Research Article

Nonlinear Stochastic H_∞ Control with Markov Jumps and (x, u, v) -Dependent Noise: Finite and Infinite Horizon Cases

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This paper is concerned with the H_∞ control problem for nonlinear stochastic Markov jump systems with state, control, and external disturbance-dependent noise. By means of inequality techniques and coupled Hamilton-Jacobi inequalities, both finite and infinite horizon H_∞ control designs of such systems are developed. Two numerical examples are provided to illustrate the effectiveness of the proposed design method.

1. Introduction

H_∞ control is one of the most important robust control approaches, which can efficiently eliminate the effect of the exogenous disturbance [1, 2]. Since Hinrichsen and Pritchard introduced H_∞ control to linear stochastic systems [3], the nonlinear stochastic H_∞ control and filtering problems have received considerable attention in both theory and practical applications [4–9]. In [4], the nonlinear stochastic H_∞ designs were first developed by solving a second-order nonlinear Hamilton-Jacobi inequality. The H_∞ filtering problems for general nonlinear continuous-time and discrete-time stochastic systems were discussed in [6] and [7], respectively. In [8], the quantized H_∞ control problem for a class of nonlinear stochastic time-delay network-based systems with probabilistic data missing is studied.

On the other hand, Itô stochastic systems with Markov jumps have attracted increasing attention due to their powerful modeling ability in many fields [10, 11]. For linear stochastic systems with Markov jumps, many important issues have been studied, such as stability and stabilization [12, 13], observability and detectability [14], optimal control [15], and H_2/H_∞ control [16]. The control issues for nonlinear stochastic Markov jump systems (NSMJSSs) have also been widely investigated. In [17], the notion of exponential dissipativity of NSMJSSs was introduced, and it was used to estimate

the possible variations of the output feedback control. In [18], the stabilization of nonlinear Markov jump systems with partly unknown transition probabilities was studied via fuzzy control. In [19], the H_∞ control problems of NSMJSSs were studied, which extended the results of [4] to stochastic systems with Markovian jump parameters.

Most of the existing literature was concerned with stochastic Markov jump systems with state-dependent noise or both state and disturbance dependent noise ((x, v) -dependent noise for short) [16, 19]. However, for most natural phenomena described by Itô stochastic systems, not only state but also control input or external disturbance maybe corrupted by noise. By introducing three coupled Hamilton-Jacobi equations (HJEs), the finite and infinite horizon H_∞ control problems were solved for Itô stochastic systems with all system state, control, and disturbance-dependent noise ((x, u, v) -dependent noise for short) in [20] and [21], respectively. In [22], the finite/infinite horizon H_∞ control of nonlinear stochastic systems with (x, u, v) -dependent noise was solved by means of a Hamilton-Jacobi inequality (HJI) instead of three coupled Hamilton-Jacobi equations (HJEs). However, the control problems of nonlinear stochastic systems with Markov jumps and (x, u, v) -dependent noise have never been tackled and deserved further research.

In this paper, the finite and infinite horizon H_∞ control problems are studied for nonlinear stochastic Markov jumps

systems with (x, u, v) -dependent noise. Firstly, a very useful elementary identity is proposed. Then, by using the technique of squares completion, a sufficient condition for finite/infinite horizon H_∞ control of NSMJSs is presented based on a set of coupled HJIs. By means of linear matrix inequalities (LMIs), a sufficient condition for infinite horizon H_∞ control of linear stochastic Markov jump systems is derived. Finally, two numerical examples are provided to show the effectiveness of our obtained results.

For conveniences, we make use of the following notations throughout this paper. $\mathcal{R}^n$ is the n -dimensional Euclidean space. $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices. $A > 0$ ($A \geq 0$): A is a positive definite (positive semi-definite) symmetric matrix. A' is the transpose of a matrix A . I is the identity matrix. $\|x\|$ is the Euclidean norm of a vector x . $L^2_{\mathcal{F}}([0, T]; \mathcal{R}^l)$ (resp., $L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^l)$) is the space of nonanticipative stochastic processes $y(t) \in \mathcal{R}^l$ with respect to increasing σ -algebras $\mathcal{F}_t$ ($t \geq 0$) satisfying $\|y(t)\|_{L^2_{\mathcal{F}}([0, T]; \mathcal{R}^l)} = (E \int_0^T \|y(t)\|^2 dt)^{1/2} < \infty$ (resp., $\|y(t)\|_{L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^l)} = (E \int_0^\infty \|y(t)\|^2 dt)^{1/2} < \infty$).

2. Definitions and Preliminaries

Consider the following time-varying nonlinear stochastic Markov jump systems with (x, u, v) -dependent noise:

$$\begin{aligned} dx(t) &= [f(t, x, r_t) + g(t, x, r_t)u(t) + h(t, x, r_t)v(t)]dt \\ &\quad + [l(t, x, r_t) + q(t, x, r_t)u(t) \\ &\quad + s(t, x, r_t)v(t)]dw(t), \\ z(t) &= \text{col}(m(t, x, r_t), u(t)) := \begin{bmatrix} m(t, x, r_t) \\ u(t) \end{bmatrix}, \\ x(0) &= x_0 \in \mathcal{R}^n, \end{aligned} \quad (1)$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^{n_u}$, $v(t) \in \mathcal{R}^{n_v}$, and $z(t) \in \mathcal{R}^{n_z}$ represent the system state, control input, exogenous input, and regulated output, respectively. $w(t)$ is the one-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$, with the natural filter $\mathcal{F}_t$ generated by $w(\cdot)$ and $r(\cdot)$ up to time t . The jumping process $\{r_t, t \geq 0\}$ is a continuous-time discrete-state Markov process taking values in a finite set $\mathcal{T} = \{1, \dots, N\}$. The transition probabilities for the process r_t are defined as

$$P(r_{t+h} = j \mid r_t = k) = \begin{cases} \pi_{kl}h + o(h), & \text{if } k \neq j, \\ 1 + \pi_{kk}h + o(h), & \text{if } k = j, \end{cases} \quad (2)$$

where $h > 0$, $\lim_{h \rightarrow 0} (o(h)/h) = 0$, and $\pi_{kl} \geq 0$ ($k, j \in \mathcal{T}, k \neq j$) denotes the switching rate from mode k at time t to mode j at time $t + h$ and $\pi_{kk} = -\sum_{j=1, j \neq k} \pi_{kj}$ for all $k \in \mathcal{T}$. In this paper, the processes r_t and $w(t)$ are assumed to be independent. For every $r_t = k \in \mathcal{T}$, $f(t, x, k)$, $g(t, x, k)$, $h(t, x, k)$, $l(t, x, k)$, $q(t, x, k)$, $s(t, x, k)$, and $m(t, x, k)$ are Borel measurable functions of suitable dimensions, which guarantee that system (1) has a unique strong solution [23].

