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Research Article

A Regularization Process for Electrical Impedance Equation Employing Pseudoanalytic Function Theory

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The electrical impedance equation is considered an ill-posed problem where the solution to the forward problem is more easy to achieve than the inverse problem. This work tries to improve convergence in the forward problem method, where the Pseudoanalytic Function Theory by means of the Taylor series in formal powers is used, incorporating a regularization method to make a solution more stable and to obtain better convergence. In addition, we include a comparison between the designed algorithms that perform proposed method with and without a regularization process and the autoadjustment parameter for this regularization process.

1. Introduction

The electrical impedance tomography (EIT) problem, which is set by employing the electrical impedance equation, was mathematically posed by Calderon [1] in 1980 proving that the solution of this problem exists as being unique and steady. This problem is exemplified by the following equation:

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0, \tag{1}$$

where σ represents the conductivity and u denotes the electric potential for a domain Ω within a boundary Γ.

This equation is considered ill-posed problem, due to its high complexity to find a solution in terms of the initial data; it means that every variation in the input data can approximate in different way the solution to this problem. Therefore, if a measurement presents any variation or noise, this variation presents difficulty to obtain good approximation; for example, when the finite element method is employed in noise presence in the electrodes, we need to introduce an extra process to correct this fault or the approximation could not be reached.

Some methods should be employed to correct these perturbations present advantages and disadvantages, such as a regularization method that permits to suppress the noise in the measurements. Due to this problem and the continuous advances in the field, the EIT is not considered a practical medical imaging procedure.

Employing (1), we can approach two types of problems; in the first one, the conductivity σ value is known and the task is to approximate the electric potential in the boundary $u|_\Gamma$. This problem is known as forward Dirichlet boundary value problem; it is also known as nonhomogeneous Laplace equation. For the second problem, the value of the electric potential in the boundary $u|_\Gamma$ is known, but the conductivity σ within the domain Ω is unknown; this problem is formulated as inverse Dirichlet boundary value problem or electrical impedance tomography problem [2].

The EIT was considered very unsteady, as well as ill-posed problem [2], but, in 2009, Kravchenko [3] and, independently in 2006, Astala and Päivärinta [4] noticed for first time that the two-dimensional case was completely equivalent to special case of Vekua equation [5].

There exist several assortment methods that try to solve forward and inverse problem for (1), where the best mathematical tool to approximate the solution for both problems by now is the finite element method (FEM). It proves to be stable and easy to employ, but it presents a difficulty when it computes the approximation; this difficulty is presented in the initial data and the variations on it [2].

The existence of different techniques and methods to reach a solution of inverse problems, in which the solutions are interpreted in terms of linear operators [6], permits analysing and studying inverse problems in order to find a solution. Other works employ the connection between pseudoholomorphic functions and conjugate Beltrami equations to deduce the well-posedness on smooth domains of the Dirichlet problem for 2D isotropic conductivity equations [7]. There exists a theory of conjugate functions to solve Dirichlet and Neumann problems for conductivity equations, in which they consider some density properties to trace solutions with a boundary approximation issues [8].

All these works contribute to analysing and reaching a solution to the forward problem and continuing working on the solution to the inverse problem of the electrical conductivity equation. Employing all these techniques and analysis the study of the forward problem could help to find a solution to the inverse problem, but the study to be done should be extensive and incorporate different techniques and methods proposed by the different authors.

For the correct understanding of the inverse problem, first, we need to analyse and study the forward problem, to determinate how the energy propagates within the domain; with the collected data we can determine that the convergence and the performance will be very important to design a method that can be employed in the medical imaging in a future.

In this study, the main purpose is to analyse the behaviour of the algorithm based on the Pseudoanalytic Function Theory, which applies the Taylor series in formal powers as the support, proposing additionally to employ a regularization process to improve the stability. The results obtained are compared with the results presented in previous paper [9, 10], in order to analyse the improvement or decrement in the convergence due to the method designed.

The remainder of this paper is organized as follows. Section 2 shows the mathematical tools for the electrical impedance equation; in Section 3, the methodology that contains the main idea of the algorithm is presented; following, a brief review of regularization procedure employed to design this algorithm is explained; Sections 4 and 5 show several experiments to compare the results and a discussion about this comparison; finally Section 6 expresses the conclusions about the behaviour of the novel method and discusses a future work.

2. Electrical Impedance Equation

2.1. Preliminaries. Employing the Pseudoanalytic Function Theory [11], let us consider a pair of complex valued functions (F, G) that fulfills the condition

$$\operatorname{Im}\left(\overline{F}G\right) > 0. \tag{2}$$

For (2), we have $\overline{F} = \operatorname{Re}(F) - i \operatorname{Im} F$ complex conjugation of F, and $i^2 = -1$ is the imaginary standard unit.

Then, any complex function W can be represented by the linear conjugation of F and G:

$$W = \phi F + \psi G, \tag{3}$$

where ϕ and ψ are indeed real valued functions. Therefore, the pair (F, G) is called *generating pair*. Following the Pseudoanalytic Function Theory posed by Bers [11], it is possible to introduce the derivative and antiderivative form of a complex valued function w, which can be reviewed in the mentioned work.

Let us suppose (F, G) *generating pair* with the form

$$F = p, \qquad G = \frac{i}{p}, \tag{4}$$

where p is a nonvanishing function within a domain $\Omega(\mathbb{R}^2)$. Then considering this (F, G) generating pair with a p separable variable function within the domain,

$$p(x, y) = p_1(x) \cdot p_2(y); \quad x, y \in \mathbb{R}. \tag{5}$$

Thereby (F, G) is embedded into a periodic sequence, in which for an m even the generating pair are

$$F_m = \frac{p_2(y)}{p_1(x)}, \qquad G_m = i\frac{p_1(x)}{p_2(y)}; \tag{6}$$

and for the odd generating pair the forms are

$$F_m = p_1(x) \cdot p_2(y), \qquad G_m = i\left(p_1(x) \cdot p_2(y)\right)^{-1}. \tag{7}$$

Consider the formal powers $Z_m^0(a_0, z_0; Z)$, associated with a (F_m, G_m) generating pair, with the formal degree 0, complex constant coefficient a_0, and center z_0, and depending upon $Z = x + iy$, are defined in agreement with the expression

$$Z_m^{(0)}(a_0, z_0; z) = \lambda F_m(z) + \mu G_m(z), \tag{8}$$

where λ and mu are complex constants which fulfills the condition

$$\lambda F_m(z_0) + \mu G_m(z_0) = a_0. \tag{9}$$

Now, let us suppose W to be a (F_m, G_m)-pseudoanalytic function. Thus, we can express it in terms of the so-called Taylor series in formal powers:

$$W = \sum_{n=0}^{\infty} Z_m^{(n)}(a_0, z_0; z). \tag{10}$$

Since every (F, G)-pseudoanalytic function W accepts this expansion, the last equation is an analytic representation of the general solution for the Vekua equation.

