Research Article

Accelerated Double Direction Method for Solving Unconstrained Optimization Problems

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An iterative method for solving a minimization problem of unconstrained optimization is presented. This multistep curve search method uses the specific form of iteration with two direction parameters, the approximation of Hessian by appropriately constructed diagonal matrix, and the inexact line search procedure. It is proved that constructed numerical process is well defined under some assumptions. Considering certain conditions, the method is linearly convergent for uniformly convex and strictly convex quadratic functions. Numerical results arising from defined algorithms are also presented and analyzed.

1. Introduction

In this paper, we derive a first-order numerical method for solving the following nonlinear unconstrained optimization problem:

$$\min f(x), \quad x \in \mathbb{R}^n, \quad (1)$$

where $f(x)$ is twice continuously differentiable function. The iteration of the form

$$x_{k+1} = x_k + \alpha_k s_k + \alpha_k^2 d_k \quad (2)$$

is considered. Here, x_{k+1} represents a new iterative point, x_k is the previous iteration, and α_k denotes the stepsize, while s_k and d_k generate search directions. Each of these directions is calculated by particular algorithms. Similar to other iterative methods for solving the unconstrained optimization problems, the crucial moment is to find the appropriate descent direction vectors s_k , d_k and the optimal step length α_k . In the proposed iteration (2), there are two direction vectors, and that was a motivation for naming this method as Accelerated Double Direction method (or shortly ADD method). The decisive point of our research stands

actually on two-direction form of the analyzed method and its implementation. Originally, the method of this particular form, but for different assumptions, is described in [1, 2]. In [1, 2], the implementation of the likely defined method is omitted so in a way in this work we extend and fulfill the similar topic. To allow the method to become suitable for the implementation, we modify choice of the vectors d_k and s_k . Another contribution of this paper is obtaining better numerical results with respect to the number of iterations among some known methods for unconstrained optimization.

Further, the following notation is used:

$$\begin{aligned} g(x) &= \nabla f(x), & G(x) &= \nabla^2 f(x), \\ g_k &= \nabla f(x_k), & G_k &= \nabla^2 f(x_k), \end{aligned} \quad (3)$$

where $\nabla f(x)$ denotes the gradient of f and $\nabla^2 f(x)$ denotes the Hessian of f . As usual, x^T denotes the transpose of x .

There are some known procedures for deriving appropriate search directions. We mention several of them (see [1, 3, 4]).

On the other hand, computation of the step length is also important. The common way to determine stepsize α_k

is the inexact line search technique. The only requirement in the line search procedure is decreasing in the objective function values. This way we calculate the step length which is appropriate enough for our iterative optimization problem (see [5–12]).

In the present paper, we use a combination of the iterative scheme (2) and the accelerated gradient descent method from [13]. More precisely, the first term, $\alpha_k s_k$, in (2) is defined using the principles from the [13] method. The second term, $\alpha_k^2 d_k$, appears as the correction factor which is defined from the Taylor expansion series.

The paper is organized as follows. In Section 2, the basic motivation and idea for deriving the accelerated gradient descent method of the form (2) are explained. The algorithm of derived Accelerated Double Direction method, in short *ADD* method, is presented in Section 3, where also the main result of this work is analyzed. The convergence of *ADD* method is proved in Section 4. Numerical tests and comparisons of derived *ADD* method with the accelerated gradient descent method with line search (so-called *SM* method) originated in [13] as well as with nonaccelerated version of *ADD* method (*NADD* method) are given in Section 5.

2. Preliminaries

The accelerated gradient descent (*AGD*) methods of the form

$$x_{k+1} = x_k - \theta_k t_k g_k \quad (4)$$

use $\theta_k > 0$ as an acceleration parameter. The first *AGD* method is originated in [14]. A type of *AGD* method is considered in [13]. This *AGD* method is called *SM* method and it is derived starting from the Newton iterative method with the line search

$$x_{k+1} = x_k - t_k S_k g_k, \quad (5)$$

where S_k is an appropriate approximation of the Hessian inverse, presented as a symmetric $n \times n$ positive definite matrix. Taking $S_k = \gamma_k^{-1} I$, $\gamma_k > 0$ as an approximation of the inverse of the Hessian, the authors in [13] derived an accelerated modified Newton scheme:

$$x_{k+1} = x_k - t_k \gamma_k^{-1} g_k, \quad (6)$$

where step length t_k is computed by means of the backtracking inexact line search procedure and γ_k is the length of the acceleration parameter given by the following expression:

$$\gamma_{k+1} = 2\gamma_k \frac{\gamma_k [f(x_{k+1}) - f(x_k)] + t_k \|g_k\|^2}{t_k^2 \|g_k\|^2}. \quad (7)$$

In this paper, we are using the motivation for calculating an accelerated parameter, γ_k , from Taylor's extension of an objective function. But, unlike *SM* method, we choose to evaluate a method which contains two direction vectors.