The finite horizon H_∞ control for system (1) is defined as follows.

Definition 1. For given $\gamma > 0$, the state feedback control $u_T^*(t) \in L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_u})$ is called a finite horizon H_∞ control of system (1), if for the zero initial state and any nonzero $v(t) \in L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_v})$, we have $\|z_T\| < \gamma$ with

$$\begin{aligned} \|z_T\| &= \sup_{\substack{v(t) \in L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_v}), \\ v(t) \neq 0, u(t) = u_T^*(t), x_0 = 0}} \frac{\|z(t)\|_{L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_z})}}{\|v(t)\|_{L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_v})}} \\ &= \sup_{\substack{v(t) \in L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_v}), \\ v(t) \neq 0, u(t) = u_T^*(t), x_0 = 0}} E \left\{ \int_0^T (\|m(t, x, r_t)\|^2 \right. \\ &\quad \left. + \|u_T^*(t)\|^2) dt \mid r_0 = k \right\}^{1/2} \\ &\quad \times \left(E \left\{ \int_0^T \|v(t)\|^2 dt \mid r_0 \right. \right. \\ &\quad \left. \left. = k \right\}^{1/2} \right)^{-1}, \\ &\quad i \in \mathcal{T}, \end{aligned} \quad (3)$$

where $\mathcal{L}_T$ is an operator associated with system (1) which is defined as

$$\begin{aligned} \mathcal{L}_T : L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_v}) &\mapsto L^2_{\mathcal{F}}([0, T]; \mathcal{R}^{n_z}), \\ \mathcal{L}_T(v(t)) &= z(t)|_{x_0=0}, \quad t \in [0, T]. \end{aligned} \quad (4)$$

Consider the time-invariant nonlinear stochastic Markov jump systems with (x, u, v) -dependent noise

$$\begin{aligned} dx(t) &= [f(x, r_t) + g(x, r_t)u(t) + h(x, r_t)v(t)]dt \\ &\quad + [l(x, r_t) + q(x, r_t)u(t) + s(x, r_t)v(t)]dw(t), \\ z(t) &= \text{col}(m(x, r_t), u(t)) := \begin{bmatrix} m(x, r_t) \\ u(t) \end{bmatrix}, \\ x(0) &= x_0 \in \mathcal{R}^n. \end{aligned} \quad (5)$$

The infinite horizon H_∞ control for system (5) is defined as follows.

Definition 2. For given $\gamma > 0$, the control $u_\infty^*(t) \in L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_u})$ is called an infinite horizon H_∞ control of system (1), if the following is considered.

- (i) For the zero initial state and any nonzero $v(t) \in L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_v})$, we have $\|\mathcal{L}_\infty\| < \gamma$ with

$$\begin{aligned} & \|\mathcal{L}_\infty\| \\ &= \sup_{\substack{v(t) \in L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_v}), \\ v(t) \neq 0, u(t) = u_\infty^*(t), x_0 = 0}} \frac{\|z(t)\|_{L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_z})}}{\|v(t)\|_{L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_v})}} \\ &= \sup_{\substack{v(t) \in L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_v}), \\ v(t) \neq 0, u(t) = u_\infty^*(t), x_0 = 0}} E \left\{ \int_0^\infty \left(\|m(x, r_t)\|^2 \right. \right. \\ & \quad \left. \left. + \|u_\infty^*(t)\|^2 \right) dt \mid r_0 = k \right\}^{1/2} \\ & \quad \times \left(E \left\{ \int_0^\infty \|v(t)\|^2 dt \mid r_0 \right. \right. \\ & \quad \left. \left. = k \right\}^{1/2} \right)^{-1}, \quad i \in \mathcal{T}, \end{aligned} \quad (6)$$

where $\mathcal{L}_\infty$ is an operator associated with system (5) which is defined as

$$\begin{aligned} \mathcal{L}_\infty : L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_v}) &\mapsto L^2_{\mathcal{F}}(\mathcal{R}_+; \mathcal{R}^{n_z}), \\ \mathcal{L}_\infty(v(t)) &= z(t) \mid_{x_0=0}, \quad t \in \mathcal{R}_+. \end{aligned} \quad (7)$$

- (ii) System (5) is internally stable; that is, the following system

$$\begin{aligned} dx(t) &= [f(x, r_t) + g(x, r_t)u_\infty^*(x, r_t)]dt \\ &+ [l(x, r_t) + q(x, r_t)u_\infty^*(x, r_t)]dw(t), \end{aligned} \quad (8)$$

is globally asymptotically stable in probability [10].

To give our main results, we need the following lemmas.

Lemma 3 (see [10]). (Generalized Itô formula). Let $\alpha(t, x, k)$ and $\beta(t, x, k)$ be given $\mathcal{R}^n$ -valued, $\mathcal{F}_t$ -adapted process, $k \in \mathcal{T}$, and $dx(t) = \alpha(t, x(t), r_t)dt + \beta(t, x(t), r_t)dw(t)$. Then for given $V(t, x, k) \in \mathcal{C}^{1,2}([0, T]; \mathcal{R}^n)$, $k \in \mathcal{T}$, we have

$$\begin{aligned} & E \{ V(T, x(T), r_T) - V(s, x(s), r_s) \mid r_s = k \} \\ &= E \left\{ \int_s^T \mathcal{L}_T V(t, x(t), r_t) dt \mid r_s = k \right\}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \mathcal{L}_T V(t, x, k) &= \frac{\partial V(t, x, k)}{\partial t} + \frac{\partial V'(t, x, k)}{\partial x} \alpha(t, x, k) \\ &+ \sum_{j=1}^N \pi_{kj} V(t, x, k) \\ &+ \frac{1}{2} \beta'(t, x, k) \frac{\partial^2 V(t, x, k)}{\partial x^2} \beta(t, x, k). \end{aligned} \quad (10)$$

Lemma 4. If $x, b \in \mathcal{R}^n$, $A \in \mathcal{R}^{n \times n}$ is a symmetric matrix and A^{-1} exists, we have

$$x'Ax + x'b + b'x = (x + A^{-1}b)'A(x + A^{-1}b) - b'A^{-1}b. \quad (11)$$

Proof. This lemma is very easily proved by using completing squares technique, so the proof is omitted. $\square$

3. Main Results

3.1. Finite Horizon Case. The following sufficient condition is presented for the finite horizon H_∞ control of system (1). For convenience, denote $(\cdot)_k := (\cdot)(t, x, k)$ in this subsection.

