Research Article

Number of Spanning Trees of Different Products of Complete and Complete Bipartite Graphs

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Spanning trees have been found to be structures of paramount importance in both theoretical and practical problems. In this paper we derive new formulas for the complexity, number of spanning trees, of some products of complete and complete bipartite graphs such as Cartesian product, normal product, composition product, tensor product, symmetric product, and strong sum, using linear algebra and matrix theory techniques.

1. Introduction

The number of spanning trees of a graph is an important, well-studied quantity in graph theory and appears in a number of applications. The most notable application fields are network reliability [1–4], enumerating certain chemical isomers [5], and counting of the Eulerian circuits in a graph [6]. Every connected graph has a spanning tree. A spanning tree of a graph G is a tree that (i) is a subgraph of G (i.e., that includes only edges from G) and (ii) includes every vertex of G . The most classical interest concerning a spanning tree is the number of spanning trees, also called the complexity of the graph G and denoted by $\tau(G)$. Kirchhoff [7] gave a formula for determining it, which is known as the matrix tree theorem. The spanning trees of a graph G are the value of any cofactor of the matrix $D(G) - A(G)$, where $D(G)$ is the degree matrix (the i th diagonal entry is equal to the degree of the i th vertex and the other entry is equal to zero) and $A(G)$ is the adjacency matrix of G (the entry (i, j) is equal to the number of edges between i th vertex and j th vertex), respectively. This topic is still much studied, in particular, in explicit formulas of the number of spanning trees of some special classes.

That for complete graphs is most famous among such classes; the number of spanning trees of K_n is n^{n-2} , called Cayley's formula [8]. Several proofs of Cayley's formula are known, and the most famous one is due to Prüfer [9]. The explicit formulas of the number of spanning trees are known for other classes than complete graphs: complete bipartite graphs [10–13], regular graphs [14], circulant graphs [15–19], pyramid graphs [20], and so on.

Now we introduce the following Lemma which describes a way to calculate the number of spanning trees by an extension of Kirchhoff formula.

Lemma 1 (see [21]). *Let G be a graph with n vertices. Then*

$$\tau(G) = \frac{1}{n^2} \det(nI - \overline{D} + \overline{A}), \quad (1)$$

where $\overline{A}$, $\overline{D}$ are the adjacency and degree matrices of $\overline{G}$, the complement of G , respectively, and I is the $n \times n$ unit matrix.

Lemma 2. Let $A_n(x)$ be $n \times n$ matrix, $x \geq 2$ such that

$$A_n(x) = \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix}. \quad (2)$$

Then

$$\det(A_n(x)) = (x + n - 1)(x - 1)^{n-1}. \quad (3)$$

Proof. From the definition of the circulant determinants, we have

$$\begin{aligned} \det(A_n(x)) &= \det \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix} \\ &= \prod_{j=1}^n (x + \omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}) \\ &= (x + 1 + 1 + \cdots + 1) \\ &\quad \times \prod_{j=1, \omega_j \neq 1}^n \left(x + \underbrace{\omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}}_{=-1} \right) \\ &= (x + n - 1) \times (x - 1)^{n-1}. \end{aligned} \quad (4)$$

We can generalize the above lemma as follows. $\square$

Lemma 3. Let $A, B \in F^{n \times n}$, and $\mathbb{F} \in F^{kn \times kn}$ such that

$$\mathbb{F} = \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix}. \quad (5)$$

Then

$$\det \mathbb{F} = [\det(A - B)]^{k-1} \det[A + (k-1)B]. \quad (6)$$

Lemma 4 (see [22]). Let $A \in F^{n \times n}$, $B \in F^{n \times m}$, $C \in F^{m \times n}$, and $D \in F^{m \times m}$. Assume that A and D are nonsingular matrices. Then:

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= (-1)^{nm} \det(A - BD^{-1}C) \det D \\ &= (-1)^{nm} \det A \det(D - CA^{-1}B). \end{aligned} \quad (7)$$

Formulas in Lemmas 2, 3, and 4 give some sort of symmetry in some matrices which facilitates our calculation of determinants.

2. Number of Spanning Trees of Cartesian Product of Graphs

The Cartesian product, $G_1 \times G_2$, of two graphs G_1 and G_2 is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \cup (V_1 \times E_2)]$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ [23].

