, y, z) \in \Omega$, time $t \in T$, and the total amount of heat $H(t)$ is described as

$$H(t) = \iiint c_\alpha \rho_\alpha u(x, y, z, t) d\Omega^{(\gamma)}, \quad (2)$$

where c_α is the special heat of the fractal material and ρ_α is the density of the fractal material.

The local fractional surface integral is defined as [19, 22]

$$\iint \mathbf{u}(r_p) \cdot d\mathbf{S}^{(\beta)} = \lim_{N \rightarrow \infty} \sum_{p=1}^N \mathbf{u}(r_p) \cdot \mathbf{n}_p \Delta S_p^{(\beta)}, \quad (3)$$

where N are elements of area with a unit normal local fractional vector $\mathbf{n}_p$, $\Delta S_p^{(\beta)} \rightarrow 0$ as $N \rightarrow \infty$ for $\gamma = (3/2)\beta = 3\alpha$.

From (3) the local fractional Fourier law of the material in fractal media [19, 23] was suggested as follows:

$$\frac{d^\alpha}{dt^\alpha} H(t) = \oiint_{\partial\Omega^{(\beta)}} k^{2\alpha} \nabla^\alpha u(x, y, z, t) \cdot d\mathbf{S}^{(\beta)}, \quad (4)$$

where $d\mathbf{S}^{(\beta)}$ is the fractal surface measure over $\Omega^{(\gamma)}$ and $k^{2\alpha}$ is the thermal conductivity of the fractal material.

In view of (4), the change in heat reads as follows [19, 23]:

$$\frac{d^\alpha}{dt^\alpha} H(t) = \iiint c_\alpha \rho_\alpha u_t^{(\alpha)}(x, y, z, t) d\Omega^{(\gamma)}, \quad (5)$$

where $\partial\Omega^{(\beta)}$ is the boundary of $\Omega^{(\gamma)}$.

From (2) we suggest the following source term [23]:

$$G(t) = \iiint g(x, y, z, t) d\Omega^{(\gamma)}. \quad (6)$$

Making use of (4), (5), and (6), we have

$$\begin{aligned} & \iiint c_\alpha \rho_\alpha u_t^{(\alpha)}(x, y, z, t) d\Omega^{(\gamma)} \\ &= \oiint_{\partial\Omega^{(\beta)}} k^{2\alpha} \nabla^\alpha u(x, y, z, t) \cdot d\mathbf{S}^{(\beta)} \\ &+ \iiint g(x, y, z, t) d\Omega^{(\gamma)} \end{aligned} \quad (7)$$

such that

$$\begin{aligned} & \iiint \{c_\alpha \rho_\alpha u_t^{(\alpha)}(x, y, z, t) - \nabla^\alpha \\ & \cdot [k^{2\alpha} \nabla^\alpha u(x, y, z, t)] \\ & - g(x, y, z, t)\} d\Omega^{(\gamma)} = 0, \end{aligned} \quad (8)$$

which leads to the nonhomogeneous local fractional heat equations [23]:

$$\begin{aligned} & c_\alpha \rho_\alpha u_t^{(\alpha)}(x, y, z, t) - \nabla^\alpha \cdot [k^{2\alpha} \nabla^\alpha u(x, y, z, t)] \\ &= g(x, y, z, t). \end{aligned} \quad (9)$$

From (9) we obtain the nonhomogeneous heat equations in the dimensionless case:

$$\phi_t^{(\alpha)}(x, y, z, t) - \nabla^{2\alpha} \phi(x, y, z, t) = \varphi(x, y, z, t). \quad (10)$$

The two-dimensional case is [23]

$$\phi_t^{(\alpha)}(x, y, t) - \nabla^{2\alpha} \phi(x, y, t) = \varphi(x, y, t), \quad (11)$$

and the one-dimensional case is [26]

$$\phi_t^{(\alpha)}(x, t) - \phi_x^{(2\alpha)}(x, t) = \varphi(x, t). \quad (12)$$

3. Local Fractional Laplace Variational Iteration Method

In this section, we give the idea of local fractional Laplace variational method [29, 30] in order to investigate the one-dimensional nonhomogeneous heat equations arising in fractal heat flow.

We present the following local fractional differential operator as follows:

$$L_\alpha u - R_\alpha u = 0, \quad (13)$$

where the linear local fractional differential operator denotes $L_\alpha = d^{2\alpha}/dx^{2\alpha}$ and $u(x)$ is a nondifferential function.

