Research Article

The Stability of Two-Step Runge-Kutta Methods for Neutral Delay Integro Differential-Algebraic Equations with Many Delays

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This paper studies the asymptotic stability of the two-step Runge-Kutta methods for neutral delay integro differential-algebraic equations with many delays. It proves that A-stable two-step Runge-Kutta methods are asymptotically stable for neutral delay integro differential-algebraic equations with many delays.

1. Introduction

The stability of numerical methods for delay differential equations has been intensively studied in [1–3] for many years. These equations appeared in a wide variety of scientific and engineering fields, such as circuit analysis, computer-aided design power systems, and optimal control. The structure for these, the order of convergence, and the asymptotic stability of numerical methods have been studied in [4–6]. Zhu and Petzold investigated the asymptotic stability of neutral delay differential equations with θ -methods, Runge-Kutta methods, BDF methods, and linear multistep methods [7]. Zhao et al. studied the stability of neutral delay differential equations with Rosenbrock methods [8]. Yu et al. studied the general neutral delay differential equations with multistep methods [9]. More recently, there is a growing interest in the analysis of delay integro differential equations. Baker and Ford [10] studied the asymptotic stability of a class of linear multistep (LM) methods for scalar linear delay integro differential equations; Koto [11] dealt with the linear stability of Runge-Kutta (R-K) methods for systems of delay integro differential equations; Huang and Vandewalle [12] gave sufficient and necessary stability conditions for exact and discrete solutions of linear scalar delay integro differential equations, and Luzyanina et al. [13] developed computational

procedures for determining the stability of delay integro differential equations. Zhang and Vandewalle [14] gave the stability criteria for exact and discrete solution of neutral multdelay integro differential equations. Although the stability of numerical methods for delay integro differential equations has been very intensively studied, the stability of delay integro differential equations with many delays has not been studied so far.

In this paper, we focus on the asymptotic stability of numerical methods for neutral delay integro differential-algebraic equations with many delays. This paper is structured as follows. In Section 2 we give asymptotic stability of the analytical solution and introduce two-step Runge-Kutta methods and the stability region. In Section 3, we deal with the asymptotic stability of two-step Runge-Kutta method for neutral delay integro differential-algebraic equations with many delays; the theoretical results are proved. In Section 4, an example is given to illustrate the theoretical results.

2. Asymptotic Stability of the Analytical Solution

2.1. Asymptotic Stability of the Analytical Solution of Neutral Delay Integro Differential-Algebraic Equation with Many Delays. In this section, we consider the following linear

system:

$$\begin{aligned} Au'(t) + Bu(t) + \sum_{q=1}^m C_q u'(t - \tau_q) + \sum_{q=1}^m D_q u(t - \tau_q) \\ + \sum_{q=1}^m G_q \int_{t-\tau_q}^t u(\delta) d\delta = 0, \quad t \geq 0, \\ u(t) = \varphi(t), \quad t \in [-\tau, 0), \end{aligned} \quad (1)$$

where $A, B, C_q, D_q, G_q \in R^{d \times d}$, A is a singular matrix, τ_q is a given positive delay constant ($q = 1, 2, \dots, m$), and $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_m = \tau$. $\varphi(t)$ denotes a given vector-valued function and $u(t)$ is a vector-valued unknown function to be solved for $t \geq 0$.

In order to obtain the characteristic equation of system (1), we focus on the exponential solutions $u(t) = e^{st}x$ of (1); here $x = (x_1, x_2, \dots, x_d)^T \in C^d$ denotes the unknown vector. Then we have

$$u(t) = (e^{st}x_1, e^{st}x_2, \dots, e^{st}x_d)^T, \quad (2a)$$

$$u'(t) = se^{st}x, \quad (2b)$$

$$\begin{aligned} u(t - \tau_q) = (e^{s(t-\tau_q)}x_1, e^{s(t-\tau_q)}x_2, \dots, e^{s(t-\tau_q)}x_d)^T, \\ q = 1, 2, \dots, m, \end{aligned} \quad (2c)$$

$$u'(t - \tau_q) = se^{s(t-\tau_q)}x, \quad q = 1, 2, \dots, m. \quad (2d)$$

Substituting the above results into (1), we have the following equation:

$$\begin{aligned} \left[sA + B + s \sum_{q=1}^m C_q e^{-s\tau_q} + \sum_{q=1}^m D_q e^{-s\tau_q} \right. \\ \left. + s^{-1} \sum_{q=1}^m G_q (1 - e^{-s\tau_q}) \right] x = 0. \end{aligned} \quad (2e)$$