Theorem 5. Assume that there exists a set of nonnegative functions $V(t, x, k) \in \mathcal{C}^{1,2}([0, T] \times \mathcal{R}^n \times \mathcal{T}; \mathcal{R})$, $V(0, 0, k) = 0$, and $\partial^2 V(t, x, k)/\partial x^2 \geq 0$ for all nonzero $x \in \mathcal{R}^n$, $k \in \mathcal{T}$. If $V(t, x, k)$ solves the following coupled HJIs:

$$\begin{aligned} \Delta_k &= \frac{\partial V_k}{\partial t} + \frac{\partial V'_k}{\partial x} f_k + \frac{1}{2} l'_k \frac{\partial^2 V_k}{\partial x^2} l_k + m'_k m_k + \sum_{j=1}^N \pi_{kj} V_j \\ &+ \frac{1}{4} \left(l'_k \frac{\partial^2 V_k}{\partial x^2} s_k + \frac{\partial V'_k}{\partial x} h_k \right) \left(\gamma^2 I - s'_k \frac{\partial^2 V_k}{\partial x^2} s_k \right)^{-1} \\ &\times \left(s'_k \frac{\partial^2 V_k}{\partial x^2} l_k + h'_k \frac{\partial V_k}{\partial x} \right) \\ &- \frac{1}{4} \left(l'_k \frac{\partial^2 V_k}{\partial x^2} q_k + \frac{\partial V'_k}{\partial x} g_k \right) \left(I + q'_k \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \\ &\times \left(q'_k \frac{\partial^2 V_k}{\partial x^2} l_k + g'_k \frac{\partial V_k}{\partial x} \right) < 0, \\ &\gamma^2 I - s'_k \frac{\partial^2 V_k}{\partial x^2} s_k > 0, \quad V(T, x, k) = 0, \end{aligned} \quad (12)$$

then

$$u_T^*(t, x, k) = -\frac{1}{2} \left(I + q'_k \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q'_k \frac{\partial^2 V_k}{\partial x^2} l_k + g'_k \frac{\partial V_k}{\partial x} \right) \quad (13)$$

is a finite horizon H_∞ control of system (1).

Proof. For any $T > 0$ and the initial state $x_0 = 0$, $r_0 = k$, applying Lemma 3, we have

$$\begin{aligned} & E [V(x(T), r_T) - V(0, r_0) \mid r_0 = k] \\ &= E \left\{ \int_0^T \left[\frac{\partial V_{r_t}}{\partial t} + \frac{\partial V'_{r_t}}{\partial x} (f_{r_t} + g_{r_t}u + h_{r_t}v) \right. \right. \\ & \quad + \frac{1}{2} (l_{r_t} + q_{r_t}u + s_{r_t}v)' \frac{\partial^2 V_{r_t}}{\partial x^2} (l_{r_t} + q_{r_t}u + s_{r_t}v) \\ & \quad + \|m_{r_t}\|^2 + \|u\|^2 - \gamma^2 \|v\|^2 - \|z\|^2 \\ & \quad \left. \left. + \gamma^2 \|v\|^2 + \sum_{j=1}^N \pi_{kj} V_j \right] dt \mid r_0 = k \right\} \end{aligned}$$

$$\begin{aligned}
&= E \left\{ \int_0^T \left[\frac{\partial V_{r_t}}{\partial t} - \|z\|^2 + \gamma^2 \|v\|^2 \right. \right. \\
&\quad + \sum_{j=1}^N \pi_{kj} V_j + \Theta_1(t, v, x, r_t) \\
&\quad + \Theta_2(t, x, r_t) + \Theta_3(t, u, x, r_t) \\
&\quad \left. \left. + \frac{1}{2} \left(u' q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} v \right. \right. \right. \\
&\quad \left. \left. \left. + v' s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} u \right) \right] dt \mid r_0 = k \right\}, \quad (14)
\end{aligned}$$

where $(\cdot)_{r_t}$ denotes $(\cdot)(t, x, r_t)$ and

$$\begin{aligned}
\Theta_1(t, v, x, r_t) &= v' \left(-\gamma^2 I + \frac{1}{2} s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right) v \\
&\quad + \frac{1}{2} \left(l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} + \frac{\partial V'_{r_t}}{\partial x} h_{r_t} \right) v \\
&\quad + \frac{1}{2} v' \left(s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + h'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right), \\
\Theta_2(t, x, r_t) &= \frac{\partial V_{r_t}}{\partial t} + \frac{\partial V'_{r_t}}{\partial x} f_{r_t} + \frac{1}{2} l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + m'_{r_t} m_{r_t}, \quad (15) \\
\Theta_3(t, u, x, r_t) &= u' \left(I + \frac{1}{2} q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} \right) u \\
&\quad + \frac{1}{2} \left(l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} + \frac{\partial V'_{r_t}}{\partial x} g_{r_t} \right) u \\
&\quad + \frac{1}{2} u' \left(q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + g'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right).
\end{aligned}$$

Since $\partial^2 V_k / \partial x^2 \geq 0$, $k \in \mathcal{T}$, we have

$$\frac{1}{2} (-u' q'_{r_t} + v' s'_{r_t}) \frac{\partial^2 V_{r_t}}{\partial x^2} (-q_{r_t} u + s_{r_t} v) \geq 0, \quad (16)$$

which means that

$$\begin{aligned}
&\frac{1}{2} \left(u' q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} v + v' s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} u \right) \\
&\leq \frac{1}{2} u' q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} u + \frac{1}{2} v' s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} v. \quad (17)
\end{aligned}$$