A multistep minimization iterative process (2) with two direction vectors is described in [1, 2]. This algorithm considers generally nondifferentiable functions and it consists

of three partial subalgorithms, each explaining one of the needed parameters: two direction vectors s_k and d_k and the stepsize α_k . Since this work considers uniformly convex or strictly convex quadratic functions, we modify the proposed subalgorithms in a way according to the present conditions. In a further section, we give three supplemented algorithms based on propositions originally written in [1].

3. Main Algorithm

Taking into consideration the results obtained in [1, 13], we can construct new iterative method. That process has a form predefined by (2), where the parameter s_k has properties taken from *SM* method. Using the same notation as in the previous section, the process (2) becomes

$$x_{k+1} = x_k - \alpha_k \gamma_k^{-1} g_k + \alpha_k^2 d_k, \quad (8)$$

assuming that the direction vector s_k is, according to *SM* method, defined by $-\gamma_k^{-1} g_k$. Practically, deriving the direction vector s_k is reduced to deriving the positive real number γ_k . Taking vector d_k and step length α_k defined similarly as in [1], we get

$$x_{k+1} - x_k = \alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k. \quad (9)$$

Now, from Taylor's expansion of the second rate, the approximation of $f(x_{k+1})$ can be brought as follows:

$$\begin{aligned} f(x_{k+1}) &\approx f(x_k) + g_k^T (\alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k) \\ &\quad + \frac{1}{2} (\alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k)^T \nabla^2 f(\xi) (\alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k). \end{aligned} \quad (10)$$

The matrix $\nabla^2 f(\xi)$ is, like in [13], replaced by $\gamma_{k+1} I$ and the parameter ξ fulfills the following condition:

$$\xi \in [x_k, x_{k+1}],$$

$$\xi = x_k + \delta (x_{k+1} - x_k) = x_k + \delta (\alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k), \quad (11)$$

$$0 \leq \delta \leq 1.$$

Knowing this, the expression (10) becomes

$$\begin{aligned} f(x_{k+1}) &= f(x_k) + \alpha_k g_k^T (\alpha_k d_k - \gamma_k^{-1} g_k) \\ &\quad + \frac{1}{2} \gamma_{k+1} (\alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k)^T (\alpha_k^2 d_k - \alpha_k \gamma_k^{-1} g_k). \end{aligned} \quad (12)$$

From (12), γ_{k+1} is computed in the following way:

$$\gamma_{k+1} = 2 \frac{f(x_{k+1}) - f(x_k) - \alpha_k g_k^T (\alpha_k d_k - \gamma_k^{-1} g_k)}{(\alpha_k d_k - \gamma_k^{-1} g_k)^T (\alpha_k d_k - \gamma_k^{-1} g_k)}. \quad (13)$$

It is supposed that $\gamma_{k+1} > 0$; otherwise, the second-order necessary condition and second-order sufficient condition

will not be fulfilled. If in some iterative step it happens that $\gamma_{k+1} < 0$, we take $\gamma_{k+1} = 1$.

Then, the next iterative point x_{k+2} is computed by

$$\begin{aligned} x_{k+2} &= x_{k+1} - \alpha_{k+1} g_{k+1} + \alpha_{k+1}^2 d_{k+1} \\ &= x_{k+1} - \alpha_{k+1} (g_{k+1} - \alpha_{k+1} d_{k+1}). \end{aligned} \quad (14)$$

According to the posed assumptions in this paper, original calculations for obtaining the values of step length α_k and direction vector d_k given in [1] are modified by Algorithms 1 and 2.

Supplemented further, the main contribution in this paper is presented by Algorithm 3.

Remark 1. It is possible to compare iterations (9) proposed in the present paper with the general iterative scheme proposed in [15]. The search direction in [15] is defined as a linear combination of g_{k+1} and $x_{k+1} - x_k$. On the other hand, the search direction in (9) is defined as a particular linear combination of $-g_k$ and the vector d_k defined in Algorithm 2.