The existence of a nonzero x in (2e) implies the characteristic equation of system (1) holds; that is, the following equation holds:

$$\begin{aligned} \det \left[sA + B + s \sum_{q=1}^m C_q e^{-s\tau_q} \right. \\ \left. + \sum_{q=1}^m D_q e^{-s\tau_q} \right. \\ \left. + s^{-1} \sum_{q=1}^m G_q (1 - e^{-s\tau_q}) \right] = 0. \end{aligned} \quad (3)$$

Definition 1 (see [13]). Equation (1) is said to be asymptotically stable, if for any continuous differential initial function and for any delay $\tau_q > 0$, $q = 1, 2, \dots, m$ the analytical solution to (1) satisfies $\lim_{t \rightarrow \infty} u(t) = 0$.

We know that the stability of analytical solution can be studied via the characteristic equation, so we give a criterion for the asymptotic stability of (1), which is based on the following lemmas.

Lemma 2 (see [14]). Assume

$$\sup \{ \operatorname{Re}(\lambda) : p(\lambda) = 0 \} < 0, \quad (4)$$

where $p(\lambda) = \det\{\lambda A + B + \lambda \sum_{q=1}^m C_q e^{-\lambda\tau_q} + \sum_{q=1}^m D_q e^{-\lambda\tau_q} + \sum_{q=1}^m G_q \tau_q \eta(e^{-\lambda\tau_q})\}$ is the characteristic polynomial of (1). Then, system (1) is asymptotically stable.

Where $\eta(z)$ is a complex function defined by

$$\eta(z) = \begin{cases} \frac{1-z}{\ln z}, & z \in C \setminus \{0, 1\} \\ 0, & z = 0 \\ -1, & z = 1. \end{cases} \quad (5)$$

And $\ln z = \ln |z| + i \arg z$ ($z = 0, 1; -\pi < \arg z \leq \pi$) is the principal branch of the multivalued complex natural logarithm.

Lemma 3 (see [14]). Function $\eta(z)$ is analytic in $C \setminus R_0^-$ and satisfies $|\eta(z)| \leq 1$ for $|z| \leq 1$, where $R_0^- = \{x \in R : x \leq 0\}$.

Lemma 4. If the matrix $(A + \sum_{q=1}^m C_q e^{-\lambda\tau_q})$ is invertible for $\operatorname{Re}(\lambda) \geq r$, where $r \in R$, then the function

$$\begin{aligned} \tilde{p}(\lambda) = \det \left\{ \lambda^2 I_d + \left(A + \sum_{q=1}^m C_q e^{-\lambda\tau_q} \right)^{-1} \right. \\ \left. \times \left(\lambda B + \lambda \sum_{q=1}^m D_q e^{-\lambda\tau_q} \right. \right. \\ \left. \left. + \sum_{q=1}^m G_q (1 - e^{-\lambda\tau_q}) \right) \right\} \end{aligned} \quad (6)$$

has at most a finite number of zeros for $\operatorname{Re}(\lambda) \geq r$.

Proof. When $\operatorname{Re}(\lambda) \geq r$, the function $\tilde{p}(\lambda)$ can be expanded into the following form:

$$\begin{aligned} \tilde{p}(\lambda) = \lambda^{2d} + \psi_{2d-1}(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_{m1}}) \lambda^{2d-1} \\ + \dots + \psi_0(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_{m1}}), \end{aligned} \quad (7)$$

where $\psi_i(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_{m1}})$, $i = 0, 1, \dots, 2d-1$, are rational functions for the expressions $e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_{m1}}$, and they have no poles for $\operatorname{Re}(\lambda) \geq r$.