Theorem 5. For $m, n \geq 1$ and $r \geq 2$, we have

$$\begin{aligned} \tau(K_r \times K_{m,n}) &= r^{r-2} m^{n-1} n^{m-1} (m+r)^{(r-1)(n-1)} \\ &\quad \times (n+r)^{(r-1)(m-1)} (m+n+r)^{r-1}. \end{aligned} \quad (8)$$

$$\tau(K_r \times K_{m,n})$$

$$= \frac{1}{(r(m+n))^2} \det(r(m+n)I - \overline{D} + \overline{A})$$

$$= \frac{1}{r^2(m+n)^2}$$

[illegible]

Using Lemma 3, we get

$$\begin{aligned}
 \tau(K_r \times K_{m,n}) &= \frac{1}{r^2(m+n)^2} \det \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix} \\
 &= \frac{1}{(r(m+n))^2} [\det(A-B)]^{r-1} [\det(A+(r-1)B)] \\
 &= \frac{1}{r^2(m+n)^2} \\
 &\quad \times \left(\det \begin{pmatrix} n+r & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & n+r & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & m+r & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & m+r \end{pmatrix} \right)^{r-1} \\
 &\quad \times \det \begin{pmatrix} n+r & r & \cdots & r & (r-1) & \cdots & \cdots & (r-1) \\ r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & n+r & (r-1) & \cdots & \cdots & (r-1) \\ (r-1) & \cdots & \cdots & (r-1) & m+r & r & \cdots & r \\ \vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\ (r-1) & \cdots & \cdots & (r-1) & r & \cdots & r & m+r \end{pmatrix} \\
 &= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^{r-1} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}.
 \end{aligned} \tag{10}$$

Using Lemma 4, we obtain

$$\begin{aligned}
\tau(K_r \times K_{m,n}) &= \frac{1}{r^2(m+n)^2} \times (\det A)^{r-1} (\det(C - B^T A^{-1} B))^{r-1} \times \det D \det(F - E^T D^{-1} E) \\
&= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} n+r & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+r \end{pmatrix}_{m \times m} \right)^{r-1} \\
&\quad \times \left(\det \begin{pmatrix} \frac{n(m+r) + (r-1)m + r^2}{n+r} & \frac{-m}{n+r} & \cdots & \frac{-m}{n+r} \\ \frac{-m}{n+r} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{n+r} \\ \frac{-m}{n+r} & \cdots & \frac{-m}{n+r} & \frac{n(m+r) + (r-1)m + r^2}{n+r} \end{pmatrix}_{n \times n} \right)^{r-1} \\
&\quad \times \det \begin{pmatrix} n+r & r & \cdots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & n+r \end{pmatrix}_{m \times m} \\
&\quad \times \det \begin{pmatrix} \frac{n(m+r) + rm^2 + (2r-1)m}{n+rm} & \frac{rn + (2r-1)m}{n+rm} & \cdots & \frac{rn + (2r-1)m}{n+rm} \\ \frac{rn + (2r-1)m}{n+rm} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{rn + (2r-1)m}{n+rm} \\ \frac{rn + (2r-1)m}{n+rm} & \cdots & \frac{rn + (2r-1)m}{n+rm} & \frac{n(m+r) + rm^2 + (2r-1)m}{n+rm} \end{pmatrix}_{n \times n} \\
&= \frac{1}{r^2(m+n)^2} (n+r)^{m(r-1)} \left(\frac{-m}{n+r} \right)^{n(r-1)} \\
&\quad \times \left(\det \begin{pmatrix} \frac{n(m+r) + (r-1)m + r^2}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+r) + (r-1)m + r^2}{-m} \end{pmatrix}_{n \times n} \right)^{r-1}
\end{aligned}$$

$$\begin{aligned}
& \times r^m \det \begin{pmatrix} \frac{n+r}{r} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n+r}{r} \end{pmatrix}_{m \times m} \times \left(\frac{rn + (2r-1)m}{n+rm} \right)^n \\
& \times \det \begin{pmatrix} \frac{n(m+r) + rm^2 + (2r-1)m}{rn + (2r-1)m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+r) + rm^2 + (2r-1)m}{rn + (2r-1)m} \end{pmatrix}_{n \times n}.
\end{aligned} \tag{11}$$