Consider the two-dimensional case of the electric impedance equation (1), and suppose that the conductivity σ function can be expressed in terms of a separable variable function; it follows that

$$\sigma(x, y) = \sigma_1(x) \cdot \sigma_2(y). \tag{11}$$

Introducing the notations

$$W = \sqrt{\sigma} \cdot \left(\partial_x u - \partial_y u\right),$$
$$p = \sqrt{\sigma_1 (x)^{-1} \sigma_2 (y)}, \tag{12}$$

where $\partial_x = \partial/\partial_x$ and $\partial_y = \partial/\partial_y$, then (1) turns into special case of Vekua equation with the form

$$\partial_{\overline{z}} W - \frac{\partial_{\overline{z}} p}{p} \overline{W} = 0, \tag{13}$$

where $\partial_{\overline{z}} = \partial_x - i\partial_y$ and $i^2 = -1$ for the standard imaginary unit.

Employing the last statement presented in (13), we have to introduce some expressions that were presented in the Pseudoanalytic Function Theory [3] and in assortment works [9, 10, 12], explaining the mathematical basis of the Taylor series in formal powers:

$$W = \sum_{n=0}^{\infty} Z_m^{(n)} \left(a_n, z_0; z\right);$$
$$a_n = \frac{1}{n!} \partial_{(F_m,G_m)}^{(n)} W \left(z_0\right). \tag{14}$$

Equation (14) was proved in [11], showing that any (F_m, G_m)-pseudoanalytic function can be represented in Taylor series in formal powers. So, from this point of view, (14) is an analytical representation of the general solution of the Vekua equation (13).

In (14), we used the Pseudoanalytic Function Theory to approximate the special case of Vekua equation in Taylor series in formal powers, where we can read in the last statement that it has a center in z_n, depending upon z and approximating the coefficient a_n, like a Taylor series.

For this case, we also remember that Taylor series in formal powers are performed by introducing a generating pair (F_m, G_m) and the real constants λ_m and μ_m that fulfill the condition presented before in (9) for the expression (8).

The admittance of the integral was also introduced in [11], taking place when the next notation is written according the recursive formula:

$$Z_{m+1}^{(n+1)} \left(a_{n+1}, z_0; z\right) = (n+1) \int_{z_0}^{z} Z_m^{(n)} \left(a_0, z_0; z\right) d_{(F_m,G_m)} z, \tag{15}$$

where (15) is employed to approximate higher exponents of the formal powers.

Employing the structure of (8), (9), and (15), the special case of Vekua equation presented in (13) can be approached by the next statement introducing the factors from (14) to (16), obtaining the next equation:

$$\int_{z_0}^{z} W \, d_{(F_m,G_m)} z = F_m \operatorname{Re} \int_{z_0}^{z} G_m^* W \, dz + G_m \operatorname{Re} \int_{z_0}^{z} F_m^* W \, dz.$$
$$F_m = p; \quad G_m = \frac{i}{p}; \quad p = p_1 (x) \cdot p_2 (y); \quad p = \sqrt{\sigma}. \tag{16}$$

Equation (16) is the definition of a (F, G)-integral of a complex valued function W. Specifically, since (F_m, G_m)-integral of the (F_m, G_m)-derivative of W reaches

$$\int_{z_0}^{z} \partial_{(F_0,G_0)} W \, d_{(F_0,G_0)} z = W - \phi \left(z_0\right) F_0 - \psi \left(z_0\right) G_0, \tag{17}$$

taking into account that

$$\partial_{(F_0,G_0)} F_0 = \partial_{(F_0,G_0)} G_0 = 0, \tag{18}$$

the integral expression (17) can be considered the (F_0, G_0)-antiderivative of the function $\partial_{(F_0,G_0)} W$.

For the correct approximation of the electric potential $u|_\Gamma$ for (1), using the Taylor series in formal powers, let us note that this electric potential can be found by the next statements:

$$u^n (1, 0; z) = \operatorname{Re} Z_m^{(n)} (1, 0; z)\Big|_\Gamma;$$
$$u^n (i, 0; z) = \operatorname{Re} Z_m^{(n)} (i, 0; z)\Big|_\Gamma. \tag{19}$$

In this case, if the boundary condition $u|_\Gamma$ is provided by (19), we can always approximate asymptotically the experimental electric potential $u|_{\text{app}}$ by the next expression:

$$u_{\text{app}} = \lim_{N \to \infty} \sum_{n=0}^{N} \left(\alpha_n u^n (1, 0; z) + \beta_n u^n (i, 0; z)\right), \tag{20}$$

where α_n and β_n are real numbers. This procedure has proven its effectiveness in assortment works [9, 10, 12], where the numerical approximation achieved highly accurate results.

The analysis of the problem can include samples in piecewise and nonpiecewise separable variable functions for the conductivity σ; to understand this situation we have to employ the conjecture exposed in [10].

Conjecture 1. *Suppose a σ arbitrary conductivity function defined within a bounded domain $\Omega(\mathbb{R}^2)$. This function can be approximated by means of a piecewise separable variables function in the form*

$$\sigma_{pw} = \begin{cases} \dfrac{x+g}{(\rho_1 - \rho_0) + g} \cdot f_1 (y), & x \in [\rho_0, \rho_1); \\ \dfrac{x+g}{(\rho_2 - \rho_1) + g} \cdot f_2 (y), & x \in [\rho_1, \rho_2); \\ \vdots & \vdots \\ \dfrac{x+g}{(\rho_K - \rho_{K-1}) + g} \cdot f_K (y), & x \in [\rho_{K-1}, \rho_K); \end{cases} \tag{21}$$

here g is a real constant such that $x + g \neq 0 : x \in \Omega(\mathbb{R}^2)$ and $\{f_k\}_{k=1}^K$ are the constructed interpolation functions with a finite number of samples $\mathscr{M}$ of the conductivity function σ, valued in the y-axis that is parallel line within the subdomain Ω, created by tracing $\{\rho\}_{k=0}^K$ parallel lines.

These piecewise separable variable functions can be employed for numerically approximating the set of formal powers.

Proposition 2. *Consider an arbitrary conductivity function σ defined within a domain $\Omega(\mathbb{R}^2)$. It can be considered the limiting case of a piecewise separable variables function, expressed in the form of the Conjecture (21), when K subdomains and $\mathscr{M}$ number of samples are every subdomain tending to infinity:*

$$\lim_{K,\mathscr{M}\to\infty} \sigma_{pw} = \sigma. \tag{22}$$

A regularization process is needed to increase the precision of the algorithm and obtain a better solution to be analysed for further research in the inverse problem; in the next section we introduce a simple idea to employ a regularization process in the Taylor series in formal powers and iterative method using this idea to improve the results, or at least to obtain a better convergence.