4. Convergence Analysis

In this section, the convergence analysis of constructed method is discussed. We will first analyze a set of uniformly convex functions and afterwards a subset of strictly convex functions. We will start with the following proposition and lemma that can be found in [16, 17].

Proposition 2 (see [16, 17]). *If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable and uniformly convex on $\mathbb{R}^n$, then*

- (1) *the function f has a lower bound on $L_0 = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$, where $x_0 \in \mathbb{R}^n$ is available;*
- (2) *the gradient g is Lipschitz continuous in an open convex set B which contains L_0 ; that is, there exists $L > 0$ such that*

$$\|g(x) - g(y)\| \leq L \|x - y\|, \quad \forall x, y \in B. \quad (15)$$

Lemma 3 (see [16, 17]). *Under the assumptions of Proposition 2, there exist real numbers m, M satisfying*

$$0 < m \leq 1 \leq M, \quad (16)$$

such that $f(x)$ has a unique minimizer x^ and*

$$\begin{aligned} m\|y\|^2 &\leq y^T \nabla^2 f(x) y \leq M\|y\|^2, \quad \forall x, y \in \mathbb{R}^n; \\ \frac{1}{2}m\|x - x^*\|^2 &\leq f(x) - f(x^*) \leq \frac{1}{2}M\|x - x^*\|^2, \\ &\forall x \in \mathbb{R}^n; \\ m\|x - y\|^2 &\leq (g(x) - g(y))^T (x - y) \leq M\|x - y\|^2, \\ &\forall x, y \in \mathbb{R}^n. \end{aligned} \quad (17)$$

The estimation of decreasing of a given uniformly convex function in each iteration is described in the following lemma taken from [13].

Lemma 4 (see [13]). *For a twice continuously differentiable and uniformly convex function f on $\mathbb{R}^n$ and for sequence $\{x_k\}$ generated by Algorithm 3 the following inequality is valid:*

$$f(x_k) - f(x_{k+1}) \geq \mu \|g_k\|^2, \quad (18)$$

where

$$\mu = \left\{ \frac{\sigma}{M}, \frac{\sigma(1-\sigma)}{L} \beta \right\}. \quad (19)$$

Proof. The proof follows directly from the proof of Lemma 4.2 in [13] using $s_k = -\gamma_k^{-1} g_k$ instead of d_k . $\square$

The following theorem guarantees a linear convergence of Accelerated Double Direction method. The proof is the same as the proof of Theorem (4.1) in [13].

Theorem 5 (see [13]). *If the objective function f is twice continuously differentiable and uniformly convex on $\mathbb{R}^n$ and the sequence $\{x_k\}$ is generated by Algorithm 3, then*

$$\lim_{k \rightarrow \infty} \|g_k\| = 0 \quad (20)$$

and the sequence $\{x_k\}$ converges to x^ at least linearly.*

We now consider the case of strictly convex quadratic function which has the form

$$f(x) = \frac{1}{2} x^T A x - b^T x, \quad (21)$$

where A is real $n \times n$ symmetric positive definite matrix and $b \in \mathbb{R}^n$. This particular case is observed since the convergence of gradient methods is generally difficult and nonstandard. In the following analysis, we will use some known assumptions taken from [18–20]. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be eigenvalues of the matrix A . In [20], the Q-linear rate of convergence is presented for BB method under the assumption $\lambda_n < 2\lambda_1$.

Lemma 6. *Let f be a strictly convex quadratic function given by the expression (21) which involves symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ and the gradient descent method (8). Let λ_1 and λ_n be, respectively, the smallest and the largest eigenvalues of A . Let the parameters γ_k , α_k , and d_k be determined according to (13) and Algorithm 3. Then, the following holds:*

$$\lambda_1 \leq \frac{\gamma_{k+1}}{\alpha_{k+1}} \leq \frac{2\lambda_n}{\sigma}, \quad k \in \mathbb{N}. \quad (22)$$

Proof. According to (21), the difference between the values of the objective strictly convex quadratic function in current and previous iterative point is

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= \frac{1}{2} x_{k+1}^T A x_{k+1} - b^T x_{k+1} \\ &\quad - \frac{1}{2} x_k^T A x_k + b^T x_k. \end{aligned} \quad (23)$$

Require: Objective function $f(x)$, the direction d_k of the search at the point x_k and numbers $0 < \sigma < 0.5$ and $\beta \in (\sigma, 1)$.