Using Lemma 2, we have

$$\begin{aligned}
& \tau(K_r \times K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} \times (n+r)^{m(r-1)} \\
& \times \left(\frac{-m}{n+r} \right)^{n(r-1)} \left[-\frac{nm + rn + (r-1)m + r^2}{m} + n - 1 \right]^{r-1} \\
& \times \left[-\frac{nm + rn + (r-1)m + r^2}{m} - 1 \right]^{(r-1)(n-1)} \\
& \times r^m \left(\frac{n+r}{r} + m - 1 \right) \left(\frac{n+r}{r} - 1 \right)^{m-1} \\
& \times \left(\frac{nr + (2r-1)m}{n+rm} \right)^n \\
& \times \left[\frac{n(m+r) + rm^2 + (2r-1)m}{rn + (2r-1)m} + n - 1 \right] \\
& \times \left[\frac{n(m+r) + rm^2 + (2r-1)m}{rn + (2r-1)m} - 1 \right]^{(n-1)} \\
&= \frac{1}{r^2(m+n)^2} \times r^{r-1} (n+r)^{(m-n)(r-1)} \times (m+n+r)^{r-1} \\
& \times (rm + rn + mn + r^2)^{(r-1)(n-1)} \\
& \times (n+rm) \times n^{m-1} \times \frac{1}{(n+rm)^n} \times r(m+n)^2 \\
& \times (n+rm)^{n-1} \times m^{n-1} \\
&= r^{r-2} \times (n+r)^{(m-n)(r-1)} \times m^{n-1} \times n^{m-1} \\
& \times (m+n+r)^{r-1} \times (rm + rn + mn + r^2)^{(r-1)(n-1)} \\
&= r^{r-2} \times m^{n-1} \times n^{m-1} \times (m+n+r)^{(r-1)(n-1)} \\
& \times (n+r)^{(r-1)(m-1)} \times (m+n+r)^{r-1}.
\end{aligned} \tag{12}$$

Specially,

$$\begin{aligned}
& \tau(K_r \times K_{n,n}) = r^{r-2} \times n^{2n-2} \times (2n+r)^{r-1} \\
& \times (n+r)^{2(r-1)(n-1)}; \quad n \geq 1.
\end{aligned} \tag{13}$$

□

3. Number of Spanning Trees of Normal Product of Graphs

The normal product, or the strong product, $G_1 \circ G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 , or u_1 is adjacent to v_1 and $u_2 = v_2$, or u_1 is adjacent to v_1 and u_2 is adjacent to v_2 , [24].

Theorem 6. For $m, n \geq 1$ and $r \geq 2$, we have

$$\begin{aligned}
& \tau(K_r \circ K_{m,n}) = r^{rm+rn-2} m^{n-1} n^{m-1} \\
& \times (m+1)^{n(r-1)} (n+1)^{m(r-1)}.
\end{aligned} \tag{14}$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}
 & \tau(K_r \circ K_{m,n}) \\
 &= \frac{1}{(r(m+n))^2} \det(r(m+n)I - \overline{D} + \overline{A}) \\
 &= \frac{1}{r^2(m+n)^2} \\
 & \quad \times \det \begin{pmatrix}
 r(n+1) & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
 1 & \cdots & 1 & r(n+1) & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
 0 & \cdots & \cdots & 0 & r(m+1) & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
 \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & r(m+1) & 0 & \cdots & \cdots & 0 \\
 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0
 \end{pmatrix}
 \end{aligned}$$