Since $\tau_i > 0$, we have that

$$|e^{-\lambda\tau_i}| = e^{-\tau_i \operatorname{Re}(\lambda)} \leq e^{-\tau_i r}, \quad \text{for } \operatorname{Re}(\lambda) \geq r. \quad (8)$$

Hence, there exist constants $K_i > 0$ such that

$$|\psi_i(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_{m1}})| \leq K_i, \quad i = 0, 1, \dots, 2d-1. \quad (9)$$

Let M be a positive number large enough such that

$$\frac{K_{2d-1}}{M} + \frac{K_{2d-2}}{M^2} + \cdots + \frac{K_0}{M^{2d}} < 1, \quad (10)$$

which implies that, for $\operatorname{Re}(\lambda) \geq r$ and $|\lambda| \geq M$,

$$|\tilde{p}(\lambda)| \geq |\lambda|^{2d} \left[1 - \frac{K_{2d-1}}{M} - \frac{K_{2d-2}}{M^2} - \cdots - \frac{K_0}{M^{2d}} \right] > 0. \quad (11)$$

That is, $\tilde{p}(\lambda) \neq 0$ in the set $\{\lambda : \operatorname{Re} \lambda \geq r, |\lambda| \geq M\}$.

By the isolation property of the zeros for analytic functions, $\tilde{p}(\lambda)$ has at most a finite number of zeros in the set $\{\lambda : \operatorname{Re} \lambda \geq r, |\lambda| < M\}$; this proves the lemma. $\square$

In the following, we denote the spectrum of a square matrix A by $\sigma(A)$ and introduce the set

$$C^- = \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\}. \quad (12)$$

Theorem 5. System (1) is asymptotically stable if the following conditions are satisfied:

- (a) $\det(A + \sum_{q=1}^m \xi_q C_q) \neq 0$ for $|\xi_q| \leq 1$,
- (b) $\sigma(G(\xi)) \subseteq C^-$ for $\xi = (\xi_1, \xi_2, \dots, \xi_m)^T$ with $|\xi_q| \leq 1$, where

$$G(\xi) = \left(A + \sum_{q=1}^m \xi_q C_q \right)^{-1} \times \left(-B - \sum_{q=1}^m \xi_q D_q - \sum_{q=1}^m \eta(\xi_q) G_q \tau_q \right). \quad (13)$$

Proof. When $|\xi_q| \leq 1$, $q = 1, 2, \dots, m$, condition (a) leads to

$$\begin{aligned} \hat{P}(\lambda, \xi_1, \xi_2, \dots, \xi_m) &= \det \left(\lambda A + \lambda \sum_{q=1}^m \xi_q C_q + B \right. \\ &\quad \left. + \sum_{q=1}^m \xi_q D_q \right. \\ &\quad \left. + \sum_{q=1}^m \eta(\xi_q) G_q \tau_q \right) \\ &= \det \left(A + \sum_{q=1}^m \xi_q C_q \right) \det(\lambda I_d - G(\xi)). \end{aligned} \quad (14)$$

Condition (b) leads to

$$P(\lambda) = \hat{P}(\lambda, e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, \dots, e^{-\lambda \tau_m}) \neq 0 \quad \text{for } \operatorname{Re}(\lambda) \geq 0. \quad (15)$$

Hence

$$\sup \{\operatorname{Re}(\lambda) : P(\lambda) = 0\} \leq 0. \quad (16)$$

Now we will show that the strict inequality in (16) holds. Define

$$F(\xi_1, \xi_2, \dots, \xi_m) = \det \left(A + \sum_{q=1}^m \xi_q C_q \right), \quad (17)$$

and then $F(\xi_1, \xi_2, \dots, \xi_m)$ is a multivariate polynomial and is nonzero on the compact domain defined by $|\xi_q| \leq 1$, $q = 1, 2, \dots, m$, and equal to 1 at the origin. Hence, its modulus is bounded; that is,

$$|F(\xi_1, \xi_2, \dots, \xi_m)| \geq \varepsilon > 0, \quad \text{when } |\xi_q| \leq 1, \quad \text{for } q = 1, 2, \dots, m. \quad (18)$$

By the continuity of F , there exists a $\delta > 0$ such that

$$|F(\xi_1, \xi_2, \dots, \xi_m)| > 0, \quad \text{when } |\xi_q| \leq 1 + \delta, \quad \text{for } q = 1, 2, \dots, m. \quad (19)$$