2.2. Regularization Process. Regularization methods are employed in many fields, introducing additional information with the purpose to solve ill-posed problems. Also, they are employed to prevent overfitting and are usually presented in form of penalty for complexity, such as restrictions for smoothness or bounds in the space norm for a vector. In particular case, the including of these methods to the algorithm designed to compute a solution for inverse problems is essential to fitting the data in order to reduce a norm of the solution.

For current problem, the imposed electric potential $u|_\Gamma$ represents a finite number of current densities $j_k|_{k=0}^m$ for a domain Ω with a boundary Γ and the knowledge of the Neumann-Dirichlet map [13] where

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0, \qquad \sigma\partial_\Gamma u = j. \tag{23}$$

Remember the notations introduced before in (1); we know that σ represents the conductivity and u denotes the electric potential in the electric conductivity equation, where $\partial_\Gamma u = \partial u/\partial_\Gamma$, being all the values of the electric potential u in the boundary domain Γ, and j is the current density.

For the forward Dirichlet boundary problem, we know the value of the conductivity σ but the electric potential $u|_\Gamma$ is unknown. In this work, the electric potential $u|_\Gamma$ is imposed, and, for the purpose of this paper, the approximation of an experimental value u is computed comparing the results by means of the Lebesgue measure and determining if the method employed is being corrected [14].

The Tikhonov regularization process can be presented in such form

$$\min_x \left(\|Ax - b\|^2 + \gamma^2 \|x\|^2\right), \tag{24}$$

where $\|\cdot\|$ denotes the L_2 norm, A can be a square matrix or a matrix of $r \times s$ dimension, b denotes a vector with r rows, γ is the regularization parameter, and the problem turns into a minimization of the parameter x.

Algorithm 1, which we propose in this study, is based on the Pseudoanalytic Function Theory, employing the Taylor series in formal powers, and it approximates the forward Dirichlet boundary value problem for (1), by means of S total number of radii, N maximum number of formal powers, and K total number of points.

(1) $S \leftarrow 500$; $N \leftarrow 20$; $K+1 \leftarrow 501$;
(2) **while** $s = 1 \rightarrow S$
(3) **while** $n = 1 \rightarrow N$
(4) **while** $k = 0 \rightarrow K$
(5) $Z^{(n+1)}[k] = \mathscr{B}\left[Z^{(n)}[k]\right]$;
(6) **Function** *Orthonormalization*
(7) Classical Gram-Schmidt Orthonormalization Process
(8) **Function** R*egularization*
(9) Tikhonov Regularization Process
(10) **Function** *Approach_Boundary_Condition*
(11) $u_{\text{app}} = \sum_{n=0}^{N}\left(\alpha_n u^{(n)}(1,0,z) + \beta_n u^{(n)}(i,0,z)\right)$;
(12) **Function** *Save Orthonormal_Sytem*;
(13) **Function** *Save Coefficients*;

ALGORITHM 1: Boundary value with regularization approximation.

Thus, (24) has an explicit solution expressed as follows:

$$x = \left(A^T A + \gamma^2 I\right)^{-1} A^T b, \tag{25}$$

where I is the identity matrix with the same dimension of matrix A, the regularization parameter $\gamma > 0$, in this case $A = Z^{(n+1)}[k]$ represents the orthonormalized system, which is the result of the formal powers employing the conductivity σ and the imposed boundary condition, and b is the electric potential $u|_\Gamma$. The main problem is to choose the correct parameter γ, which approximates the solution for (25).

To choose a parameter γ, the next expression should be used:

$$\gamma = \sum_{k=1}^{m} \frac{A_{kk}^T A_{kk}}{u_k^T \cdot u_k}, \tag{26}$$

where A, as it was mentioned before, is the orthonormalized Taylor series in formal powers and u represents the electric potential [15]. We employ this regularization process to understand the behaviour of the problem leaving for the next section the methodology of the main process for approximation of the solution.

3. Procedure

The main goal is to develop an effective stable method, which let us to obtain better approximations for the forward problem by the employment of a regularization procedure for better convergence. The results obtained will be analysed in order to understand the problem presented in (1).

For this analysis, to find the forward Dirichlet boundary value problem using the methodology shown before, we use the algorithm from [9, 10]; as a result we obtain an accurate solution with a lower computer cost.

The principal difficulty, when the algorithm is being employed, lies in the instability of the method when the approximations are taking place. So, in order to improve the convergence of the result, a regularization process is included in the approximation process.

$S \leftarrow 500$; $N \leftarrow 20$; $K+1 \leftarrow 501$;
while $s = 1 \rightarrow S$
  while $n = 1 \rightarrow N$
    while $k = 0 \rightarrow K$
      $Z^{(n+1)}[k] = \mathscr{B}\left[Z^{(n)}[k]\right]$;
while $\gamma > \gamma_e$
  Function *Orthonormalization*
    Classical Gram-Schmidt Orthonormalization Process
  Function *Regularization*
    Tikhonov Regularization Process
  Function *Approach_Boundary_Condition*
    $u_{\text{app}} = \sum_{n=0}^{N} \left(\alpha_n u^{(n)}(1,0,z) + \beta_n u^{(n)}(i,0,z)\right)$;
  $\gamma_e = \sum_{k=1}^{m} \dfrac{A_{kk}^T A_{kk}}{u_{\text{app}}^T u_{\text{app}}}$;
Function *Save Orthonormal_Sytem*;
Function *Save Coefficients*;

Algorithm 2: Boundary value with autoadjustment regularization approximation.

A procedure for the approximation consists of the following.

(1) Choose a domain.
(2) Select a conductivity.
(3) Apply Taylor series in formal powers approximation.
(4) Apply regularization process.
(5) Perform the computation of the electric potential.

First, a domain should be chosen, employing Algorithm 1. According to [9], there can be used smooth and nonsmooth domains, below; to analyse the results a unitary disk domain is chosen. Then, a conductivity function has to be chosen, and, in this study, the conductivity function is taken from the mathematical analysis and geometrical distribution; both cases are within the domain Ω.

Once the domain Ω and the conductivity function sigma are selected, the process continues by selecting the N maximum number of formal powers, S number of radii. and K number of points per radius, the approximation of the Taylor series in formal powers is performed, and the result obtained from this process should pass through the algorithm to the Gram-Schmidt orthonormalization method.

In this part of the procedure, the boundary solution can be computed, but we propose to modernize the procedure using a regularization method, in order to obtain better convergence of the method. Figure 1(a) shows graphically the procedure explained before.

The main idea is to confirm that novel regularized algorithm is able to compute more stable solution for the electric potential in the boundary and to express numerically the convergence of the method.

Additionally, we propose a modification of the explained algorithm where an autoadjustment should be performed before the error estimation process. Here, we use iterative scheme that can estimate the regularization parameter improving the convergence property of the algorithm.