$$\begin{pmatrix}
 0 & \dots & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \vdots & \backslash & & \vdots & \vdots & \backslash & \vdots & \vdots & \backslash & \vdots \\
 \vdots & \backslash & & \vdots & \vdots & \backslash & \vdots & \vdots & \backslash & \vdots \\
 0 & \dots & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 1 & \dots & 1 & \dots & \dots & \dots & \dots & \dots & \dots \\
 1 & \backslash & & \vdots & \vdots & \backslash & \vdots & \vdots & \backslash & \vdots \\
 \vdots & \backslash & & 1 & \vdots & \backslash & \vdots & \vdots & \backslash & \vdots \\
 1 & \dots & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\
 \dots & \dots & \dots & \dots & 0 & 1 & \dots & 1 & 0 & \dots & \dots & 0 \\
 \vdots & \backslash & & \vdots & 1 & \backslash & \vdots & \vdots & \backslash & \vdots \\
 \vdots & \backslash & & \vdots & \vdots & \backslash & 1 & \vdots & \backslash & \vdots \\
 \dots & \dots & \dots & \dots & 1 & \dots & 1 & 0 & 0 & \dots & \dots & 0 \\
 \dots & \dots & \dots & \dots & 0 & \dots & \dots & 0 & 0 & 1 & \dots & 1 \\
 \vdots & \backslash & & \vdots & \vdots & \backslash & \vdots & 1 & \backslash & \vdots \\
 \vdots & \backslash & & \vdots & \vdots & \backslash & \vdots & \vdots & \backslash & 1 \\
 \dots & \dots & \dots & \dots & 0 & \dots & \dots & 0 & 1 & \dots & 1 & 0 \\
 0 & \dots & \dots & 0 & r(n+1) & 1 & \dots & 1 & 0 & \dots & \dots & 0 \\
 \vdots & \backslash & & \vdots & 1 & \backslash & \vdots & \vdots & \backslash & \vdots \\
 \vdots & \backslash & & \vdots & \vdots & \backslash & 1 & \vdots & \backslash & \vdots \\
 0 & \dots & \dots & 0 & 1 & \dots & 1 & r(n+1) & 0 & \dots & \dots & 0 \\
 0 & 1 & \dots & 1 & 0 & \dots & \dots & 0 & r(m+1) & 1 & \dots & 1 \\
 1 & \backslash & & \vdots & \vdots & \backslash & \vdots & 1 & \backslash & \vdots \\
 \vdots & \backslash & & 1 & \vdots & \backslash & \vdots & \vdots & \backslash & 1 \\
 1 & \dots & 1 & 0 & 0 & \dots & \dots & 0 & 1 & \dots & 1 & r(m+1)
 \end{pmatrix}.$$

(15)

Using Lemma 3, we get

$$\begin{aligned}
 & \tau(K_r \circ K_{m,n}) \\
 &= \frac{1}{r^2(m+n)^2} \times \det \begin{pmatrix}
 A & B & \dots & \dots & \dots & B \\
 B & A & \backslash & & \backslash & \vdots \\
 \vdots & \backslash & \backslash & & \backslash & \vdots \\
 \vdots & \backslash & \backslash & & \backslash & \vdots \\
 \vdots & \backslash & \backslash & & \backslash & \vdots \\
 B & \dots & \dots & \dots & A & B \\
 B & \dots & \dots & \dots & B & A
 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(r(m+n))^2} [\det(A-B)]^{r-1} [\det(A+(r-1)B)] \\
&= \frac{1}{r^2(m+n)^2} \det \left(\begin{pmatrix} r(n+1) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & r(n+1) & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & r(m+1) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & r(m+1) \end{pmatrix} \right)^{r-1} \\
&\quad \times \det \left(\begin{pmatrix} r(n+1) & r & \cdots & r & 0 & \cdots & 0 \\ r & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r & \vdots & \ddots & \vdots \\ r & \cdots & r & r(n+1) & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & r(m+1) & r & \cdots & r \\ \vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\ 0 & \cdots & \cdots & 0 & r & \cdots & r & r(m+1) \end{pmatrix} \right) \\
&= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} r(n+1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r(n+1) \end{pmatrix}_{m \times m} \right)^{r-1} \\
&\quad \times \left(\det \begin{pmatrix} r(m+1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r(m+1) \end{pmatrix}_{n \times n} \right)^{r-1} \\
&\quad \times \det \begin{pmatrix} r(n+1) & r & \cdots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & r(n+1) \end{pmatrix}_{m \times m} \times \det \begin{pmatrix} r(m+1) & r & \cdots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & r(m+1) \end{pmatrix}_{n \times n}.
\end{aligned} \tag{16}$$

Using Lemma 2, we obtain

$$\begin{aligned}
&\tau(K_r \circ K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} (r(n+1))^{m(r-1)} (r(m+1))^{n(r-1)} \\
&\quad \times (r^m \times (n+m) \times n^{m-1}) (r^n \times (n+m) \times m^{n-1}) \\
&= r^{r(m+n)-2} m^{n-1} n^{m-1} (m+1)^{n(r-1)} (n+1)^{m(r-1)}.
\end{aligned} \tag{17}$$

Specially,

$$\tau(K_r \circ K_{n,n}) = r^{2(rn-1)} \times n^{2(n-1)} \times (n+1)^{2n(r-1)}; \quad n \geq 1. \tag{18}$$

4. Number of Spanning Trees of Composition Product of Graphs

The composition, or lexicographic product, $G_1[G_2]$, of two graphs G_1 and G_2 is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 [24].