It follows from this that

$$\det \left(A + \sum_{q=1}^m \xi_q C_q \right) \neq 0, \quad \text{when } |e^{-\lambda \tau_q}| \leq 1 + \delta, \quad \text{for } q = 1, 2, \dots, m. \quad (20)$$

Let r be the strictly positive number $r = \ln(1 + \delta)/\tau$; then

$$\det \left(A + \sum_{q=1}^m e^{-\lambda \tau_q} C_q \right) \neq 0 \quad \text{for } \operatorname{Re}(\lambda) \geq -r. \quad (21)$$

Thus, the equation $\tilde{P}(\lambda) = 0$ has only a finite number of roots when $\operatorname{Re}(\lambda) \geq -r$, and it holds true for the equation $p(\lambda) = 0$ by condition (a). Combined with (16) we get that the characteristic equation has at most a finite number of roots in the region $\{\lambda : -r \leq \operatorname{Re}(\lambda) < 0\}$.

Let

$$-\gamma = \max_{-r \leq \operatorname{Re}(\lambda) < 0} \{\operatorname{Re}(\lambda)\}; \quad (22)$$

then $\gamma > 0$.

When $\operatorname{Re}(\lambda) > -\gamma$, the characteristic equation $p(\lambda) = 0$ has no root. Hence, a strict inequality holds in (16). By Lemma 2, the proof is completed. $\square$

2.2. The Two-Step Runge-Kutta Methods and the Stability Region. Consider the two-step Runge-Kutta method:

$$\begin{aligned} Y^{(n)} &= hC_{11}F(t_n, Y^{(n)}) + C_{12}y^{(n-1)} \\ y^{(n)} &= hC_{21}F(t_n, Y^{(n)}) + C_{22}y^{(n-1)} \end{aligned} \quad (23)$$

for solving the initial value problem (1).

In order to simplify the analysis, we consider two-step Runge-Kutta method (TSRK) of the form

$$\begin{aligned} u_{n+1} &= (1 - \theta) u_n + \theta u_{n-1} \\ &+ h \sum_{j=1}^s \widehat{b}_j f(t_j, U_n^j) \\ &+ h \sum_{j=1}^s \widetilde{b}_j f(\widetilde{t}_j, U_{n-1}^j), \end{aligned} \quad (24a)$$

$$U_n^i = u_n + h \sum_{j=1}^s \widetilde{a}_{i,j} f(t_j, U_n^j) \quad i = 1, 2, \dots, s, \quad (24b)$$

where $t_j = t_n + c_j h$, $\widetilde{t}_j = t_{n-1} + c_j h$, u_i is an approximation to $u(t_i)$, h is a fixed step-size, θ , $\widehat{b}_j$, $\widetilde{b}_j$, $\widetilde{a}_{i,j}$, and c_j are coefficients of the method, $0 \leq \theta \leq 1$.

These methods are a subclass of general linear methods introduced by Butcher [15] and could be possibly also referred to as two-step hybrid methods. They generalize k -step collocation methods (with $k = 2$) for ordinary differential equations (ODEs) studied by Lie and Nørsett [16] and Lie [17] and two-step Runge-Kutta methods for ODEs investigated by Byrne and Lambert [18]. The variable stepsize continuous two-step Runge-Kutta methods for ODEs were investigated by Jackiewicz and Tracogna [19]. Here we will represent (24a) and (24b) by the following table of the coefficients:

$$\begin{array}{c|c} C & A \\ \hline \theta & \begin{array}{c} \widehat{b}^T \\ \widetilde{b}^T \end{array} \end{array} = \frac{\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_s \end{array}}{\theta} = \frac{\begin{array}{c} \widetilde{a}_{11} \quad \widetilde{a}_{12} \quad \cdots \quad \widetilde{a}_{1s} \\ \widetilde{a}_{21} \quad \widetilde{a}_{22} \quad \cdots \quad \widetilde{a}_{2s} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \widetilde{a}_{s1} \quad \widetilde{a}_{s2} \quad \cdots \quad \widetilde{a}_{ss} \end{array}}{\begin{array}{c} \widehat{b}_1 \quad \widehat{b}_2 \quad \cdots \quad \widehat{b}_s \\ \widetilde{b}_1 \quad \widetilde{b}_2 \quad \cdots \quad \widetilde{b}_s \end{array}}, \quad (25)$$

where $c_i = \sum_{j=1}^s \widetilde{a}_{ij}$ and $\sum_{j=1}^s (\widehat{b}_j + \widetilde{b}_j) = 1 + \theta$.