Figure 2 illustrates the diagram of the algorithm proposed. The procedure analyses the convergence and the stability when an automatic adjustment takes place.

The modification made to Algorithm 1 presents a condition that should be satisfied, presenting an iterative method to approximate the solution of (1). In this algorithm, the condition is the same as presented before in (26), where the u_k in the iterative algorithm is the approximation of the electric potential u_{app}, as it is exposed in Algorithm 2. This autoadjustment introduced in the algorithm computes the solution of the forward problem increasing the computational cost but permits obtaining a better solution that is more stable and convergent.

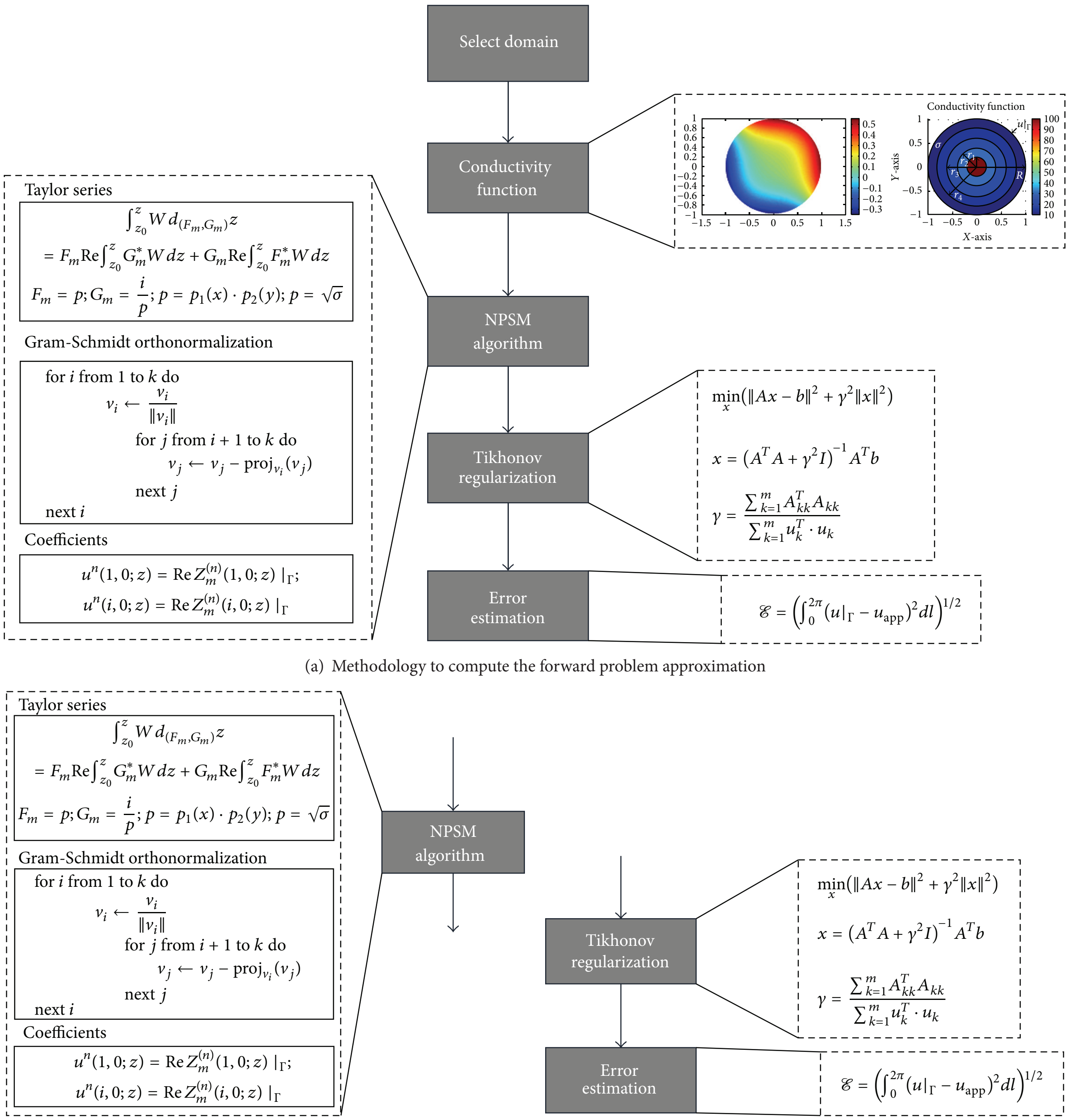
Applying Algorithms 1 and 2 shown above, we perform the numerical experiments using mathematical expressions, such as exponential and sinusoidal function, and geometrical distributions such as disk center, and five disks' structure at the center.

This set of examples gives us a full perspective of the behaviour of the method, because we employed some samples to analyse the convergence of the method and then make experiments with the geometric distributions to analyse the results that are reviewed in the next section.

The examples investigated in the next section are designed to compare the results when a regularization method is used in the process versus the actual method, which does not employ a regularization procedure.

4. Results

As it is exposed in the works [9, 10], the possibility to employ geometrical distributions functions and mathematical expression in the algorithm, which use the Taylor series in formal powers, let us analyse the electrical impedance equation (1), emphasizing the possibility to approximate the forward problem to this equation.



(a) Methodology to compute the forward problem approximation

(b) NPSM algorithm block

(c) Tikhonov and error estimation blocks

FIGURE 1: Methodology to compute the forward problem approximation.

Applying this analysis, we used an exponential and sinusoidal function coming from the mathematical analysis, circle at center, and the five disks' structure at center geometrical distributions. These cases are computed within the domain, and the results of the approximation are passed to regularization process in order to analyse the approximation and stability of the method, obtaining the electric potential in the boundary.

The solution for error, in order to analyse behaviour of the solution, is used in form of the Lebesgue measure

$$\mathscr{E} = \left(\int_0^{2\pi} \left(u|_\Gamma - u_{app}\right)^2 dl\right)^{1/2}, \tag{27}$$

where $u|_\Gamma$ is the imposed boundary condition and u_{app} denotes the approximation by using the original algorithm