Theorem 7. For $m, n \geq 1$ and $r \geq 2$, we get

$$\times (rm + rn - n)^{r(n-1)}. \quad (19)$$

$$\tau(K_r[K_{m,n}]) = r^{2(r-1)}(m+n)^{2(r-1)}(rm+rn-m)^{r(m-1)}$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} & \tau(K_r[K_{m,n}]) \\ &= \frac{1}{r^2(m+n)^2} \\ & \times \left(\det \begin{pmatrix} rn+(r-1)m+1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & rn+(r-1)m+1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & rm+(r-1)n+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & rm+(r-1)n+1 \end{pmatrix} \right)^r \\ &= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} rn+(r-1)m+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & rn+(r-1)m+1 \end{pmatrix}_{m \times m} \right)^r \\ & \times \left(\det \begin{pmatrix} rm+(r-1)n+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & rm+(r-1)n+1 \end{pmatrix}_{n \times n} \right)^r. \end{aligned} \quad (20)$$

Using Lemma 2, we obtain

$$\begin{aligned} \tau(K_r[K_{m,n}]) &= r^{2(r-1)}(m+n)^{2(r-1)} \\ & \times (rm+rn-m)^{r(m-1)}(rm+rn-n)^{r(n-1)}. \end{aligned} \quad (21)$$

Specially,

$$\tau(K_r[K_{n,n}]) = (2r)^{2(r-1)} n^{2(rn-1)} (2r-1)^{2r(n-1)}; \quad n \geq 1. \quad (22)$$

□

5. Complexity of Tensor Product of Graphs

The tensor product, or Kronecker product, $G_1 \otimes G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [24].

Theorem 8. For $m, n \geq 1$ and $r \geq 2$, we have

$$\tau(K_r \otimes K_{m,n}) = r^{r-2}(r-2)^{r-1}(r-1)^{r(m+n-2)+1} m^{rn-1} n^{rm-1}. \quad (23)$$

$$\begin{aligned} & \tau(K_r \otimes K_{m,n}) \\ &= \frac{1}{(r(m+n))^2} \det(r(m+n)I - \overline{D} + \overline{A}) \\ &= \frac{1}{r^2(m+n)^2} \end{aligned}$$

[illegible]

$$\begin{pmatrix}
 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
 \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
 0 & \cdots & \cdots & 0 & (r-1)n+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
 \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots \\
 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & (r-1)n+1 & 1 & \cdots & \cdots & 1 \\
 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & (r-1)m+1 & 1 & \cdots & 1 \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & (r-1)m+1
 \end{pmatrix}.$$

(24)

Using Lemma 3, we obtain

$$\begin{aligned}
 & \tau(K_r \otimes K_{m,n}) \\
 &= \frac{1}{r^2(m+n)^2} \det \begin{pmatrix}
 A & B & \cdots & \cdots & \cdots & B \\
 B & A & \ddots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & A & B \\
 B & \cdots & \cdots & \cdots & B & A
 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(r(m+n))^2} [\det(A-B)]^{r-1} [\det(A+(r-1)B)] \\
&= \frac{1}{r^2(m+n)^2} \\
&\quad \times \left(\det \begin{pmatrix} (r-1)n & 0 & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (r-1)n & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & (r-1)m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 & (r-1)m \end{pmatrix} \right)^{r-1} \\
&\quad \times \det \begin{pmatrix} (r-1)n+r & r & \cdots & r & 1 & \cdots & \cdots & 1 \\ r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & (r-1)n+r & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & (r-1)m+r & r & \cdots & r \\ \vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\ 1 & \cdots & \cdots & 1 & r & \cdots & r & (r-1)m+r \end{pmatrix} \\
&= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^{r-1} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}.
\end{aligned} \tag{25}$$