- (1) Set $\alpha = 1$
- (2) while $f(x_k + \alpha d_k) > f(x_k) + \sigma \alpha g_k^T d_k$ take $\alpha := \alpha \beta$
- (3) Return $\alpha_k = \alpha$.

ALGORITHM 1: Calculation of the step length α_k by reducing the curve search rule from [1] to the backtracking line search starting from $\alpha = 1$.

Require: The stepsize α_k .

(1) $\alpha = \alpha_k$.

$$(2) \quad d_k(\alpha) = \begin{cases} d_k^*, & k \leq m-1 \\ \sum_{i=2}^m \alpha^{i-1} d_{k-i+1}^*, & k \geq m, \end{cases}$$

where d_k^* is the solution of the problem $\min_{x \in \mathbb{R}} \Phi_k(d)$, and $\Phi_k(d)$ is defined by

$$\Phi_k(d) = g_k^T d + \frac{1}{2} \gamma_{k+1} I.$$

ALGORITHM 2: Calculation of direction vector d_k .

Knowing the fact that $x_{k+1} = x_k - \alpha_k \gamma_k^{-1} g_k + \alpha_k^2 d_k$, it follows that

Further grouping gives

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= \frac{1}{2} (x_k - \alpha_k \gamma_k^{-1} g_k + \alpha_k^2 d_k)^T A (x_k - \alpha_k \gamma_k^{-1} g_k + \alpha_k^2 d_k) \\ &\quad - b^T (x_k - \alpha_k \gamma_k^{-1} g_k + \alpha_k^2 d_k) - \frac{1}{2} x_k^T A x_k + b^T x_k \\ &= \frac{1}{2} x_k^T A x_k - \frac{1}{2} \alpha_k \gamma_k^{-1} g_k^T A x_k \\ &\quad + \frac{1}{2} \alpha_k^2 d_k^T A x_k - \frac{1}{2} \alpha_k \gamma_k^{-1} x_k^T A g_k + \frac{1}{2} \alpha_k^2 \gamma_k^{-2} g_k^T A g_k \\ &\quad - \frac{1}{2} \alpha_k^3 \gamma_k^{-1} d_k^T A g_k + \frac{1}{2} \alpha_k^2 x_k^T A d_k - \frac{1}{2} \alpha_k^3 \gamma_k^{-1} g_k^T A d_k \\ &\quad + \frac{1}{2} \alpha_k^4 d_k^T A d_k - b^T x_k + \alpha_k \gamma_k^{-1} b^T g_k - \alpha_k^2 b^T d_k \\ &\quad - \frac{1}{2} x_k^T A x_k + b^T x_k. \end{aligned} \quad (24)$$

Replacing the equivalence $g_k = Ax_k - b$, we obtain

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= -\alpha_k \gamma_k^{-1} x_k^T A g_k + \alpha_k^2 x_k^T A g_k - \alpha_k^3 \gamma_k^{-1} d_k^T A g_k \\ &\quad + \frac{1}{2} \alpha_k^2 \gamma_k^{-2} g_k^T A g_k + \frac{1}{2} \alpha_k^4 d_k^T A d_k + \alpha_k \gamma_k^{-1} b^T g_k - \alpha_k^2 b^T d_k. \end{aligned} \quad (25)$$

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= \alpha_k \gamma_k^{-1} (b^T g_k - x_k^T A g_k) + \alpha_k^2 (x_k^T A d_k - b^T d_k) \\ &\quad - \alpha_k^3 \gamma_k^{-1} d_k^T A g_k + \frac{1}{2} \alpha_k^2 \gamma_k^{-2} g_k^T A g_k + \frac{1}{2} \alpha_k^4 d_k^T A d_k \end{aligned} \quad (26)$$

and this leads to

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= -\alpha_k \gamma_k^{-1} g_k^T g_k + \alpha_k^2 g_k^T d_k - \alpha_k^3 \gamma_k^{-1} d_k^T A g_k \\ &\quad + \frac{1}{2} \alpha_k^2 \gamma_k^{-2} g_k^T A g_k + \frac{1}{2} \alpha_k^4 d_k^T A d_k. \end{aligned} \quad (27)$$