Considering the above inequality and (14), we have

$$\begin{aligned}
&E[V(x(T), r_T) - V(0, r_0) \mid r_0 = k] \\
&\leq E \left\{ \int_0^T \left[\Theta_1(t, v, x, r_t) + \Theta_2(t, x, r_t) + \Theta_3(t, u, x, r_t) \right. \right. \\
&\quad + \sum_{j=1}^N \pi_{kj} V(x, j) - \|z\|^2 + \gamma^2 \|v\|^2 \\
&\quad + \frac{1}{2} \left(u' q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} v \right. \\
&\quad \left. \left. + v' s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} u \right) \right] dt \mid r_0 = k \right\} \\
&= E \left[\int_0^T \left(\bar{\Theta}_1(t, v, x, r_t) + \Theta_2(t, x, r_t) + \bar{\Theta}_3(t, u, x, r_t) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^N \pi_{kj} V(x, j) - \|z\|^2 + \gamma^2 \|v\|^2 \right) dt \mid r_0 = k \right], \quad (18)
\end{aligned}$$

where

$$\begin{aligned}
\bar{\Theta}_1(v, x, r_t) &= v' \left(-\gamma^2 I + s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right) v \\
&\quad + \frac{1}{2} \left(l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} + \frac{\partial V'_{r_t}}{\partial x} h_{r_t} \right) v \\
&\quad + \frac{1}{2} v' \left(s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + h'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right), \\
\bar{\Theta}_3(t, u, x, r_t) &= u' \left(I + q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} \right) u \\
&\quad + \frac{1}{2} \left(l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} + \frac{\partial V'_{r_t}}{\partial x} g_{r_t} \right) u \\
&\quad + \frac{1}{2} u' \left(q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + g'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right). \quad (19)
\end{aligned}$$

Applying Lemma 4 to $\bar{\Theta}_1(t, v, x, r_t)$ and $\bar{\Theta}_3(t, u, x, r_t)$, we have

$$\begin{aligned}
\bar{\Theta}_1(t, v, x, r_t) &= (v + \Gamma_1)' \left(-\gamma^2 I + s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right) (v + \Gamma_1) \\
&\quad - \frac{1}{4} \left(l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} + \frac{\partial V'_{r_t}}{\partial x} h_{r_t} \right) \\
&\quad \times \left(-\gamma^2 I + s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right)^{-1} \\
&\quad \times \left(s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + h'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right),
\end{aligned}$$

$$\begin{aligned}
\bar{\Theta}_3(t, u, x, r_t) &= (u + \Gamma_2)' \left(I + q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} \right) (u + \Gamma_2) \\
&\quad - \frac{1}{4} \left(l'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} + \frac{\partial V'_{r_t}}{\partial x} g_{r_t} \right) \\
&\quad \times \left(I + q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} \right)^{-1} \\
&\quad \times \left(q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + g'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\Gamma_1 &= \frac{1}{2} \left(\gamma^2 I + s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right)^{-1} \left(s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + h'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right), \\
\Gamma_2 &= \frac{1}{2} \left(I + q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} \right)^{-1} \left(q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} l_{r_t} + g'_{r_t} \frac{\partial V_{r_t}}{\partial x} \right).
\end{aligned} \tag{21}$$

Substituting (20) into (18), and considering (12), it yields

$$\begin{aligned}
&E[V(x(T), r_T) - V(0, r_0) \mid r_0 = k] \\
&< E \left\{ \int_0^T \left[(v + \Gamma_1)' \left(-\gamma^2 I + s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right) (v + \Gamma_1) \right. \right. \\
&\quad \left. \left. + (u + \Gamma_2)' \left(I + q'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} q_{r_t} \right) (u + \Gamma_2) \right. \right. \\
&\quad \left. \left. + \gamma^2 \|v\|^2 - \|z\|^2 \right] dt \mid r_0 = k \right\}.
\end{aligned} \tag{22}$$

Taking $u = u_T^* = -\Gamma_2$ and considering the second item of (12), (22) leads to

$$\begin{aligned}
&E \left(\int_0^T \|z(t)\|^2 dt \mid r_0 = k \right) \\
&< \gamma^2 E \left(\int_0^T \|v(t)\|^2 dt \mid r_0 = k \right) \\
&\quad - E \left[\int_0^T (v(t) + \Gamma_1)' \right. \\
&\quad \times \left(\gamma^2 I - s'_{r_t} \frac{\partial^2 V_{r_t}}{\partial x^2} s_{r_t} \right) \\
&\quad \times (v(t) + \Gamma_1) dt \mid r_0 = k \Big],
\end{aligned} \tag{23}$$

which means $\|\mathcal{L}_T\| < \gamma$ in Definition 1. The theorem is proved. $\square$

Remark 6. The proof of Theorem 5 is based on an elementary identity (11), which avoids using stochastic dissipative theory

as done in [4, 5]. We believe that the identity (11) will have many other applications in system analysis and synthesis.

3.2. Infinite Horizon Case. In contrast to the finite horizon case, the infinite horizon H_∞ control exhibits more complexity due to the additional requirement of stabilizing the closed-loop system internally. The following sufficient condition is derived for the infinite horizon H_∞ control of system (5). In this subsection, denote $(\cdot)_k := (\cdot)(x, k)$.

Theorem 7. Assume that there exists a set of nonnegative functions $V(x, k) \in \mathcal{C}^2(\mathcal{R}^n \times \mathcal{T}; \mathcal{R})$, $V(0, k) = 0$, and $\partial^2 V(x, k)/\partial x^2 \geq 0$ for all nonzero $x \in \mathcal{R}^n$, $k \in \mathcal{T}$. If $V(x, k)$ solves the following coupled HJIs:

$$\begin{aligned}
\Lambda_k &= \frac{\partial V'_k}{\partial x} f_k + \frac{1}{2} l'_k \frac{\partial^2 V_k}{\partial x^2} l_k + m'_k m_k + \sum_{j=1}^N \pi_{kj} V_j \\
&\quad + \frac{1}{4} \left(l'_k \frac{\partial^2 V_k}{\partial x^2} s_k + \frac{\partial V'_k}{\partial x} h_k \right) \left(\gamma^2 I - s'_k \frac{\partial^2 V_k}{\partial x^2} s_k \right)^{-1} \\
&\quad \times \left(s'_k \frac{\partial^2 V_k}{\partial x^2} l_k + h'_k \frac{\partial V_k}{\partial x} \right) \\
&\quad - \frac{1}{4} \left(l'_k \frac{\partial^2 V_k}{\partial x^2} q_k + \frac{\partial V'_k}{\partial x} g_k \right) \left(I + q'_k \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \\
&\quad \times \left(q'_k \frac{\partial^2 V_k}{\partial x^2} l_k + g'_k \frac{\partial V_k}{\partial x} \right) < 0, \\
&\quad \gamma^2 I - s'_k \frac{\partial^2 V_k}{\partial x^2} s_k > 0,
\end{aligned} \tag{24}$$

then

$$u_\infty^*(x, k) = -\frac{1}{2} \left(I + q'_k \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q'_k \frac{\partial^2 V_k}{\partial x^2} l_k + g'_k \frac{\partial V_k}{\partial x} \right) \tag{25}$$

is an infinite horizon H_∞ control of system (5).