Apply (24a) and (24b) to the basic test equation

$$u'(t) = au(t) \quad t \geq 0, \quad \text{Re } a < 0, \quad (26)$$

which gives the following equations:

$$u_{n+1} = (1 - \theta) u_n + \theta u_{n-1} + \alpha \left(\sum_{j=1}^s \widehat{b}_j U_n^j + \widetilde{b}_j U_{n-1}^j \right) \quad (27)$$

$$U_n^i = u_n + \alpha \sum_{j=1}^s \widetilde{a}_{i,j} U_n^j.$$

Rewriting (27) we obtain

$$u_{i+1} = R(\alpha, \theta) u_i + S(\alpha, \theta) u_{i-1}, \quad (28)$$

where

$$\begin{aligned} R(\alpha, \theta) &= 1 - \theta + \alpha \widehat{b}^T (I - \alpha \widetilde{A})^{-1} e, \\ S(\alpha, \theta) &= \theta + \alpha \widetilde{b}^T (I - \alpha \widetilde{A})^{-1} e, \\ \alpha &= ah, \quad e = [1, 1, \dots, 1]_s^T. \end{aligned} \quad (29)$$

To investigate the stability properties of (24a) and (24b) with (26), we must investigate the asymptotic behaviors of the solution to (28). This is determined by the location of roots of the characteristic polynomial

$$\varphi(z) = z^2 - R(\alpha, \theta)z - S(\alpha, \theta). \quad (30)$$

The stability region of the two-step Runge-Kutta methods (24a) and (24b) is the set of all points α for which the roots of $\varphi(z)$ are inside or on the unit circle with those on the unit circle being simple. If $\varphi(z)$ is a Schur polynomial for any α with $\text{Re } \alpha < 0$, the stability of the two-step Runge-Kutta method contains the negative half plane; the method is said to be A-stable for ODEs.

3. Asymptotic Stability of TSRK Methods for Neutral Delay Integro Differential-Algebraic Equation with Many Delays

In this section, we will confine our discussion to neutral delay integro differential-algebraic equation with commensurate delays, that is, systems of the form (1) with $\tau_q = q\tau$, $\tau = Mh$, M is a positive integer, $q = 1, 2, \dots, m$.

Definition 6 (see [20]). A numerical method for asymptotically stable system (1) is called asymptotically stable if the numerical solution satisfies

$$\lim_{n \rightarrow \infty} u_n = 0. \quad (31)$$

Applying the two-step method (24a) and (24b) to (1), we have

$$u_{n+1} = (1 - \theta) u_n + \theta u_{n-1} + \sum_{j=1}^s \widehat{b}_j K_{n,j} + \sum_{j=1}^s \widetilde{b}_j K_{n-1,j}, \quad (32)$$

$$\begin{aligned} AK_{n,i} + hB \left(u_n + \sum_{j=1}^s \widetilde{a}_{ij} K_{n,j} \right) + \sum_{q=1}^m C_q K_{n-qM,i} \\ + h \sum_{q=1}^m D_q \left(u_{n-qM} + \sum_{j=1}^s \widetilde{a}_{ij} K_{n-qM,j} \right) \\ + h \sum_{q=1}^m \sum_{\gamma=0}^{qM} \gamma_r G_q \cdot \left(u_{n-r} + \sum_{j=1}^s \widetilde{a}_{ij} K_{n-r,j} \right) = 0 \end{aligned} \quad (33)$$

for $i = 1, 2, \dots, s$,

where $K_{n,i} = [K_{n,i}^1, K_{n,i}^2, \dots, K_{n,i}^d]^T$, $i = 1, 2, \dots, s$, are stage derivatives multiplied by h .