Using the symmetry properties of matrix A implies

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= \alpha_k g_k^T (\alpha_k d_k - \gamma_k^{-1} g_k) - \frac{1}{2} \alpha_k^2 \gamma_k^{-1} g_k^T A (\alpha_k d_k - \gamma_k^{-1} g_k) \\ &\quad + \frac{1}{2} \alpha_k^3 d_k^T A (\alpha_k d_k - \gamma_k^{-1} g_k) \\ &= \left(\alpha_k g_k^T - \frac{1}{2} \alpha_k^2 \gamma_k^{-1} g_k^T A + \frac{1}{2} \alpha_k^3 d_k^T A \right) (\alpha_k d_k - \gamma_k^{-1} g_k). \end{aligned} \quad (28)$$

Require: $0 < \rho < 1, 0 < \tau < 1, x_0$

- (1) Set $k = 0$, compute $f(x_0)$, g_0 and take $\gamma_0 = 1$.
- (2) If $\|g_k\| < \epsilon$, then go to Step 9, else continue by the next Step.
- (3) Find the stepsize α_k applying by Algorithm 1.
- (4) Compute d_k by Algorithm 2.
- (5) Compute x_{k+1} using (8), $f(x_{k+1})$ and g_{k+1} .
- (6) Determine the scalar γ_{k+1} using (13).
- (7) If $\gamma_{k+1} < 0$, then take $\gamma_{k+1} = 1$.
- (8) Set $k := k + 1$, go to Step 2.
- (9) Return x_{k+1} and $f(x_{k+1})$.

ALGORITHM 3: Accelerated Double Direction method (ADD method).

Switching the last equivalence into (13), γ_{k+1} becomes

$$\gamma_{k+1} = 2 \left[\frac{(\alpha_k g_k^T - (1/2) \alpha_k^2 \gamma_k^{-1} g_k^T A + (1/2) \alpha_k^3 d_k^T A)(\alpha_k d_k - \gamma_k^{-1} g_k)}{\alpha_k^2 (\alpha_k d_k - \gamma_k^{-1} g_k)^T (\alpha_k d_k - \gamma_k^{-1} g_k)} - \frac{\alpha_k g_k^T (\alpha_k d_k - \gamma_k^{-1} g_k)}{\alpha_k^2 (\alpha_k d_k - \gamma_k^{-1} g_k)^T (\alpha_k d_k - \gamma_k^{-1} g_k)} \right] \quad (29)$$

and further

$$\begin{aligned} \gamma_{k+1} &= 2 \left(\left(\alpha_k g_k^T - \frac{1}{2} \alpha_k^2 \gamma_k^{-1} g_k^T A + \frac{1}{2} \alpha_k^3 d_k^T A - \alpha_k g_k^T \right) \right. \\ &\quad \times (\alpha_k d_k - \gamma_k^{-1} g_k) \left. \right) \\ &\quad \times \left(\alpha_k^2 (\alpha_k d_k - \gamma_k^{-1} g_k)^T (\alpha_k d_k - \gamma_k^{-1} g_k) \right)^{-1}. \end{aligned} \quad (30)$$

Finally,

$$\gamma_{k+1} = 2 \cdot \frac{\frac{1}{2} \alpha_k^2 (\alpha_k d_k - \gamma_k^{-1} g_k)^T A (\alpha_k d_k - \gamma_k^{-1} g_k)}{\alpha_k^2 (\alpha_k d_k - \gamma_k^{-1} g_k)^T (\alpha_k d_k - \gamma_k^{-1} g_k)}, \quad (31)$$

which implies definitive expression for γ_{k+1} :

$$\gamma_{k+1} = \frac{(\alpha_k d_k - \gamma_k^{-1} g_k)^T A (\alpha_k d_k - \gamma_k^{-1} g_k)}{(\alpha_k d_k - \gamma_k^{-1} g_k)^T (\alpha_k d_k - \gamma_k^{-1} g_k)}. \quad (32)$$

Since A is a real symmetric positive definite matrix and since the previous expression for γ_{k+1} presents the Rayleigh quotient of the real symmetric matrix A at the vector $\alpha_k d_k - \gamma_k^{-1} g_k$, it can be concluded that

$$\lambda_1 \leq \gamma_{k+1} \leq \lambda_n, \quad k \in \mathbb{N}. \quad (33)$$