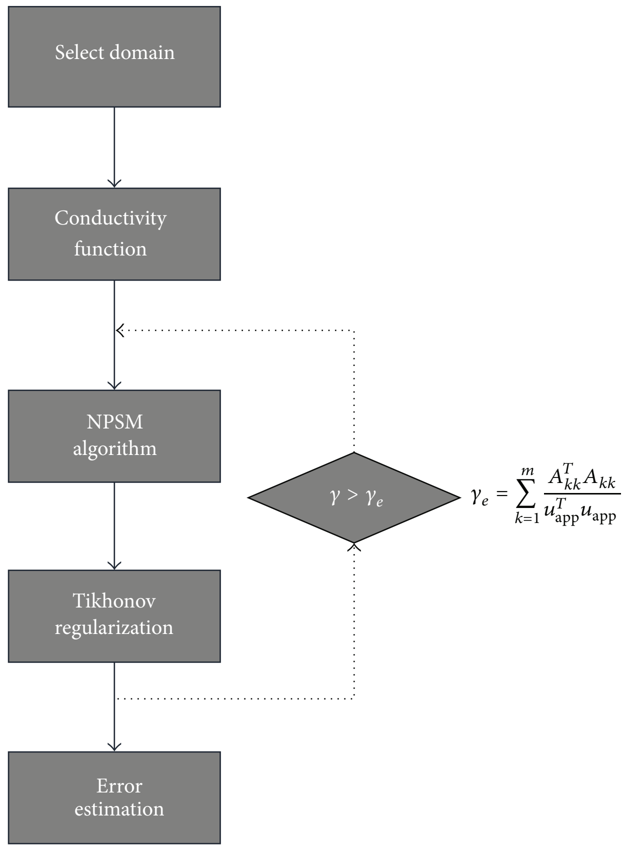


FIGURE 2: Methodology to compute the forward problem approximation employing autoadjustment regularization.

and the new modification employing the regularization process of everything within the unitary disk. We show the results in tables to compare and analyse the results obtained by these methods.

The next sections show the results for analytical and geometrical cases, in which the analytical cases are representatives due to the consideration of nonseparable conductivity functions in which the results can be achieved; for the geometrical case, the conductivity employed does not possess a boundary condition, imposing an artificial condition to compare the results and determine the accuracy of the method.

4.1. Exponential Conductivity Function. Let us use conductivity function, which fulfills (12) and possesses the form

$$\sigma = e^{-\mu xy}, \tag{28}$$

and this expression is shown in Figure 3(a), and for the boundary condition the expression to be imposed is

$$u = e^{\mu xy}, \tag{29}$$

where μ denotes the coefficient employed to change the behaviour of the function employed.

Table 1 expresses the behaviour when the μ coefficient increases and demonstrates the stability property of the method in two cases when the regularization method is employed and when this regularization process is not used.

The results in Table 1 show the known behaviour; meanwhile the number of formal powers increases and the error decreases, demonstrating a better convergence when the regularization method is employed.

The comparison between two algorithms that use the regularization procedure with and without the autoadjustment algorithms has been proved, the convergence and stability in the autoadjustment procedure being better than for comparison methods, but the computational cost considerably increases due to the iterative procedure performed.

This comparison is presented in Figure 3(b); in this graphic the three methods show the behaviour of the expression and in this case all methods reached a good approximation; since the three methods possess a low error, the difference can not be appreciated in the graphic matter, but Table 1 resumes correctly the full error analysis between all the methods.

4.2. Sinusoidal Conductivity Function. Let us employ a conductivity function, with the form

$$\sigma = 1 + \sin(\mu xy). \tag{30}$$

Figure 4(a) shows the expression within the unit disk, and, imposing a boundary condition,

$$u = \frac{1}{\tan(\mu xy/2) + 1}, \tag{31}$$

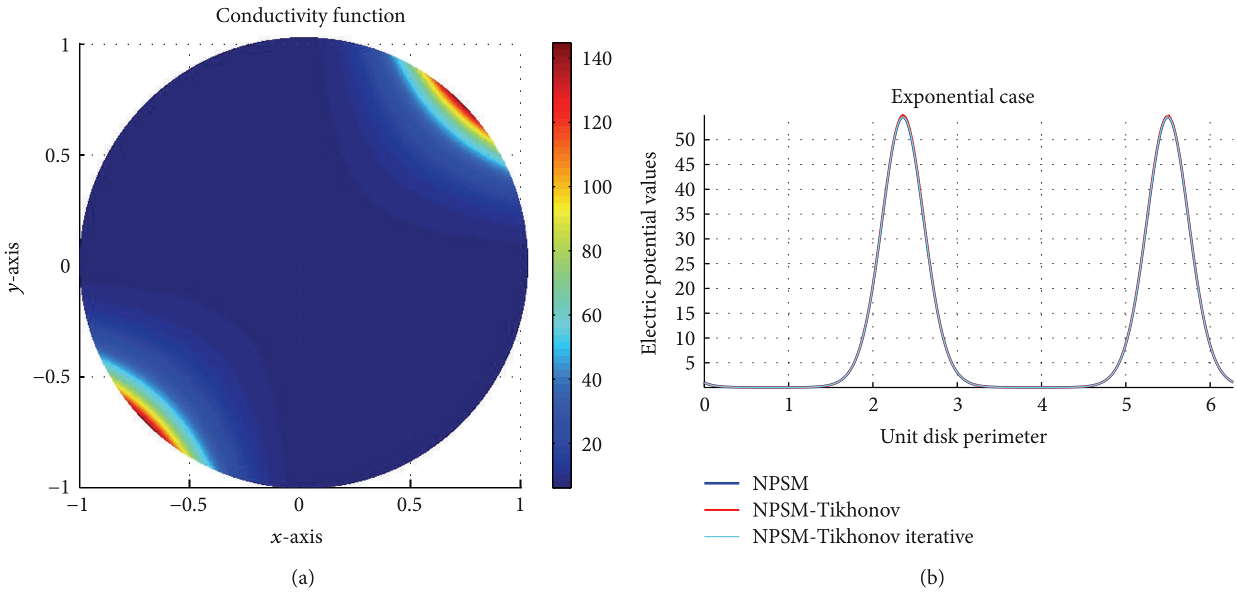


FIGURE 3: (a) Exponential conductivity function. (b) Comparison.