Proof. Similar to the proof of Theorem 5, it is easy to show $\|\mathcal{L}_\infty\| < \gamma$ under condition (24). Next, we need to prove system (8) to be globally asymptotically stable in probability. Let $u_k^* = u_\infty^*(x, k)$ and let $\mathcal{L}_\infty$ be the infinitesimal generator of the system (8) which is similar to $\mathcal{L}_T$ in Lemma 3; then

$$\begin{aligned}
\mathcal{L}_\infty V_k &= \frac{\partial V'_k}{\partial x} (f_k + g_k u_k^*) + \sum_{j=1}^N \pi_{kj} V_j \\
&\quad + \frac{1}{2} (l_k + q_k u_k^*)' \frac{\partial^2 V_k}{\partial x^2} (l_k + q_k u_k^*) \\
&= \frac{\partial V'_k}{\partial x} f_k + \frac{\partial V'_k}{\partial x} g_k u_k^* + \sum_{j=1}^N \pi_{kj} V_j \\
&\quad + \frac{1}{2} l'_k \frac{\partial^2 V_k}{\partial x^2} l_k + \frac{1}{2} l'_k \frac{\partial^2 V_k}{\partial x^2} q_k u_k^*
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(q_k u_k^*)' \frac{\partial^2 V_k}{\partial x^2} l_k + \frac{1}{2}(q_k u_k^*)' \frac{\partial^2 V_k}{\partial x^2} (q_k u_k^*) \\
& = \frac{\partial V_k'}{\partial x} f_k + \frac{1}{2} l_k' \frac{\partial^2 V_k}{\partial x^2} l_k + \sum_{j=1}^N \pi_{kj} V_j + Y_{1k} + Y_{2k},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
Y_{1k} & = \frac{\partial V_k'}{\partial x} g_k u_k^* + \frac{1}{2} l_k' \frac{\partial^2 V_k}{\partial x^2} q_k u_k^* + \frac{1}{2} (q_k u_k^*)' \frac{\partial^2 V_k}{\partial x^2} l_k \\
& = -\frac{1}{2} \frac{\partial V_k'}{\partial x} g_k \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \\
& \quad \times \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right) - \frac{1}{4} l_k' \frac{\partial^2 V_k}{\partial x^2} q_k \\
& \quad \times \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right) \\
& \quad - \frac{1}{4} \left(\frac{\partial V_k'}{\partial x} g_k + l_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right) \\
& \quad \times \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} q_k' \frac{\partial^2 V_k}{\partial x^2} l_k, \\
Y_{2k} & = \frac{1}{2} (q_k u_k^*)' \frac{\partial^2 V_k}{\partial x^2} (q_k u_k^*) \\
& = \frac{1}{8} \left(\frac{\partial V_k'}{\partial x} g_k + l_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right) \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \\
& \quad \times q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \\
& \quad \times \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right).
\end{aligned} \tag{27}$$

It can be checked that

$$\begin{aligned}
Y_{1k} & = -\frac{1}{2} \left(l_k' \frac{\partial^2 V_k}{\partial x^2} q_k + \frac{\partial V_k'}{\partial x} g_k \right) \\
& \quad \times \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right),
\end{aligned} \tag{28}$$

$$\begin{aligned}
Y_{2k} & \leq \frac{1}{8} \left(l_k' \frac{\partial^2 V_k}{\partial x^2} q_k + \frac{\partial V_k'}{\partial x} g_k \right) \\
& \quad \times \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right).
\end{aligned} \tag{29}$$

The following inequality is used during the calculation of (29):

$$\begin{aligned}
(I + X)^{-1} X (I + X)^{-1} & \leq (I + X)^{-1} (I + X) (I + X)^{-1} \\
& = (I + X)^{-1}, \quad X > 0.
\end{aligned} \tag{30}$$

Implementing (28) and (29) into (26) and considering (24), it yields

$$\begin{aligned}
\mathcal{L}_\infty V_k & \leq \frac{\partial V_k'}{\partial x} f_k + \frac{1}{2} l_k' \frac{\partial^2 V_k}{\partial x^2} l_k + \sum_{j=1}^N \pi_{kj} V_j \\
& \quad - \frac{3}{8} \left(l_k' \frac{\partial^2 V_k}{\partial x^2} q_k + \frac{\partial V_k'}{\partial x} g_k \right) \\
& \quad \times \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right) \\
& \leq \frac{\partial V_k'}{\partial x} f_k + \frac{1}{2} l_k' \frac{\partial^2 V_k}{\partial x^2} l_k + \sum_{j=1}^N \pi_{kj} V_j \\
& \quad - \frac{1}{4} \left(l_k' \frac{\partial^2 V_k}{\partial x^2} q_k + \frac{\partial V_k'}{\partial x} g_k \right) \\
& \quad \times \left(I + q_k' \frac{\partial^2 V_k}{\partial x^2} q_k \right)^{-1} \left(q_k' \frac{\partial^2 V_k}{\partial x^2} l_k + g_k' \frac{\partial V_k}{\partial x} \right) \\
& < -m_k' m_k - \frac{1}{4} \left(l_k' \frac{\partial^2 V_k}{\partial x^2} s_k + \frac{\partial V_k'}{\partial x} h_k \right) \\
& \quad \times \left(\gamma^2 I - s_k' \frac{\partial^2 V_k}{\partial x^2} s_k \right)^{-1} \left(s_k' \frac{\partial^2 V_k}{\partial x^2} l_k + h_k' \frac{\partial V_k}{\partial x} \right) \\
& \leq 0
\end{aligned} \tag{31}$$

which implies that (8) is globally asymptotically stable in probability from [10]. This theorem is completed. $\square$

Remark 8. The methods proposed in [19, 24] cannot be applied to study the H_∞ problem of NSMJSS with (x, u, v) -dependent noise, although they are suitable for NSMJSS with (x, v) -dependent noise. One reason for this is that u and v are no longer separable in the conditions, when they enter the diffusion term simultaneously. In the proofs of Theorems 5 and 7, we resort to Lemma 4 to solve this problem.