The fact $0 \leq \alpha_{k+1} \leq 1$ implies the left hand side of inequality (22). The right hand side of the same inequality arises from the inequality

$$\alpha_k > \frac{\beta(1-\sigma)\gamma_k}{L}, \quad (34)$$

which is proved in [13], in Lemma 4. The direct consequence of the previous expression is

$$\frac{\gamma_{k+1}}{\alpha_{k+1}} < \frac{L}{\beta(1-\sigma)}. \quad (35)$$

We know that A is symmetric and $g(x) = A(x) - b$. Considering these two relevant facts, we can conclude that

$$\begin{aligned} \|g(x) - g(y)\| &= \|Ax - Ay\| \\ &= \|A(x - y)\| \leq \|A\| \|x - y\| = \lambda_n \|x - y\|, \end{aligned} \quad (36)$$

which means that in the last expression the largest eigenvalue λ_n of matrix A has the property of Lipschitz constant L . In the backtracking algorithm, we chose that parameters σ and β take the values $0 < \sigma < 0.5$ and $\beta \in (\sigma, 1)$. As a resulting inequality, we have

$$\frac{\gamma_{k+1}}{\alpha_{k+1}} < \frac{L}{\beta(1-\sigma)} = \frac{\lambda_n}{\beta(1-\sigma)} < \frac{2\lambda_n}{\sigma} \quad (37)$$

and with this expression we are ending the proof. $\square$

Theorem 7. For the strictly convex quadratic function f given by (21) and the gradient descent method (2), under the additional assumptions $\lambda_n < 2\lambda_1$ for the eigenvalues of matrix A , one has

$$(p_i^{k+1})^2 \leq \delta^2 (p_i^k)^2, \quad (q_i^{k+1})^2 \leq \lambda_n^2 (q_i^k)^2, \quad (38)$$

where

$$\delta = \max \left\{ 1 - \frac{\sigma\lambda_1}{2\lambda_n}, \frac{\lambda_n}{\lambda_1} - 1 \right\}, \quad (39)$$

as well as

$$\lim_{k \rightarrow \infty} \|g_k\| = 0. \quad (40)$$

Proof. Suppose that $\{v_1, v_2, \dots, v_n\}$ are orthonormal eigenvectors of symmetric positive definite matrix A and let $\{x_k\}$ be the sequence of values constructed by Algorithm 3. For some k and value x_k , $g_k = Ax_k - b$. On the other hand,

$$g_k = \sum_{i=1}^n p_i^k v_i, \quad d_k = \sum_{i=1}^n q_i^k v_i \quad (41)$$

for some real constants $p_1^k, p_2^k, \dots, p_n^k$ and $q_1^k, q_2^k, \dots, q_n^k$.

From (2), it follows that

$$\begin{aligned} g_{k+1} &= A(x_k - \alpha_k \gamma_k^{-1} g_k + \alpha_k^2 d_k) - b \\ &= (I - \alpha_k \gamma_k^{-1} A) g_k + \alpha_k^2 A d_k, \end{aligned} \quad (42)$$

which together with (41) gives

$$\begin{aligned} g_{k+1} &= \sum_{i=1}^n p_i^{k+1} v_i \\ &= \sum_{i=1}^n ((1 - \alpha_k \gamma_k^{-1} \lambda_i) p_i^k + \alpha_k^2 \lambda_i d_i^k) v_i \\ &= \sum_{i=1}^n (1 - \alpha_k \gamma_k^{-1} \lambda_i) p_i^k v_i + \alpha_k^2 \sum_{i=1}^n \lambda_i d_i^k v_i. \end{aligned} \quad (43)$$

To prove (38), it is enough to show that $|1 - (\lambda_i / \gamma_k \alpha_k^{-1})| \leq \delta$ since $|\lambda_i| \leq \lambda_n$ for all $i \in \{1, 2, \dots, n\}$. There are two cases. First, if $\lambda_i \leq \gamma_k / \alpha_k$ implying (22), we can conclude the following:

$$1 > \frac{\lambda_i}{\gamma_k \alpha_k^{-1}} \geq \frac{\sigma \lambda_1}{2 \lambda_n} \implies 1 - \frac{\lambda_i}{\gamma_k \alpha_k^{-1}} \leq 1 - \frac{\sigma \lambda_1}{2 \lambda_n} \leq \delta. \quad (44)$$

Now, let us examine another case $\gamma_k / \alpha_k < \lambda_i$. Since

$$1 < \frac{\lambda_i}{\gamma_k \alpha_k^{-1}} \leq \frac{\lambda_n}{\lambda_1}, \quad (45)$$

we have

$$\left| 1 - \frac{\lambda_i}{\gamma_k \alpha_k^{-1}} \right| \leq \frac{\lambda_n}{\lambda_1} - 1 \leq \delta. \quad (46)$$