Let

$$\begin{aligned} \widehat{b}^T &= [\widehat{b}_1, \widehat{b}_2, \dots, \widehat{b}_s], \quad \widetilde{b}^T = [\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_s], \\ \widetilde{A} &= (\widetilde{a}_{ij}). \end{aligned} \quad (34)$$

We assume that all the eigenvalues of $\tilde{A}$ have positive real part. Rearrange the variables of the stage derivatives as

$$K_n = [K_{n,1}^1, K_{n,2}^1, \dots, K_{n,s}^1, K_{n,1}^2, K_{n,2}^2, \dots, K_{n,s}^2, \dots, K_{n,1}^d, K_{n,2}^d, \dots, K_{n,s}^d]^T. \quad (35)$$

Define

$$Y_n = (K_n^T, u_{n+1}^T)^T, \quad \bar{B} = hB, \quad \bar{D}_q = hD_q, \quad \bar{\bar{G}}_q = h^2 G_q. \quad (36)$$

Rewrite (32) and (33) in the form

$$\begin{aligned} & \begin{bmatrix} A \otimes I_s + \bar{B} \otimes \tilde{A} & 0 \\ -I_d \otimes \tilde{b}^T & I_d \end{bmatrix} Y_n \\ & + \begin{bmatrix} 0 & \bar{B} \otimes e \\ -I_d \otimes \tilde{b}^T & -(1-\theta)I_d \end{bmatrix} Y_{n-1} \\ & + \begin{bmatrix} 0 & 0 \\ 0 & -\theta I_d \end{bmatrix} Y_{n-2} \\ & + \sum_{q=1}^m \begin{bmatrix} C_q \otimes I_s + \bar{D}_q \otimes \tilde{A} & 0 \\ 0 & 0 \end{bmatrix} Y_{n-qM} \\ & + \sum_{q=1}^m \begin{bmatrix} 0 & \bar{D}_q \otimes e \\ 0 & 0 \end{bmatrix} Y_{n-qM-1} \\ & + \sum_{q=1}^m \sum_{r=0}^{Mq} \begin{bmatrix} C_q \otimes I_s + \bar{D}_q \otimes \tilde{A} & 0 \\ 0 & 0 \end{bmatrix} Y_{n-r} \\ & + \sum_{q=1}^m \sum_{r=0}^{Mq} \begin{bmatrix} 0 & \gamma_r \bar{\bar{G}}_q \otimes e \\ 0 & 0 \end{bmatrix} Y_{n-r-1} = 0. \end{aligned} \quad (37)$$

The characteristic polynomial of (37) is given by

$$p(z) = \det \begin{bmatrix} T_1(z) & T_2(z) \\ T_3(z) & T_4(z) \end{bmatrix}, \quad z \in C, \quad (38)$$

where

$$\begin{aligned} T_1(z) &= z^2 \left[(A \otimes I_s + \bar{B} \otimes \tilde{A}) \right. \\ &+ \sum_{q=1}^m (C_q \otimes I_s + \bar{D}_q \otimes \tilde{A}) z^{-qM} \\ &+ \left. \sum_{q=1}^m \sum_{r=0}^{Mq} \gamma_r \bar{\bar{G}}_q \otimes \tilde{A} z^{-r} \right], \end{aligned}$$

$$T_2(z) = z \left[\bar{B} \otimes e + \sum_{q=1}^m \bar{D}_q \otimes e z^{-qM} + \sum_{q=1}^m \sum_{r=0}^{Mq} \gamma_r \bar{\bar{G}}_q \otimes e z^{-r} \right],$$

$$T_3(z) = -z^2 I_d \otimes \tilde{b}^T - z I_d \otimes \tilde{b}^T,$$

$$T_4(z) = z^2 I_d - z(1-\theta)I_d - \theta I_d.$$

(39)

Following from the theorem on difference equations, we get that if all the zeros z of (38) satisfy $|z| < 1$, then

$$\lim_{n \rightarrow \infty} Y_n = 0. \quad (40)$$

Hence, we formulate the following lemmas.