We can write the local fractional functional formula as

$$\begin{aligned} & u_{n+1}(x) = u_n(x) \\ & + {}_0I_x^{(\alpha)} \left\{ \frac{\lambda(x-t)^\alpha}{\Gamma(1+\alpha)} [L_\alpha u_n(t) - R_\alpha u_n] \right\}. \end{aligned} \quad (14)$$

The local fractional Laplace transform is given as [29–32]

$$\begin{aligned} & \tilde{L}_\alpha \{f(x)\} \\ &= f_s^{\tilde{L}_\alpha}(s) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha, \\ & 0 < \alpha \leq 1, \end{aligned} \quad (15)$$

and the inverse formula of local fractional Laplace transform is suggested as [29–32]

$$\begin{aligned} f(x) &= \tilde{L}_\alpha^{-1} \{f_s^{L,\alpha}(s)\} \\ &= \frac{1}{(2\pi)^\alpha} \int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{\tilde{L},\alpha}(s) (ds)^\alpha, \end{aligned} \quad (16)$$

where $f(x)$ is a local fractional continuous function, $s^\alpha = \beta^\alpha + i^\alpha \infty^\alpha$, $\text{Re}(s^\alpha) = \beta^\alpha$, and the local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is given as [23]

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha, \end{aligned} \quad (17)$$

with the partitions of the interval $[a, b]$ which is (t_j, t_{j+1}) , for $\Delta t_j = t_{j+1} - t_j$, $t_0 = a$, $t_N = b$, and $\Delta t = \max\{\Delta t_0, \Delta t_1, \Delta t_j, \dots\}$, $j = 0, \dots, N-1$.

From (15) the local fractional convolution of two functions is defined as [29–32]

$$f_1(x) * f_2(x) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f_1(t) f_2(x-t) (dt)^\alpha, \quad (18)$$

and we have

$$F_\alpha \{f_1(x) * f_2(x)\} = \tilde{L}_\alpha \{f_1(x)\} \tilde{L}_\alpha \{f_2(x)\}. \quad (19)$$

From (19) we obtain

$$\begin{aligned} \tilde{L}_\alpha \{u_{n+1}(x)\} &= \tilde{L}_\alpha \{u_n(x)\} \\ &+ \tilde{L}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} \tilde{L}_\alpha \{L_\alpha u_n(x) - R_\alpha u_n(x)\}. \end{aligned} \quad (20)$$

By the local fractional variation [23, 27, 29, 30], we obtain

$$\begin{aligned} \delta^\alpha \{\tilde{L}_\alpha \{u_{n+1}(x)\}\} &= \delta^\alpha \{\tilde{L}_\alpha \{u_n(x)\}\} \\ &+ \delta^\alpha \left\{ \tilde{L}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} \tilde{L}_\alpha \{L_\alpha u_n(x) - R_\alpha u_n(x)\} \right\}, \end{aligned} \quad (21)$$

which leads to

$$\begin{aligned} \delta^\alpha \{\tilde{L}_\alpha \{u_{n+1}(x)\}\} &= \delta^\alpha \{\tilde{L}_\alpha \{u_n(x)\}\} \\ &+ \tilde{L}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} \{\delta^\alpha \{\tilde{L}_\alpha \{L_\alpha u_n(x)\}\}\} = 0. \end{aligned} \quad (22)$$

From (22) we have

$$\begin{aligned} \delta^\alpha \{\tilde{L}_\alpha \{L_\alpha u_n(x)\}\} &= \delta^\alpha \{s^{2\alpha} \tilde{L}_\alpha \{u_n(x)\} - s^\alpha u_n(0) - u_n^{(\alpha)}(0)\} \\ &= s^{2\alpha} \delta^\alpha \tilde{L}_\alpha \{u_n(x)\} \end{aligned} \quad (23)$$

such that

$$1 + \tilde{L}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} s^{2\alpha} = 0. \quad (24)$$