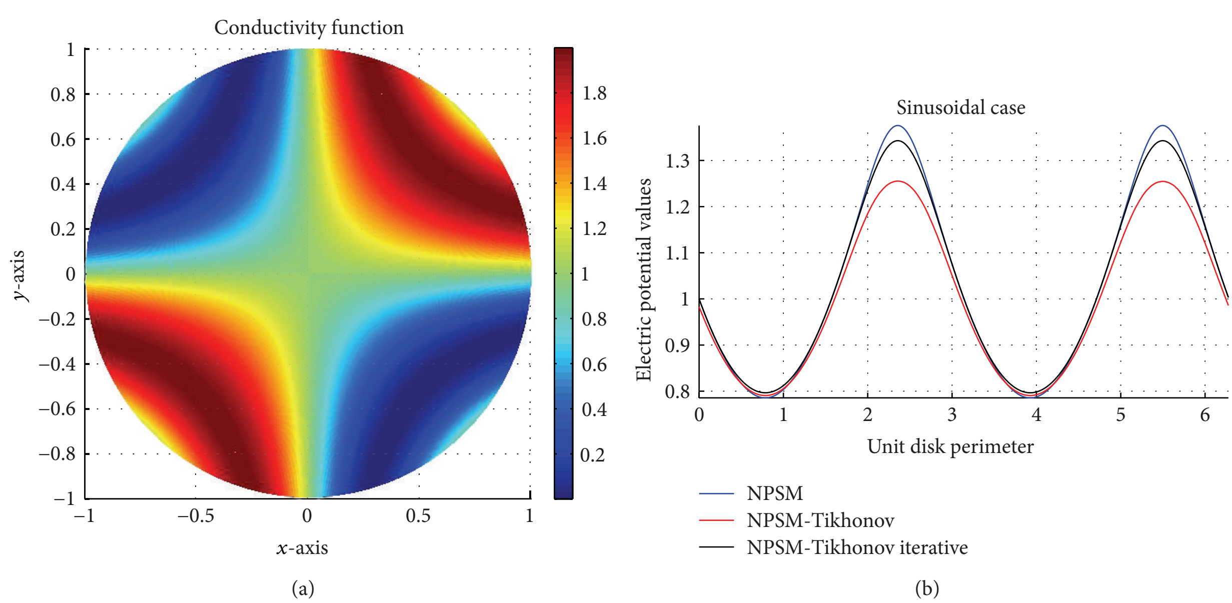


FIGURE 4: (a) Sinusoidal conductivity function. (b) Comparison.

where μ denotes the coefficient used to change the behaviour of the function employed.

Table 2 shows the behaviour of the method when the μ coefficient increases its value, including the simulation with and without the regularization process.

The results in this case are better with a regularization procedure; due to the conductivity that presents so many vibrations in the domain, the algorithm tries to reach a solution but its behaviour is bad; differently, when the regularization process is introduced, the results expose better performance because the perturbations or vibrations within the domain are smoothed and it is more easy to compute an approximation to (1).

In comparison with the other methods, the behaviour of the proposed algorithms is similar: using more formal powers the error is decreased and by the employing a regularization method the stability and convergence are improved considerably, but due to the iterative method used the computational cost is increased.

In Figure 3(b), the behaviour of all methods is shown; in this case the approximation can be appreciated because of the values of its error; for this case, the error can be analysed in

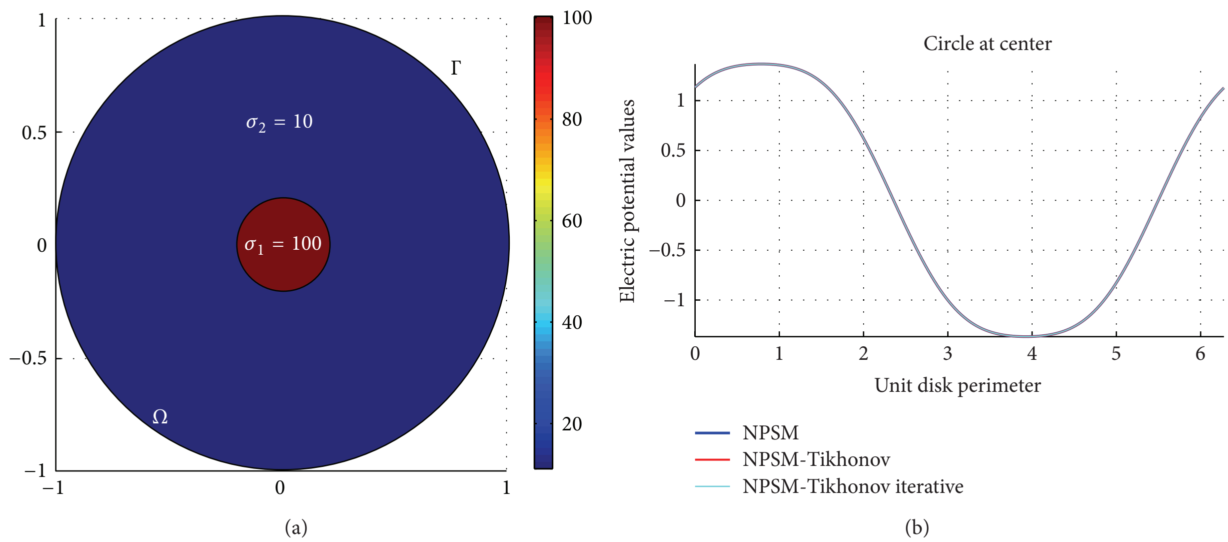


FIGURE 5: (a) Disk at center conductivity function, $\sigma_1 = 100$; $\sigma_2 = 10$. (b) Comparison.

TABLE 1: Exponential conductivity function $\sigma = e^{-\mu xy}$.

N	S	K	μ	$\mathscr{E}$ NPSM	$\mathscr{E}$ NPSM-Tikhonov	$\mathscr{E}$ NPSM-Tikhonov autoadjustment
20	500	500	10	7.9269×10^{-4}	4.3572×10^{-7}	9.5152×10^{-9}
40	500	500	10	1.2168×10^{-12}	3.2525×10^{-7}	9.4462×10^{-9}
60	500	500	10	9.9732×10^{-13}	3.0521×10^{-7}	9.4187×10^{-9}
80	500	500	10	9.9623×10^{-13}	3.0021×10^{-7}	9.3702×10^{-9}
100	500	500	10	9.9450×10^{-13}	3.0015×10^{-7}	9.3651×10^{-9}
120	500	500	10	9.9230×10^{-13}	3.0005×10^{-7}	9.3551×10^{-9}
20	500	500	20	4.8450	12.5271×10^{-7}	3.5451×10^{-9}
40	500	500	20	2.6416×10^{-6}	11.1423×10^{-7}	3.4570×10^{-9}
60	500	500	20	7.5986×10^{-10}	11.0912×10^{-7}	3.3721×10^{-9}
80	500	500	20	7.5983×10^{-10}	11.0510×10^{-7}	3.2987×10^{-9}
100	500	500	20	7.5985×10^{-10}	11.0031×10^{-7}	3.1067×10^{-9}
120	500	500	20	7.5979×10^{-10}	10.9998×10^{-7}	3.0923×10^{-9}

terms of the method employed, proving that a regularization method is needed to approximate and reach a better solution.

4.3. Circle at Center. For the next case, we employed a disk at the center with conductivity $\sigma_1 = 100$ and the conductivity in the rest of the domain Ω is $\sigma_2 = 10$, as is shown in Figure 5(a).

For this case, we imposed the following boundary condition:

$$u = \frac{1}{3}\left(x^3 + y^3\right) + 0.5\left(x + y\right). \tag{32}$$

The results are shown in Table 3, where the behaviour of the method with and without the regularization process is presented.

In this case, the figure inside the domain is a circle with a variable radius (in concrete case, radius is 0.5); we know that if the disk within the domain is bigger, the approximation will be good, but if the inner conductivity is smaller, the algorithm could not detect the conductivity good; differently, when a regularization process is introduced, it warrants that the conductivity will be employed to compute the solution and the regularization process permits smoothing the conductivity to reach results.