Lemma 7 (see [21]). *If all the zeros z of (38) satisfy $|z| < 1$, then numerical method ((32) and (33)) satisfies*

$$\lim_{n \rightarrow \infty} u_n = 0. \quad (41)$$

Lemma 8. *Assume that condition (a) of Theorem 5 holds and assume that $[I_s - \lambda_l(r(z))\tilde{A}]$ are invertible for $|z| \geq 1$, where*

$$\begin{aligned} r(z) &= \left(A + \sum_{q=1}^m z^{-qM} C_q \right)^{-1} \\ &\times \left(-\bar{B} - \sum_{q=1}^m z^{-qM} \bar{D}_q - \sum_{q=1}^m \sum_{r=0}^{qM} \gamma_r z^{-r} \bar{\bar{G}}_q \right); \end{aligned} \quad (42)$$

then, $\det[T_1(z)] \neq 0$, for $|z| \geq 1$.

Proof. Condition (a) in Theorem 5 implies that the matrix $(A + \sum_{q=1}^m z^{-qM} C_q)$ is invertible for $|z| \geq 1$; then $T_1(z) = z^2[(A + \sum_{q=1}^m z^{-qM} C_q) \otimes I_s][I_d \otimes I_s - r(z) \otimes \tilde{A}]$.

We have that

$$\begin{aligned} \det[T_1(z)] &= z^{2sd} \left[\det \left(A + \sum_{q=1}^m z^{-qM} C_q \right) \right]^s \\ &\times \det[I_d \otimes I_s - r(z) \otimes \tilde{A}] \\ &= z^{2sd} \left[\det \left(A + \sum_{q=1}^m z^{-qM} C_q \right) \right]^s \\ &\times \prod_{l=1}^d \prod_{j=1}^s [1 - \lambda_l(r(z)) \lambda_j(\tilde{A})]. \end{aligned} \quad (43)$$

The matrix $I_s - \lambda_l(r(z))\tilde{A}$ is invertible meaning that $\lambda_l(r(z))\lambda_j(\tilde{A}) \neq 1$ for all l, j .

Hence, $\det[T_1(z)] \neq 0$, for $|z| \geq 1$. $\square$

Theorem 9. *If the system ((32) and (33)) satisfies Lemma 8 and the following conditions,*

(a) $\det(A + \sum_{q=1}^m \xi_q C_q) \neq 0$, for $|\xi_q| \leq 1$;

(b) $|1 - \theta - z^{-1}\theta - (\hat{b}^T + z^{-1}\tilde{b}^T)[I_d \otimes I_s - r(z) \otimes \tilde{A}]^{-1}[e_s \otimes r(z)]| < 1$, for $|z| \geq 1$,

then the solution of the TSRK methods for (1) is asymptotically stable.

Proof. By Lemma 7, we need to prove that all the zeros of (38) satisfy $|z| < 1$.

If these were not true, there would exist a $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$, such that

$$\det \begin{bmatrix} T_1(z_0) & T_2(z_0) \\ T_3(z_0) & T_4(z_0) \end{bmatrix} = 0. \quad (44)$$

By Lemma 8, we have that $\det[T_1(z_0)] \neq 0$.

Hence, (44) is equivalent to

$$\det [T_4(z_0) - T_3(z_0) T_1^{-1}(z_0) T_2(z_0)] = 0. \quad (45)$$

Using the Kronecker product [5, chapter 4], we have that

$$\begin{aligned} & \det [T_4(z_0) - T_3(z_0) T_1^{-1}(z_0) T_2(z_0)] \\ &= \det \left\{ z_0^2 I_d - z_0(1 - \theta) I_d - \theta I_d \right. \\ & \quad \left. - (z_0^2 I_d \otimes \hat{b}^T + z_0 I_d \otimes \tilde{b}^T) \right. \\ & \quad \times z_0^{-2} \left[\left(A + \sum_{q=1}^m z_0^{-qM} C_q \right) \otimes I_s \right]^{-1} \\ & \quad \cdot [I_d \otimes I_s - r(z_0) \otimes \tilde{A}]^{-1} \\ & \quad \cdot z_0 \left[\bar{B} \otimes e + \sum_{q=1}^m \bar{D}_q \otimes e z_0^{-qM} \right. \\ & \quad \left. + \sum_{q=1}^m \sum_{r=0}^{Mq} \gamma_r \bar{G}_q \otimes e z_0^{-r} \right] \left. \right\} \end{aligned} \quad (46)$$

$$\begin{aligned} &= z_0^d \det \left\{ z_0 I_d - [(1 - \theta) I_d - z_0^{-1} \theta I_d \right. \\ & \quad \left. - (I_d \otimes \hat{b}^T + z_0^{-1} I_d \otimes \tilde{b}^T) \right. \\ & \quad \times [I_d \otimes I_s - r(z_0) \otimes \tilde{A}]^{-1} \\ & \quad \times [e \otimes r(z_0)]] \left. \right\}. \end{aligned}$$