Using Lemma 4, we obtain

$$\begin{aligned}
&\tau(K_r \otimes K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} \times (\det A)^{r-1} (\det(C - B^T A^{-1} B))^{r-1} \times \det D \det(F - E^T D^{-1} E) \\
&= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} (r-1)n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (r-1)n \end{pmatrix}_{m \times m} \right)^{r-1}
\end{aligned}$$

$$\times \det \left(\begin{array}{ccc} \frac{[(r-1)^2 m + (r^2 - r)] n + (r^2 - r) m^2 + (r^2 - 1) m}{rm + (r-1)n} & \frac{(r^2 - 1) m + (r^2 - r) n}{rm + (r-1)n} & \\ \frac{(r^2 - 1) m + (r^2 - r) n}{rm + (r-1)n} & & \ddots \\ \vdots & & \ddots \\ \frac{(r^2 - 1) m + (r^2 - r) n}{rm + (r-1)n} & & \dots \\ \dots & \frac{(r^2 - 1) m + (r^2 - r) n}{rm + (r-1)n} & \\ \ddots & \vdots & \\ \ddots & \frac{(r^2 - 1) m + (r^2 - r) n}{rm + (r-1)n} & \\ \frac{(r^2 - 1) m + (r^2 - r) n}{rm + (r-1)n} & \frac{[(r-1)^2 m + (r^2 - r)] n + (r^2 - r) m^2 + (r^2 - 1) m}{rm + (r-1)n} & \end{array} \right)_{n \times n}$$

$$\times \left(\det \left(\begin{array}{ccc} \frac{m[(r-1)^2 n - 1]}{(r-1)n} & \frac{-m}{(r-1)n} & \dots & \frac{-m}{(r-1)n} \\ \frac{-m}{(r-1)n} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{(r-1)n} \\ \frac{-m}{(r-1)n} & \dots & \frac{-m}{(r-1)n} & \frac{m[(r-1)^2 n - 1]}{(r-1)n} \end{array} \right)_{n \times n} \right)^{r-1}$$

$$\times \det \left(\begin{array}{ccc} (r-1)n + r & r & \dots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \dots & r & (r-1)n + r \end{array} \right)_{m \times m}$$

$$= \frac{1}{r^2(m+n)^2} ((r-1)n)^{m(r-1)}$$

$$\times \left(\frac{-m}{(r-1)n} \right)^{n(r-1)} \left(\det \left(\begin{array}{ccc} -[(r-1)^2 n - 1] & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -[(r-1)^2 n - 1] \end{array} \right)_{n \times n} \right)^{r-1}$$

$$\begin{aligned}
& \times r^m \det \left(\begin{array}{cccc} \frac{(r-1)n+r}{r} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{(r-1)n+r}{r} \end{array} \right)_{m \times m} \times \left(\frac{(r^2-1)m + (r^2-r)n}{rm + (r-1)n} \right)^n \\
& \times \det \left(\begin{array}{cccc} \frac{[(r-1)^2m + (r^2-r)]n + (r^2-r)m^2 + (r^2-1)m}{(r^2-1)m + (r^2-r)n} & 1 & \cdots & 1 \\ & 1 & & \vdots \\ & \vdots & & 1 \\ & 1 & \cdots & \frac{[(r-1)^2m + (r^2-r)]n + (r^2-r)m^2 + (r^2-1)m}{(r^2-1)m + (r^2-r)n} \end{array} \right)_{n \times n}.
\end{aligned} \tag{26}$$

Using Lemma 2 yields

$$\begin{aligned}
& \tau(K_r \otimes K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} \times (r-1)^{(r-1)(m-n)} \\
& \times n^{(r-1)(m-n)} \times m^{n(r-1)} \times [rn(r-2)]^{r-1} \\
& \times [n(r-1)^2]^{(r-1)(n-1)} \times [rm + (r-1)n] \\
& \times [(r-1)n]^{m-1} \times \frac{1}{(rm + (r-1)n)^n} \\
& \times [r(r-1)(m+n)^2] \\
& \times [m(r-1)(rm + (r-1)n)]^{n-1} \\
&= r^{r-2} \times (r-1)^{r(m+n)-2r+1} \times (r-2)^{r-1} \\
& \times m^{nr-1} \times n^{mr-1}.
\end{aligned} \tag{27}$$

Specially,

$$\begin{aligned}
& \tau(K_r \otimes K_{n,n}) \\
&= r^{r-2} \times (r-2)^{r-1} \times (r-1)^{2r(n-1)+1} \times n^{2(nr-1)}; \quad n \geq 1.
\end{aligned} \tag{28}$$

6. Number of Spanning Trees of Symmetric Product of Graphs

The symmetric product, $G_1 \oplus G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \oplus G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \oplus G_2$ if and only if either u_1 is adjacent to v_1 in G_1 and u_2 is not adjacent to v_2 in G_2 or u_1 is not adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [24].