Remark 9. When there is no Markov jump parameters, system (1)/(5) will reduce to general nonlinear stochastic systems with (x, u, v) -dependent noise, which was studied in [20, 21]. However, all the conditions of H_∞ control design for nonlinear stochastic systems in [20, 21] were given in terms of three coupled HJEs, which were difficult to be solved. According to this paper, the sufficient condition for H_∞ control of nonlinear stochastic systems can be derived by means of a single set of coupled HJIs, which is easier to be verified than three coupled HJEs in [20, 21].

From Theorem 7, the following nonlinear stochastic bounded real lemma will be derived for Markov jump system:

$$\begin{aligned} dx(t) &= [f(x, r_t) + h(x, r_t) v] dt \\ &\quad + [l(x, r_t) + s(x, r_t) v] dw, \end{aligned} \quad (32)$$

$$z(t) = m(x, r_t), \quad x(0) = x_0 \in \mathcal{R}^n.$$

Lemma 10. For a prescribed $\gamma > 0$, system (32) is internally stable and $\|\mathcal{L}_\infty\|_{u(t)=0} < \gamma$, if there exists a set of nonnegative functions $V(x, k) \in \mathcal{C}^2(\mathcal{R}^n \times \mathcal{T}; \mathcal{R})$, $V(0, k) = 0$, and $\partial^2 V(x, k)/\partial x^2 \geq 0$ for all nonzero $x \in \mathcal{R}^n$, $k \in \mathcal{T}$, satisfying the following coupled HJIs:

$$\begin{aligned} &\frac{\partial V'_k}{\partial x} f_k + \frac{1}{2} l'_k \frac{\partial^2 V_k}{\partial x^2} l_k + m'_k m_k + \sum_{j=1}^N \pi_{kj} V_j \\ &\quad + \frac{1}{4} \left(l'_k \frac{\partial^2 V_k}{\partial x^2} s_k + \frac{\partial V'_k}{\partial x} h_k \right) \\ &\quad \times \left(\gamma^2 I - s'_k \frac{\partial^2 V_k}{\partial x^2} s_k \right)^{-1} \left(s'_k \frac{\partial^2 V_k}{\partial x^2} l_k + h'_k \frac{\partial V_k}{\partial x} \right) \\ &< 0, \end{aligned} \quad (33)$$

$$\gamma^2 I - s'_k \frac{\partial^2 V_k}{\partial x^2} s_k > 0, \quad k \in \mathcal{T}.$$

Proof. Letting $g_k \equiv 0$, $q_k \equiv 0$, and $u \equiv 0$ in Theorem 7, we obtain (33) easily. $\square$

Next, we present a sufficient condition for the following linear stochastic Markov jump systems with (x, u, v) -dependent noise:

$$\begin{aligned} dx(t) &= [A(r_t)x + B(r_t)u + C(r_t)v] dt \\ &\quad + [A_1(r_t)x + B_1(r_t)u + C_1(r_t)v] dw, \end{aligned} \quad (34)$$

$$z(t) = \text{col}(D(r_t)x, u), \quad x(0) = x_0 \in \mathcal{R}^n.$$

Corollary 11. System (34) is internally stable and $\|\mathcal{L}_\infty\| < \gamma$ for given $\gamma > 0$, if there exist matrices $X_k > 0$, Y_k , $k \in \mathcal{T}$ satisfying the following LMIs:

$$\begin{bmatrix} \Sigma_k & C_k & X_k D'_k & X_k A'_{1k} + Y'_k B'_{1k} & 0 & \psi'_k(X) \\ * & -\gamma^2 I & 0 & C'_{1k} & C'_{1k} & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -X_k & 0 & 0 \\ * & * & * & * & -X_k & 0 \\ * & * & * & * & * & -\phi_k(X) \end{bmatrix} < 0, \quad (35)$$

where

$$\begin{aligned} \Sigma_k &= A_k X_k + B_k Y_k + X_k A'_k + Y'_k B'_k + \pi_{kk} X_k, \\ \psi_k(X) &= [\sqrt{\pi_{k1}} X_k, \dots, \sqrt{\pi_{kk-1}} X_k, \\ &\quad \sqrt{\pi_{kk+1}} X_k, \dots, \sqrt{\pi_{kN}} X_k]', \\ \phi_k(X) &= \text{diag}\{X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_N\}. \end{aligned} \quad (36)$$

Moreover, the state feedback gain matrices are given by $K_k = Y_k X_k^{-1}$.

Proof. Firstly, consider the following unforced linear stochastic Markov jump systems:

$$\begin{aligned} dx(t) &= [A(r_t)x + C(r_t)v] dt + [A_1(r_t)x + C_1(r_t)v] dw, \\ z(t) &= D(r_t)x, \quad x(0) = x_0 \in \mathcal{R}^n. \end{aligned} \quad (37)$$

Let $V_k = x' P_k x$, $P_k > 0$, $k \in \mathcal{T}$; (33) in Lemma 10 can be written as follows:

$$\begin{aligned} &2x' P_k A_k x + x' A'_{1k} P_k A_{1k} x + x' D'_k D_k x + x' \sum_{j=1}^N \pi_{kj} P_j x \\ &\quad + (x' A'_{1k} P_k C_{1k} + x' P_k C_k) \\ &\quad \times (\gamma^2 I - 2C'_{1k} P_k C_{1k})^{-1} (C'_{1k} P_k A_{1k} x + C'_k P_k x) \\ &= x' \left[P_k A_k + A'_k P_k + A'_{1k} P_k A_{1k} + D'_k D_k + \sum_{j=1}^N \pi_{kj} P_j \right. \\ &\quad \left. + (A'_{1k} P_k C_{1k} + P_k C_k) (\gamma^2 I - 2C'_{1k} P_k C_{1k})^{-1} \right. \\ &\quad \left. \times (C'_{1k} P_k A_{1k} + C'_k P_k) \right] x < 0, \end{aligned} \quad (38)$$

$$\gamma^2 I - 2C'_{1k} P_k C_{1k} > 0. \quad (39)$$

By Schur complement, (38) is equivalent to

$$\begin{bmatrix} \left\{ P_k A_k + A'_k P_k + A'_{1k} P_k A_{1k} \quad A'_{1k} P_k C_{1k} + P_k C_k \quad D'_k \right. \\ \left. + \sum_{j=1}^N \pi_{kj} P_j \right\} \\ * \\ * \\ -\gamma^2 I + 2C'_{1k} P_k C_{1k} \quad 0 \\ * \\ -I \end{bmatrix} < 0, \quad (40)$$