From (24) we get

$$\tilde{L}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} = -\frac{1}{s^{2\alpha}} \quad (25)$$

such that local fractional iteration algorithm reads as

$$\begin{aligned} \tilde{L}_\alpha \{u_{n+1}(x)\} &= \tilde{L}_\alpha \{u_n(x)\} \\ &- \frac{1}{s^{2\alpha}} \tilde{L}_\alpha \{(L_\alpha u_n(x) - R_\alpha u_n(x))\}, \end{aligned} \quad (26)$$

where the initial value is presented as follows:

$$\tilde{L}_\alpha \{u_0(x)\} = u(0). \quad (27)$$

Therefore, the local fractional series solution is given as

$$\tilde{L}_\alpha \{u\} = \lim_{n \rightarrow \infty} \tilde{L}_\alpha \{u_n\}. \quad (28)$$

From (28) we arrive at

$$u = \lim_{n \rightarrow \infty} \tilde{L}_\alpha^{-1} \{\tilde{L}_\alpha u_n\}. \quad (29)$$

4. The Nondifferentiable Solutions

In this section, we discuss the one-dimensional nonhomogeneous heat equations arising in fractal heat flow.

Example 1. The nonhomogeneous local fractional heat equation with the nondifferentiable sink term is presented as follows:

$$\begin{aligned} \frac{\partial^\alpha T(x, t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} T(x, t)}{\partial x^{2\alpha}} &= -\frac{x^\alpha E_\alpha(-t^\alpha)}{\Gamma(1+\alpha)}, \quad 0 < x < 1, \quad 0 < t \leq 1, \quad 0 < \alpha \leq 1, \end{aligned} \quad (30)$$

subject to the initial-boundary value conditions

$$\begin{aligned} \frac{\partial^\alpha T(0, t)}{\partial x^\alpha} &= E_\alpha(-t^\alpha), \\ T(0, t) &= 0. \end{aligned} \quad (31)$$

From (26) we obtain the local fractional iteration algorithm:

$$\begin{aligned}
 \tilde{L}_\alpha \{T_{n+1}(x, t)\} &= \tilde{L}_\alpha \{T_n(x, t)\} - \frac{1}{s^{2\alpha}} \tilde{L}_\alpha \\
 &\cdot \left\{ \left(\frac{\partial^\alpha T_n(x, t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} T_n(x, t)}{\partial x^{2\alpha}} \right. \right. \\
 &\quad \left. \left. + \frac{x^\alpha E_\alpha(-t^\alpha)}{\Gamma(1+\alpha)} \right) \right\} \\
 &= \tilde{L}_\alpha \{T_n(x, t)\} - \frac{1}{s^{2\alpha}} \\
 &\cdot \left\{ \frac{\partial^\alpha T_n(s, t)}{\partial t^\alpha} - s^{2\alpha} \tilde{L}_\alpha \{T_n(x, t)\} \right. \\
 &\quad \left. + s^\alpha T_n(0, t) + T_n^{(\alpha)}(0, t) \right. \\
 &\quad \left. + \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}} \right\} \\
 &= 2\tilde{L}_\alpha \{T_n(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_n(s, t)}{\partial t^\alpha} \\
 &\quad - \frac{T_n^{(\alpha)}(0, t)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}},
 \end{aligned} \quad (32)$$

where the initial value is given as

$$\tilde{L}_\alpha \{T_0(x, t)\} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}. \quad (33)$$

Using (32), we have the first approximation:

$$\begin{aligned}
 \tilde{L}_\alpha \{T_1(x, t)\} &= 2\tilde{L}_\alpha \{T_0(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_0(s, t)}{\partial t^\alpha} \\
 &\quad - \frac{T_0^{(\alpha)}(0, t)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} \\
 &= \frac{2E_\alpha(-t^\alpha)}{s^{2\alpha}} + \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}} \\
 &\quad - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}.
 \end{aligned} \quad (34)$$

In view of (32) and (34), we get the second approximation:

$$\begin{aligned}
 \tilde{L}_\alpha \{T_2(x, t)\} &= 2\tilde{L}_\alpha \{T_1(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_1(s, t)}{\partial t^\alpha} \\
 &\quad - \frac{T_1^{(\alpha)}(0, t)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} \\
 &= \frac{2E_\alpha(-t^\alpha)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}} \\
 &\quad - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}.
 \end{aligned} \quad (35)$$

Making use of (32) and (35), the third approximate term reads as follows:

$$\begin{aligned}
 \tilde{L}_\alpha \{T_3(x, t)\} &= 2\tilde{L}_\alpha \{T_2(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_2(s, t)}{\partial t^\alpha} \\
 &\quad - \frac{T_2^{(\alpha)}(0, t)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} \\
 &= \frac{2E_\alpha(-t^\alpha)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}} \\
 &\quad - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}.
 \end{aligned} \quad (36)$$