To prove (40), the representation (41) is used:

$$\|g_k\|^2 = \sum_{i=1}^n (p_i^k)^2. \quad (47)$$

Knowing that $0 < \delta < 1$ together with the proved inequalities (38) leads us to expression (40) which is the final conclusion. $\square$

5. Numerical Results

In order to numerically prove the acceleration property of *ADD* method, we constructed the nonaccelerated version of this scheme and named it Nonaccelerated Double Direction method, shortly *NADD* method. For that purpose, we had to eliminate the acceleration parameter γ_k . Since γ_k presents an approximation of the inverse of the Hessian, the question was what the adequate substitution for γ_k in iteration (9) is. The natural choice for nonaccelerated counterpart of *ADD* method is defined by taking constant value $\gamma_k = 1$ for all k in each iteration (9). Then, the Hessian is approximated by the identity matrix in each iteration. That is way the nonaccelerated form of the process is obtained and iteration (9) becomes

$$x_{k+1} - x_k = \alpha_k^2 d_k - \alpha_k g_k. \quad (48)$$

We tested the presented Accelerated Double Direction method, in short *ADD* method, on a large scale of unconstrained test problems given in 25 functions proposed in [21]. Through the execution, we investigate the number of iterative steps since our primary goal is to reduce this number. Each of 25 functions is tested for 10 numerical experiments. In order to have more general view of analyzed characteristic number of iterations, we choose to test cases of large number of variables: 1000, 2000, 3000, 5000, 7000, 8000, 10000, 15000, 20000, and 30000. *ADD* method is compared with *SM* method, since in [13] *SM* method is already compared with *AGD* method and *GD* (gradient descent) method from [14], but for a lower number of variables (500, 1000, 2000, 3000, 5000, 7000, 8000, 10000, and 15000). In the same paper, it is proved that *SM* method outperformed *AGD* and *GD* methods with respect to the number of iterative steps. Since our aim is to improve the numerical results with respect to this characteristic by using constructed *ADD* method, it is enough to show that on this matter *ADD* algorithm obtains better results than *SM*. The stopping criteria for *ADD* Algorithm 3 are like those in [13] for *SM* method:

$$\|g_k\| \leq 10^{-6}, \quad \frac{|f(x_{k+1}) - f(x_k)|}{1 + |f(x_k)|} \leq 10^{-16}. \quad (49)$$

The codes that are used for testing are written in the C++ programming language on a Workstation Intel Celeron 2.2 GHz.

The presented results in Table 1 show the enormous dominance of *ADD* method with respect to the number of iterations. Among the 25 tested functions, a very big difference in the number of iterations in favor of *ADD* method is obvious. Precisely, in cases of even 20 of the tested functions, *ADD* is significantly more effective with respect to the analyzed characteristic than *SM* method.

In Table 1, the test results of the *NADD* method are also presented. During the test procedures needed execution time for *NADD* method was evidently too long. That is why we defined an *execution time limiter parameter* t_e as follows: if the test execution is lasting more than t_e , we will stop further testing and declare that the testing is *too long*. The longest execution time in testing *ADD* method is obtained for Diagonal 7 function and it is totaled 3287 seconds. We doubled this time, approximated it, and denoted this approximation as t_e :

$$2 \cdot 3287 = 6574 = 1 \text{ h } 49 \text{ min } 34 \text{ sec} \approx 2 \text{ h} \equiv t_e. \quad (50)$$

Having this criterion included, we were able to test only 3 of 25 test functions by *NADD* scheme for the proposed number of variables (1000, 2000, 3000, 5000, 7000, 8000, 10000, 15000, 20000, and 30000). Obtained results show obvious dominance of the acceleration properties of *ADD* method comparing to its nonaccelerated version. The notation $> t_e$ in Table 1 means that the execution time exceeds t_e .

Considering the presented results in Table 1 for all 25 given functions and all 250 tests, Table 2 actually illustrates the fact that *ADD* method has approximately 66 times lower number of iterations than *SM* method.

TABLE 1: Number of iterations needed in developing *SM*, *ADD*, and *NADD* on 25 large-scale test functions.