which also implies (39). Moreover, (40) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} P_k A_k + A'_k P_k + \sum_{j=1}^N \pi_{kj} P_j & P_k C_k & D'_k \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \\ &\quad + \begin{bmatrix} A'_{1k} \\ C'_{1k} \\ 0 \end{bmatrix} P_k \begin{bmatrix} A_{1k} & C_{1k} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ C'_{1k} \\ 0 \end{bmatrix} P_k \begin{bmatrix} 0 & C_{1k} & 0 \end{bmatrix} < 0, \end{aligned} \quad (41)$$

which yields

$$\begin{bmatrix} P_k A_k + A'_k P_k + \sum_{j=1}^N \pi_{kj} P_j & P_k C_k & D'_k & A'_{1k} & 0 \\ * & -\gamma^2 I & 0 & C'_{1k} & C'_{1k} \\ * & * & -I & 0 & 0 \\ * & * & * & -P_k^{-1} & 0 \\ * & * & * & * & -P_k^{-1} \end{bmatrix} < 0, \quad (42)$$

according to Schur complement. Pre- and postmultiplying (42) by $\text{diag}\{P_k^{-1} \ I \ I \ I \ I\}$ and denoting $X_k = P_k^{-1}$, we have

$$\begin{bmatrix} A_k X_k + X_k A'_k + \pi_{kk} X_k & C_k & X_k D'_k & X_k A'_{1k} & 0 & \psi'_k(X) \\ * & -\gamma^2 I & 0 & C'_{1k} & C'_{1k} & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -X_k & 0 & 0 \\ * & * & * & * & -X_k & 0 \\ * & * & * & * & * & -\phi_k(X) \end{bmatrix} < 0, \quad (43)$$

where $\psi_k(X)$ and $\phi_k(X)$ are defined as in (35).

Considering closed-loop system (34) with state feedback control $u(t) = K_k x(t)$, $k \in \mathcal{T}$. Replacing A_k, A_{1k} by $A_k + B_k K_k, A_{1k} + B_{1k} K_k$, respectively, and setting $Y_k = K_k X_k$ in (43) yield (35). Therefore, the state feedback gain matrices can be obtained by $K_k = Y_k X_k^{-1}$. $\square$

Remark 12. Although HJIs (12) in Theorem 5 or (24) in Theorem 7 can be solved by trial and error in some simple cases (see Example 13 in Section 4), they are difficult to be dealt with for high-dimensional systems. In order to avoid solving the HJIs, the Taylor series approach [25] or fuzzy approach based on Takagi-Sugeno model [24, 26] can be considered to design the nonlinear stochastic H_∞ controller.

4. Numerical Example

In this section, two numerical examples are provided to illustrate the effectiveness of the developed results.

Example 13. Consider one-dimensional two-mode time-invariant nonlinear stochastic Markov jump systems with generator matrix $\Pi = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, and the two subsystems are as follows:

$$\begin{aligned} \text{I: } & \begin{cases} dx = (x^3 + 4xu + xv) dt + (x^2 + u + v) dw, \\ z = \text{col}(x^2, u), \quad x(0) = x_0 \in \mathcal{R}^n, \end{cases} \\ \text{II: } & \begin{cases} dx = \left(\frac{1}{2}x^3 + 3xu + \frac{1}{2}xv\right) dt + (x^2 + u + v) dw, \\ z = \text{col}\left(\frac{1}{2}x^2, u\right), \quad x(0) = x_0 \in \mathcal{R}^n. \end{cases} \end{aligned} \quad (44)$$

Set $V(x) = p_i x^2$, $i = 1, 2$, with $p_1 > 0$ and $p_2 > 0$ to be determined; then HJIs (24) become

$$\begin{aligned} & 2p_1 x \cdot x^3 + \frac{1}{2}x^2 \cdot 2p_1 \cdot x^2 + x^2 \cdot x^2 \\ & + \frac{1}{4}(x^2 \cdot 2p_1 + 2p_1 x \cdot x)^2 (\gamma^2 - 2p_1)^{-1} \\ & - \frac{1}{4}(x^2 \cdot 2p_1 + 2p_1 x \cdot 4x)^2 (1 + 2p_1)^{-1} \\ & - p_1 x^2 + p_2 x^2 < 0, \end{aligned}$$

$$\gamma^2 - 2p_1 > 0,$$

$$\begin{aligned} & 2p_2 x \cdot \frac{1}{2}x^3 + \frac{1}{2}x^2 \cdot 2p_2 \cdot x^2 + \frac{1}{2}x^2 \cdot \frac{1}{2}x^2 \\ & + \frac{1}{4}(x^2 \cdot 2p_2 + 2p_2 x \cdot \frac{1}{2}x)^2 (\gamma^2 - 2p_2)^{-1} \\ & - \frac{1}{4}(x^2 \cdot 2p_2 + 2p_2 x \cdot 3x)^2 (1 + 2p_2)^{-1} \\ & + p_1 x^2 - p_2 x^2 < 0, \end{aligned}$$

$$\gamma^2 - 2p_2 > 0.$$

(45)

For given $\gamma = \sqrt{3}$, the above inequalities have solutions $p_1 = 1$ and $p_2 = 1$. According to Theorem 7, the infinite H_∞ controllers of system (44) are $u_\infty^*(x, 1) = -(5/3)x^2$ and $u_\infty^*(x, 2) = -(4/3)x^2$.

Example 14. Consider two-dimensional two-mode linear stochastic Markov jump systems (34) with the following parameters:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}, & B_1 &= C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A_{11} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, & B_{11} &= C_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}, & B_2 &= C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} 2 & 0 \\ 1 & 11 \end{bmatrix}, & B_{12} &= C_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ D_1 &= [1 \ 0], & D_2 &= [1 \ 1], \\ \Pi &= \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}. \end{aligned} \quad (46)$$

With the choice of $\gamma = \sqrt{2}$, a possible solution of LMIs (35) in Corollary 11 can be found by using Matlab LMI control toolbox:

$$\begin{aligned} X_1 &= \begin{bmatrix} 0.8879 & -0.4074 \\ -0.4074 & 3.6974 \end{bmatrix}, & X_2 &= \begin{bmatrix} 0.5026 & 0.0464 \\ 0.0464 & 5.4520 \end{bmatrix}, \\ Y_1 &= [-3.1690 \ -5.3082], & Y_2 &= [-4.5025 \ -8.5671]. \end{aligned} \quad (47)$$

FIGURE 1: Result of the changing between modes.