From (32) and (36), the fourth approximate term can be written as follows:

$$\begin{aligned}
 \tilde{L}_\alpha \{T_4(x, t)\} &= 2\tilde{L}_\alpha \{T_3(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_3(s, t)}{\partial t^\alpha} \\
 &\quad - \frac{T_3^{(\alpha)}(0, t)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} \\
 &= \frac{2E_\alpha(-t^\alpha)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}} \\
 &\quad - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}.
 \end{aligned} \quad (37)$$

Making the best of (32) and (36), we can write the fifth approximate term as

$$\begin{aligned}
 \tilde{L}_\alpha \{T_5(x, t)\} &= 2\tilde{L}_\alpha \{T_4(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_4(s, t)}{\partial t^\alpha} \\
 &\quad - \frac{T_4^{(\alpha)}(0, t)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} \\
 &= \frac{2E_\alpha(-t^\alpha)}{s^{2\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} - \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}} \\
 &\quad - \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}.
 \end{aligned} \quad (38)$$

Hence, we obtain the final term given as

$$\tilde{L}_\alpha \{T_n(x, t)\} = \frac{E_\alpha(-t^\alpha)}{s^{2\alpha}}. \quad (39)$$

In view of (28) and (29), we suggest the exact solution of (30) as

$$\begin{aligned}
 T(x, t) &= \lim_{n \rightarrow \infty} \tilde{L}_\alpha^{-1} \{ \tilde{L}_\alpha \{T_n(x, t)\} \} \\
 &= \frac{x^\alpha}{\Gamma(1+\alpha)} E_\alpha(-t^\alpha)
 \end{aligned} \quad (40)$$

and its plot is shown in Figure 1.

FIGURE 1: The nondifferentiable solution of nonhomogeneous local fractional heat equation with nondifferentiable sink term for $\alpha = \ln 2 / \ln 3$.

Example 2. We now consider the nonhomogeneous local fractional heat equation with the nondifferentiable source term:

$$\begin{aligned} & \frac{\partial^\alpha T(x, t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} T(x, t)}{\partial x^{2\alpha}} \\ &= \frac{x^\alpha \cos_\alpha(t^\alpha)}{\Gamma(1 + \alpha)}, \quad 0 < x < 1, \quad 0 < t \leq 1, \quad 0 < \alpha \leq 1, \end{aligned} \quad (41)$$

subject to the initial-boundary value conditions

$$\begin{aligned} & \frac{\partial^{2\alpha} T(x, t)}{\partial x^{2\alpha}} = \sin_\alpha(t^\alpha), \\ & T(0, t) = 0. \end{aligned} \quad (42)$$

In view of (26), the local fractional iteration algorithm can be structured as follows:

$$\begin{aligned} \tilde{L}_\alpha \{T_{n+1}(x, t)\} &= \tilde{L}_\alpha \{T_n(x, t)\} - \frac{1}{s^{2\alpha}} \tilde{L}_\alpha \\ & \cdot \left\{ \left(\frac{\partial^\alpha T_n(x, t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} T_n(x, t)}{\partial x^{2\alpha}} \right. \right. \\ & \quad \left. \left. - \frac{x^\alpha \cos_\alpha(t^\alpha)}{\Gamma(1 + \alpha)} \right) \right\} \\ &= \tilde{L}_\alpha \{T_n(x, t)\} - \frac{1}{s^{2\alpha}} \\ & \cdot \left\{ \frac{\partial^\alpha T_n(s, t)}{\partial t^\alpha} - s^{2\alpha} \tilde{L}_\alpha \{T_n(x, t)\} \right. \\ & \quad \left. + s^\alpha T_n(0, t) + T_n^{(\alpha)}(0, t) \right. \\ & \quad \left. - \frac{\cos_\alpha(t^\alpha)}{s^{2\alpha}} \right\} \end{aligned}$$

$$\begin{aligned} &= 2\tilde{L}_\alpha \{T_n(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_n(s, t)}{\partial t^\alpha} \\ & \quad - \frac{T_n^{(\alpha)}(0, t)}{s^{2\alpha}} + \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}}. \end{aligned} \quad (43)$$