Test function	<i>SM</i>	<i>ADD</i>	<i>NADD</i>
Extended penalty	589	70	$> t_e$
Perturbed quadratic	111972	80	$> t_e$
Raydan 1	21125	86	$> t_e$
Diagonal 1	10417	81	$> t_e$
Diagonal 3	10574	80	$> t_e$
Generalized tridiagonal 1	278	110	$> t_e$
Extended tridiagonal 1	3560	120	$> t_e$
Extended three exponential terms	164	100	3551
Diagonal 4	80	100	$> t_e$
Extended Himmelblau	168	100	$> t_e$
Quadratic diagonal perturbed	53133	90	$> t_e$
Quadratic QF1	114510	80	$> t_e$
Extended quadratic penalty QP1	224	90	$> t_e$
Extended quadratic penalty QP2	162	80	$> t_e$
Quadratic QF2	118801	86	$> t_e$
Extended EP1	68	100	$> t_e$
Extended tridiagonal 2	584	120	$> t_e$
Arrwhead	10	102	$> t_e$
Almost perturbed quadratic	110121	80	$> t_e$
Engvall	185	100	$> t_e$
QUARTC	190	10	10
Generalized quartic	156	100	3258
Diagonal 7	90	3287	$> t_e$
Diagonal 8	96	3225	$> t_e$
Diagonal 9	11235	80	$> t_e$

TABLE 2: Average results for 25 tested functions on 250 numerical experiments for each method, *SM* and *ADD*, generated for large number of variables: 1000, 2000, 3000, 5000, 7000, 8000, 10000, 15000, 20000, and 30000.

Average value	<i>SM</i>	<i>ADD</i>
Number of iterations	22739.68	342.28

To get more clear comparison between *ADD* and *NADD*, we did additional tests for 100 times lower number of variables: 10, 20, 30, 50, 70, 80, 100, 150, 200, and 300. The contents of Table 3 display that in this case 9 of 25 test functions were achievable for testing by *NADD* scheme.

Remark 8. During the testing procedures, we were able to test Generalized quartic function for larger number of variables (1000, 2000, 3000, 5000, 7000, 8000, 10000, 15000, 20000, and 30000), but while applying *NADD* iteration for a 100 times lower set of variables on this function (specially for $n = 30$ and $n = 100$), time limiter parameter is $t_e \gg 2$ h. That is why this function is not displayed in Table 3.

According to the results of Table 3, obtained for *ADD* and *NADD* methods, we display average values for 90 test executions among the 9 functions that satisfy defined *time limiter condition* in Table 4. This table confirms even 1502

TABLE 3: Numerical results for 90 tests for each method, *ADD* and *NADD*, tested on 9 test functions, generated for a lower number of variables.

Test function	Number of iterations	
	<i>ADD</i>	<i>NADD</i>
Perturbed quadratic	80	7917
Extended tridiagonal 1	120	901248
Extended three exponential terms	100	3438
Diagonal 4	100	9036
Quadratic diagonal perturbed	102	224031
Quadratic QF1	92	8709
Almost perturbed quadratic	90	7939
Quartic	10	10
Diagonal 9	80	339

TABLE 4: Average results for 9 tested functions on 90 numerical experiments for each method, *NADD* and *ADD*, for a lower number of variables: 10, 20, 30, 50, 70, 80, 100, 150, 200, and 300.

Average value	<i>NADD</i>	<i>ADD</i>
Number of iterations	129185.22	86

times better results in favor of *ADD* method compared to its nonaccelerated dual method *NADD*.

6. Conclusion

We used the proposed form of the iteration for unconstrained optimization problems from [1] to define in similar way a double direction method for uniformly convex functions and for some strictly convex quadratic functions. The presented iterative method is an acceleration gradient descent method, constructed from the Newton method with the line search, approximating the Hessian by appropriate diagonal matrix.

The aim of constructing Accelerated Double Direction method, in short *ADD* method, is to improve the number of iterations for chosen test functions from [21] for a large number of parameters and this goal is successfully obtained. Also, important contribution of *ADD* method is the implementation of specific form of iteration introduced originated in [1].

It is proved that *ADD* is linearly convergent method for the uniformly convex functions and for the special subset of strictly convex quadratic functions.

In order to confirm the advantages of accelerated properties of *ADD* iteration, a nonaccelerated representation of *ADD* scheme, *NADD* method, is constructed. Comparative tests substantiate the enormous benefits of *ADD* method. Derived accelerated method is also compared with *SM* method whose dominance among the *AGD* method and *GD* method has already been proven in [13]. *ADD* algorithm generates multiple times better numerical results with respect to the number of iterations compared to *SM* method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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