This example provides unexpected results, because the convergence of the NPSM-Tikhonov and NPSM-Tikhonov with autoadjustment process does not obtain the convergence that the NPSM method can obtain. This case presents the best convergence and stability obtained by the NPSM algorithm, and the method with regularization process appears to obtain worse convergence but is still acceptable due to the error estimated.

This example gives us different ideas to improve and to analyse the problem for the optimized method. The first idea about the bad convergence in the solution is that the imposed

Table 2: Sinusoidal conductivity function $\sigma = 1 + \sin(\mu x y)$.

N	S	K	μ	$\mathscr{E}$ NPSM	$\mathscr{E}$ NPSM-Tikhonov	$\mathscr{E}$ NPSM-Tikhonov autoadjustment
20	500	500	6	47.8026	1.2558×10^{-3}	4.9995×10^{-5}
40	500	500	6	49.9321	1.2143×10^{-3}	4.9301×10^{-5}
60	500	500	6	54.7917	1.2013×10^{-3}	4.9015×10^{-5}
80	500	500	6	61.9715	1.1970×10^{-3}	3.8084×10^{-5}
100	500	500	6	70.3529	1.1905×10^{-3}	3.7970×10^{-5}
120	500	500	6	80.2248	1.2013×10^{-3}	3.6523×10^{-5}
20	500	500	12	42.5009	1.0078×10^{-3}	7.2012×10^{-5}
40	500	500	12	42.3868	1.0009×10^{-3}	7.1562×10^{-5}
60	500	500	12	42.5782	2.2561×10^{-4}	7.1132×10^{-5}
80	500	500	12	42.9195	1.2013×10^{-3}	7.0999×10^{-5}
100	500	500	12	43.1998	1.1817×10^{-3}	7.0741×10^{-5}
120	500	500	12	43.6383	1.1098×10^{-3}	7.0594×10^{-5}
20	500	500	18	54.2971	5.2045×10^{-4}	9.5051×10^{-5}
40	500	500	18	54.6205	5.1270×10^{-4}	9.5005×10^{-5}
60	500	500	18	56.3661	5.0976×10^{-4}	9.4980×10^{-5}
80	500	500	18	59.4942	5.0081×10^{-3}	9.4953×10^{-5}
100	500	500	18	64.2429	4.9954×10^{-3}	9.4032×10^{-5}
120	500	500	18	69.9109	4.8013×10^{-3}	9.3511×10^{-5}

Table 3: Disk at center with $\sigma_1 = 10$ and $\sigma_2 = 100$.

N	S	K	$\mathscr{E}$ NPSM	$\mathscr{E}$ NPSM-Tikhonov	$\mathscr{E}$ NPSM-Tikhonov autoadjustment
20	500	500	5.3404×10^{-15}	5.8084×10^{-7}	3.1415×10^{-9}
40	500	500	6.8485×10^{-15}	5.8590×10^{-7}	3.1010×10^{-9}
60	500	500	7.7718×10^{-15}	5.9150×10^{-7}	3.0578×10^{-9}
80	500	500	8.5317×10^{-15}	6.1232×10^{-7}	3.0020×10^{-9}
100	500	500	9.3593×10^{-15}	6.2140×10^{-7}	3.0001×10^{-9}
120	500	500	9.8723×10^{-15}	6.2491×10^{-7}	2.9982×10^{-9}

boundary condition is not the correct condition for this case; another idea is to employ a different approximation process based on mesh and finally to determine if the regularization process can improve; a better solution can be imposed in the boundary when it comes from the approximation in the same method, which comes from the Taylor series.

Figure 5(b) shows the behaviour of the solution in terms of the error computed; in this case, due to the low error, the difference in the graphic cannot be observed, but the difference exists and can be appreciated in the analysis of Table 3. For this case, the error increases a little, but it possesses a steady behaviour.

4.4. Concentric Disk at Center. In this experiment, we used a disk with four rings at the center with different radius and conductivities: $\sigma_1 = 10$ for the disk, $\sigma_2 = 15$ for the first ring, $\sigma_3 = 20$ for the second ring, $\sigma_4 = 30$ for the third ring, and $\sigma = 100$ for the last ring, as is shown in Figure 6(a).

For this case, we imposed the boundary condition (32).

The simulation results are shown in Table 4, presenting the behaviour of the method with and without the regularization process.

This case is difficult due to the disk structure inside the domain; both methods proved to be good when they approximate the electric potentials, in which the algorithm without regularization presents better convergence, compared to the one which used a regularization process presenting smoothing within the domain to find an approximation with more stability.

The best approximation is obtained for the NPSM algorithm, and the NPSM-Tikhonov with and without autoadjustment presents worse approximation but is useful for the problem to be solved. The behaviours of the methods are similar: a high number of formal powers give a better convergence, the number of radii and points per radius do not affect the convergence, and the regularization process

TABLE 4: Disk at center conductivity function, $\sigma_1 = 10$, $\sigma_2 = 15$, $\sigma_3 = 20$, $\sigma_4 = 30$, and $\sigma_5 = 100$.

N	S	K	$\mathscr{E}$ NPSM	$\mathscr{E}$ NPSM-Tikhonov	$\mathscr{E}$ NPSM-Tikhonov autoadjustment
20	500	500	5.7828×10^{-15}	3.1309×10^{-7}	1.8012×10^{-8}
40	500	500	5.9170×10^{-15}	3.2404×10^{-7}	1.7925×10^{-8}
60	500	500	6.0634×10^{-15}	3.2602×10^{-7}	1.6841×10^{-8}
80	500	500	6.2671×10^{-15}	3.2702×10^{-7}	1.5165×10^{-8}
100	500	500	6.3172×10^{-15}	4.0112×10^{-7}	1.4409×10^{-8}
120	500	500	6.3476×10^{-15}	4.0201×10^{-7}	1.3001×10^{-8}