Applying (43) gives the first approximate term:

$$\begin{aligned} \tilde{L}_\alpha \{T_1(x, t)\} &= 2\tilde{L}_\alpha \{T_0(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_0(s, t)}{\partial t^\alpha} \\ & \quad - \frac{T_0^{(\alpha)}(0, t)}{s^{2\alpha}} + \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ &= \frac{2\sin_\alpha(t^\alpha)}{s^{2\alpha}} - \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ & \quad - \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}} + \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}}. \end{aligned} \quad (44)$$

In view of (43) and (44), the second approximate term reads as

$$\begin{aligned} \tilde{L}_\alpha \{T_2(x, t)\} &= 2\tilde{L}_\alpha \{T_1(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_1(s, t)}{\partial t^\alpha} \\ & \quad - \frac{T_1^{(\alpha)}(0, t)}{s^{2\alpha}} + \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ &= \frac{2\sin_\alpha(t^\alpha)}{s^{2\alpha}} - \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} - \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}} \\ & \quad + \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}}. \end{aligned} \quad (45)$$

Making use of (43) and (45), we arrive at the third approximate term:

$$\begin{aligned} \tilde{L}_\alpha \{T_3(x, t)\} &= 2\tilde{L}_\alpha \{T_2(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_2(s, t)}{\partial t^\alpha} \\ & \quad - \frac{T_2^{(\alpha)}(0, t)}{s^{2\alpha}} + \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ &= \frac{2\sin_\alpha(t^\alpha)}{s^{2\alpha}} - \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ & \quad - \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}} + \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}}. \end{aligned} \quad (46)$$

From (43) and (46) we give the fourth approximation:

$$\begin{aligned} \tilde{L}_\alpha \{T_4(x, t)\} &= 2\tilde{L}_\alpha \{T_3(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_3(s, t)}{\partial t^\alpha} \\ & \quad - \frac{T_3^{(\alpha)}(0, t)}{s^{2\alpha}} + \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ &= \frac{2\sin_\alpha(t^\alpha)}{s^{2\alpha}} - \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} - \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}} \\ & \quad + \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}}. \end{aligned} \quad (47)$$

FIGURE 2: The nondifferentiable solution of the nonhomogeneous local fractional heat equation with the nondifferentiable sink term for $\alpha = \ln 2 / \ln 3$.

In view of (43) and (47), the fifth approximate term is presented as

$$\begin{aligned} \tilde{L}_\alpha \{T_5(x, t)\} &= 2\tilde{L}_\alpha \{T_4(x, t)\} - \frac{1}{s^{2\alpha}} \frac{\partial^\alpha T_4(s, t)}{\partial t^\alpha} \\ &\quad - \frac{T_4^{(\alpha)}(0, t)}{s^{2\alpha}} + \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} \\ &= \frac{2\sin_\alpha(t^\alpha)}{s^{2\alpha}} - \frac{\cos_\alpha(t^\alpha)}{s^{4\alpha}} - \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}} \\ &\quad + \frac{E_\alpha(-t^\alpha)}{s^{4\alpha}} = \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}}. \end{aligned} \quad (48)$$

Hence, we finally have

$$\tilde{L}_\alpha \{T_n(x, t)\} = \frac{\sin_\alpha(t^\alpha)}{s^{2\alpha}} \quad (49)$$

so that the exact solution of nonhomogeneous local fractional heat equation with nondifferentiable source term is

$$\begin{aligned} T(x, t) &= \lim_{n \rightarrow \infty} \tilde{L}_\alpha^{-1} \{\tilde{L}_\alpha \{T_n(x, t)\}\} \\ &= \frac{x^\alpha}{\Gamma(1 + \alpha)} \sin_\alpha(t^\alpha). \end{aligned} \quad (50)$$

For the fractal dimension $\alpha = \ln 2 / \ln 3$, the plot of the nondifferentiable solution of the nonhomogeneous local fractional heat equation with the nondifferentiable source term is shown in Figure 2.

5. Conclusions

At the present work, the nonhomogeneous heat equations arising in the fractal heat flow were investigated. The local fractional Laplace variational iteration method was applied to obtain the nondifferentiable solutions for the nonhomogeneous local fractional heat equations with the nondifferentiable source and sink terms. Finally, the graphs of the obtained solutions are also shown.

Conflict of Interests

The authors declare that they have no conflict of interests in this paper.

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