FIGURE 2: The state responses of unforced system (37).

Then, the H_∞ control gain matrices of system (34) are as follows:

$$\begin{aligned} K_1 &= Y_1 X_1^{-1} = [-4.4531 \quad -1.9263], \\ K_2 &= Y_2 X_2^{-1} = [-8.8209 \quad -1.4963]. \end{aligned} \quad (48)$$

Figure 1 shows the result of the changing between modes during the simulation with the initial mode at mode 1. The initial condition is chosen as $x_0 = [-0.2 \quad 0.3]'$ and exogenous input $v(t) = e^{-t}$. By means of Euler-Maruyama method [27], the state responses of unforced system (37) and controlled system (34) are shown in Figures 2 and 3, respectively. From Figure 3, one can find that the controlled system (34) can achieve stability and attenuation performance in the sense of mean square by using the proposed H_∞ controller.

FIGURE 3: The state responses of controlled system (34).

5. Conclusions

In this paper, we have studied the H_∞ control problem for NSMJSS with (x, u, v) -dependent noise. A sufficient condition for finite/infinite horizon H_∞ control has been derived in terms of a set of coupled HJIs. It can be found that Lemma 4 plays an essential role during the proof of Theorems 5 and 7. The validity of the obtained results has been verified by two examples.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] G. Zames, "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 301–320, 1981.
- [2] I. R. Petersen, V. A. Ugrinovskii, and A. V. Savkin, *Robust Control Design Using H_∞ -Methods*, Springer, Berlin, Germany, 2000.

- [3] D. Hinrichsen and A. J. Pritchard, "Stochastic H_∞ ," *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pp. 1504–1538, 1998.
- [4] W. Zhang and B.-S. Chen, "State feedback H_∞ control for a class of nonlinear stochastic systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 1973–1991, 2006.
- [5] N. Berman and U. Shaked, " H_∞ -like control for nonlinear stochastic systems," *Systems & Control Letters*, vol. 55, no. 3, pp. 247–257, 2006.
- [6] W. Zhang, B.-S. Chen, and C.-S. Tseng, "Robust H_∞ filtering for nonlinear stochastic systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 2, pp. 589–598, 2005.
- [7] B. Shen, Z. Wang, H. Shu, and G. Wei, "On nonlinear H_∞ filtering for discrete-time stochastic systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2170–2180, 2008.
- [8] Z. Wang, B. Shen, H. Shu, and G. Wei, "Quantized H_∞ control for nonlinear stochastic time-delay systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1431–1444, 2012.
- [9] B. Shen, S. X. Ding, and Z. Wang, "Finite-horizon H_∞ fault estimation for linear discrete time-varying systems with delayed measurements," *Automatica*, vol. 49, no. 1, pp. 293–296, 2013.
- [10] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, UK, 2006.
- [11] V. Dragan, T. Morozan, and A.-M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, Springer, New York, NY, USA, 2010.
- [12] Z.-Y. Li, B. Zhou, Y. Wang, and G.-R. Duan, "On eigenvalue sets and convergence rate of Itô stochastic systems with Markovian switching," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1118–1124, 2011.
- [13] L. Sheng, M. Gao, and W. Zhang, "Spectral characterisation for stability and stabilisation of linear stochastic systems with Markovian switching and its applications," *IET Control Theory & Applications*, vol. 7, no. 5, pp. 730–737, 2013.
- [14] L. Shen, J. Sun, and Q. Wu, "Observability and detectability of discrete-time stochastic systems with Markovian jump," *Systems & Control Letters*, vol. 62, no. 1, pp. 37–42, 2013.
- [15] O. L. V. Costa and A. de Oliveira, "Optimal mean-variance control for discrete-time linear systems with Markovian jumps and multiplicative noises," *Automatica*, vol. 48, no. 2, pp. 304–315, 2012.
- [16] L. Sheng, W. Zhang, and M. Gao, "Relationship between Nash equilibrium strategies and H_2/H_∞ control of stochastic Markov jump systems with multiplicative noise," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2955–2962, 2014.
- [17] P. V. Pakshin, "Exponential dissipativeness of the random-structure diffusion processes and problems of robust stabilization," *Automation and Remote Control*, vol. 68, no. 10, pp. 1852–1870, 2007.
- [18] L. Sheng and M. Gao, "Stabilization for Markovian jump nonlinear systems with partly unknown transition probabilities via fuzzy control," *Fuzzy Sets and Systems*, vol. 161, no. 21, pp. 2780–2792, 2010.
- [19] Z. Lin, Y. Lin, and W. Zhang, "A unified design for state and output feedback H_∞ control of nonlinear stochastic Markovian jump systems with state and disturbance-dependent noise," *Automatica*, vol. 45, no. 12, pp. 2955–2962, 2009.
- [20] W. Zhang, H. Zhang, and B.-S. Chen, "Stochastic H_2/H_∞ control with (x, u, v) -dependent noise: finite horizon case," *Automatica*, vol. 42, no. 11, pp. 1891–1898, 2006.
- [21] W. Zhang and G. Feng, "Nonlinear stochastic H_2/H_∞ control with (x, u, v) -dependent noise: infinite horizon case," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1323–1328, 2008.
- [22] W. Zhang, B.-S. Chen, H. Tang, L. Sheng, and M. Gao, "Some remarks on general nonlinear stochastic H_∞ control with state, control, and disturbance-dependent noise," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 237–242, 2014.
- [23] V. Dragan, T. Morozan, and A.-M. Stoica, *Mathematical Methods in Robust Control of Linear Stochastic Systems*, Springer, New York, NY, USA, 2006.
- [24] Z. Lin, Y. Lin, and W. Zhang, " H_∞ stabilisation of non-linear stochastic active fault-tolerant control systems: fuzzy-interpolation approach," *IET Control Theory & Applications*, vol. 4, no. 10, pp. 2003–2017, 2010.
- [25] M. Abu-Khalaf, J. Huang, and F. L. Lewis, *Nonlinear H_2/H_∞ Constrained Feedback Control*, Springer, London, UK, 2006.
- [26] L. Sheng, M. Gao, and W. Zhang, "Dissipative control for Markov jump non-linear stochastic systems based on T-S fuzzy model," *International Journal of Systems Science*, vol. 45, no. 5, pp. 1213–1224, 2014.
- [27] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," *SIAM Review*, vol. 43, no. 3, pp. 525–546, 2001.