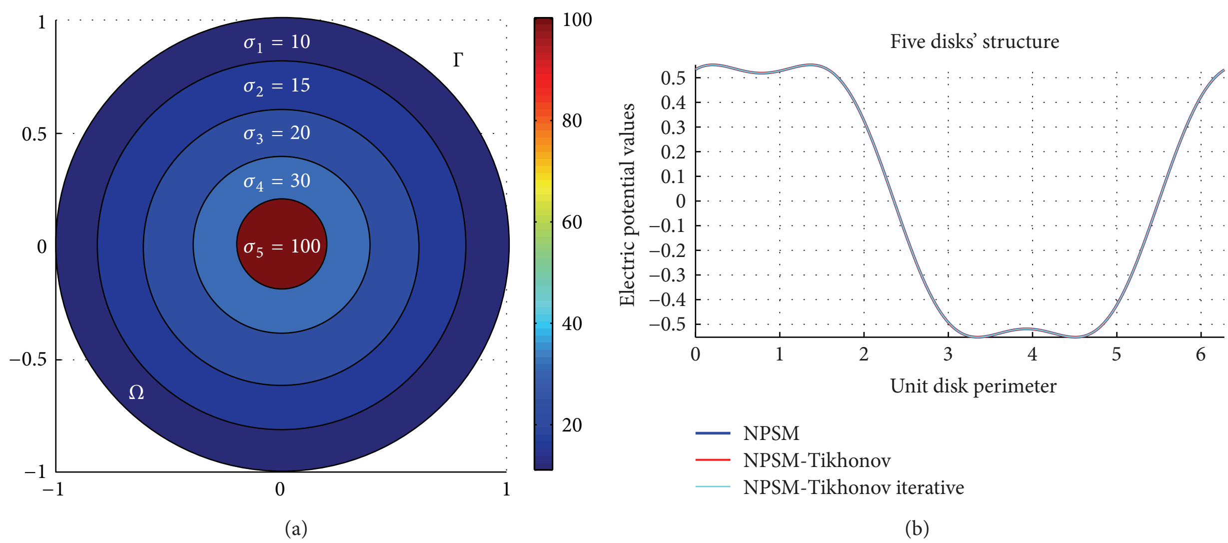


FIGURE 6: (a) Five disks' structure at center conductivity function, $\sigma_1 = 10$, $\sigma_2 = 15$, $\sigma_3 = 20$, $\sigma_4 = 30$, and $\sigma_5 = 100$. (b) Comparison.

with and without an autoadjustment method only affects the convergence and stability in the solution. In this case, the approximation that is obtained by these methods is considered good, proving that the regularization method can be used for this problem.

This example is useful to understand the behaviour and, thanks to this analysis, the main idea is preserved; a complex study needs to be performed paying more attention to the boundary condition and employing the results obtained in the Taylor series instead of employing the imposed boundary condition in order to determine if the boundary condition which is employed is the best condition to fulfill the domain and the conductivity within; this analysis could be demonstrated if the inverse problem can be approached by means of the Taylor series in formal powers with and without a different interpolation procedure including the regularization process.

Figure 6(b) shows the error analysis of the example employed, in which the behaviour is the same; the method proves to be steady and to reach the solution in terms of the error computed. For this case, the solution shown in the graphic does not express a considerable difference and fits correctly, but in terms Table 4 shows that the error increases and decreases depending upon the data introduced in the problem.

5. Discussion

The examples exposed in Tables 1, 2, 3, and 4 illustrate the behaviour of the algorithms with and without the regularization process. In the tables, the columns with the title E NPSM, proved this behavior; meanwhile the formal powers increase, the convergence of the approximation is better, and the number of points and radii do not affect considerably the approximation in this method.

Then, the regularization process, which is employed in all cases, showed better stability and in some examples a better convergence; this is the proof that a regularization process could be employed in order to approximate forward problems when the conductivity function presents a nonsmooth conductivity function, giving the possibility to approximate the forward Dirichlet boundary value problem for (1) when the regularization is employed.

All the cases analyzed and discussed in the last section show better convergence and stability. The reasons for these approximations are due to the conductivity employed in every case, where the exponential conductivity case does not present problem to approximate employing any of the methods; the case when the conductivity function is a disk at center presents a better convergence without the regularization method, but the results are more stable when regularization is employed.

In cases such as sinusoidal function, where the conductivity function presents vibrations and the possibility to obtain an approximation is more difficult, the regularization process presents better convergence and stability here; the reason for it is the smoothing phase for the conductivity function within the domain, because the regularization process is added to the algorithm.

The last case that is analysed exposes the structure of the five disks at center; a Gibbs phenomenon is presented, provoking disaster in the computing process and presenting a better convergence in the method without the regularization process, but, oppositely, the proposed method exposes more stability with regularization process.

All these examples have shown that a regularization process can be involved to stabilise the process and to obtain better approximation. Such performance of the proposed method is crucial in developing a novel numerical tool for a solution to inverse problem for (1).

The proposed autoadjustment algorithm with a regularization process shows better stability due to the regularization parameter chosen experimentally. This advantage is presented when employing mentioned autoadjustment parameter, which permits the stability and convergence to reach their maximum value generating a reliable approximation. Drawback of the method is a considerable increment in the computational cost. So, we can obtain the solution more slowly comparing with the basic algorithm without regularization procedure.

These examples let us analyze the behaviour in analytical and geometrical cases near to finding a new idea to improve the algorithm and optimize the process. This study arouses several questions that need to be analysed in future work. In the interpolation procedure where the method employed is a radial basis interpolation, the approximation by means of a regularization procedure is a good idea but thanks to this analysis it should be analysed; what happens if the imposed boundary condition does not fulfill the conductivity within the domain and its boundary? Some ideas for future work in order to solve this query come, beginning with the employment of a mesh grid instead of a radial base function of interpolation and the use of a Taylor series approximation instead of the full boundary condition imposed in the domain with boundary.

6. Conclusions

The regularization method employed in the approximation for the forward Dirichlet boundary value problem for (1) gives a stability to the process permitting better convergence by smoothing the conductivity functions employed within the domain.

The conductivity is employed in the analytical cases, in which the conductivity function is presented in nonseparable variables; the convergence of the method permits employing also separable variables conductivity function presenting the same convergence. As it is shown, independently of the conductivity function, both algorithms reached a good accuracy, and let us analyse their behaviour to reach a possible solution for the inverse problem.

The results obtained in this study have proven that a regularization method is important to solve these problems, when considered as an ill-posed task. The incorporating of this regularization process presents smoothing in the conductivity functions within the domain to approximate the solution, independently of the domain that we employ.

For the nonsmooth domain, the regularization method can approximate the solution independently of domain; it means that if the boundary of the domain is not smooth, the regularization method smoothing it approaches the solution, but the convergence it is not the best.

Both methods have proven good approximation of the solution for the forward Dirichlet boundary value problem, where the autoadjustment regularization algorithm shows that it is possible to obtain a better convergence and stability than other proposed and analysed algorithms.

More experiments should be performed to determine if the novel algorithm can approximate different class of conductivity functions also. In future investigations, it should be checked if the algorithm can approximate real medical images obtaining the estimation of the conductivity employing image processing techniques.

At the light of the results and the discussion generated by means of both proposed algorithms, based on regularization process, it seems that novel proposal can be useful for ill-posed problems as it presents (1).

We leave for a future work the analysis of the data and the proposition of the method, which can be used for the approximation of the forward problem to obtain a solution for the inverse problem. This is a hard task and still open problem, due to the complexity of the electrical impedance tomography problem. Analysing the computer complexity of the algorithm employed, we ensure that it can be parallelized in order to compute the solution faster, and a characterization of the formal power should be performed to determine if the imposed boundary condition is correct or not for all cases used before.

Conflict of Interests

All the authors declare that there is no conflict of interests regarding the publication of this article, due to the numerical methods used along this work were fully developed in GNUC/C++ Compiler.

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