Combining (45) and (46) gives that

$$\begin{aligned} & \det \left\{ z_0 I_d - [(1 - \theta) I_d - z_0^{-1} \theta I_d \right. \\ & \quad \left. - (I_d \otimes \hat{b}^T + z_0^{-1} I_d \otimes \tilde{b}^T) \right. \\ & \quad \cdot [I_d \otimes I_s - r(z_0) \otimes \tilde{A}]^{-1} [e \otimes r(z_0)]] \left. \right\} = 0, \end{aligned} \quad (47)$$

which implies $|1 - \theta - z_0^{-1}\theta - (\hat{b}^T + z_0^{-1}\tilde{b}^T)[I_d \otimes I_s - r(z_0) \otimes \tilde{A}]^{-1}[e \otimes r(z_0)]| \geq 1$.

This contradicts the assumption that (b) for $|z_0| \geq 1$. Hence, the theorem is proved. $\square$

4. Numerical Experiments

Example 1. Consider the following linear system:

$$\begin{aligned} & Au'(t) + Bu(t) + \sum_{q=1}^m C_q u'(t - \tau_q) \\ & + \sum_{q=1}^m D_q u(t - \tau_q) \\ & + \sum_{q=1}^m G_q \int_{t-\tau_q}^t u(\delta) d\delta = 0, \quad t \geq 0, \\ & u(t) = \varphi(t), \quad t \in [-\tau, 0), \end{aligned} \quad (48)$$

with $m = 2$, $\tau_1 = 1$, $\tau_2 = 2$, and $\varphi(t) = (\cos(t), \sin(t), \cos(t))^T$ for $t \in [-2, 0]$, where

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0.8 & 0.5 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0.2 & -0.2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0.15 & 0.8 \end{bmatrix}, \\ G_1 &= \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & -0.2 \\ 0 & 0.1 & 0.25 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & -0.2 \\ 0 & 0.05 & 0.25 \end{bmatrix}, \end{aligned} \quad (49)$$

and $C_1 = -0.3A$ and $C_2 = -0.5A$.

Here the matrix coefficients satisfy Theorem 9. Hence, the system is asymptotically stable.

We choose the A-stable TSRK methods as follows [22]:

C	A	$\frac{51}{32}$	$\frac{5151}{9760}$	$\frac{2601}{2440}$
	$\hat{b}^T$	$\frac{103}{256}$	$\frac{10609}{156160}$	$\frac{73439}{156160}$
θ	$\tilde{b}^T$	$\frac{16977449}{36697976}$	$\frac{636886846889}{1074516737280}$	$\frac{61448158637}{134314592160}$
		$\frac{21872982199}{1074516737280}$	$\frac{52658918227}{134314592160}$	$\frac{52658918227}{134314592160}$

where

$$c_i = \sum_{j=1}^s \tilde{a}_{ij}, \quad \sum_{j=1}^s (\hat{b}_j + \tilde{b}_j) = 1 + \theta. \quad (51)$$

It can be easily seen that the A-stable TSRK method is asymptotically stable, which illuminates the conclusion of Theorem 9.

5. Conclusions

This paper develops the asymptotic stability of the two-step Runge-Kutta methods for neutral delay integro differential-algebraic equations with many delays. It studies the asymptotic stability of the analytical solution and introduces two step Runge-Kutta methods and the stability region. It also deals with the asymptotic stability of two-step Runge-Kutta method for neutral delay integro differential-algebraic equations with many delays and proves that the A-stable two-step Runge-Kutta methods are asymptotically stable for neutral delay integro differential-algebraic equations with many delays.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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