Theorem 9. For $m, n \geq 1$ and $r \geq 2$, we have

$$\begin{aligned}
& \tau(K_r \oplus K_{m,n}) \\
&= r^{r-2} \times ((r-1)m + n)^{r(m-1)} \\
& \times ((r-1)n + m)^{r(n-1)} \times (m^2 + n^2 + rmn)^{r-1}.
\end{aligned} \tag{29}$$

$$\tau(K_r \oplus K_{m,n})$$

$$= \frac{1}{r^2(m+n)^2}$$

$$\times \det \left(\begin{array}{ccccccc} (r-1)m+n+1 & 1 & \cdots & 1 & 0 & \cdots \cdots & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & (r-1)m+n+1 & 0 & \cdots \cdots & 0 \\ 0 & \cdots \cdots & 0 & (r-1)n+m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots \cdots & 0 & 1 & \cdots & 1 & (r-1)n+m+1 \\ 0 & \cdots \cdots & 0 & 1 & \cdots \cdots & 1 & \cdots \cdots \cdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots \cdots & 0 & 1 & \cdots \cdots & 1 & \cdots \cdots \cdots \\ 1 & \cdots \cdots & 1 & 0 & \cdots \cdots & 0 & \cdots \cdots \cdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots \cdots & 1 & 0 & \cdots \cdots & 0 & \cdots \cdots \cdots \\ \cdots & \cdots \cdots & \cdots & \cdots & \cdots \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots \cdots & \cdots & \cdots & \cdots \cdots & \cdots & 0 \\ \cdots & \cdots \cdots & \cdots & \cdots & \cdots \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots \cdots & \cdots & \cdots & \cdots \cdots & \cdots & 1 \end{array} \right)$$

[illegible]

Using Lemma 3, we obtain

$$\tau(K_r \oplus K_{m,n}) = \frac{1}{r^2(m+n)^2} \det \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix}$$

$$= \frac{1}{(r(m+n))^2} [\det(A-B)]^{r-1} [\det(A+(r-1)B)]$$

$$= \frac{1}{r^2(m+n)^2}$$

$$\times \left(\det \begin{pmatrix} (r-1)m+n+1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & (r-1)m+n+1 & -1 & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & (r-1)n+m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & \cdots & \cdots & -1 & 1 & \cdots & 1 & (r-1)n+m+1 \end{pmatrix} \right)^{r-1}$$

$$\times \det \begin{pmatrix} (r-1)m+n+1 & 1 & \cdots & 1 & (r-1) & \cdots & (r-1) \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & (r-1)m+n+1 & (r-1) & \cdots & (r-1) \\ (r-1) & \cdots & \cdots & (r-1) & (r-1)n+m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ (r-1) & \cdots & \cdots & (r-1) & 1 & \cdots & 1 & (r-1)n+m+1 \end{pmatrix}$$

$$= \frac{1}{r^2(m+n)^2} \left(\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^{r-1} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}.$$

(31)

Using Lemma 4, we get

$$\begin{aligned} & \tau(K_r \oplus K_{m,n}) \\ &= \frac{1}{r^2(m+n)^2} \times (\det A)^{r-1} (\det(C - B^T A^{-1} B))^{r-1} \times \det D \det(F - E^T D^{-1} E) \\ &= \frac{(rm+n)^{r-1} ((r-1)m+n)^{(r-1)(m-1)}}{r^2(n+m)^2} \end{aligned}$$

$$\begin{aligned}
& \times \left(\det \begin{pmatrix} \frac{(r-1)n^2 + [(r^2 - r + 1)m + 1]n + rm^2 + (r-1)m}{n + rm} & \frac{n + (r-1)m}{n + rm} \\ \frac{n + (r-1)m}{n + rm} & \ddots \\ \vdots & \ddots \\ \frac{n + (r-1)m}{n + rm} & \dots \end{pmatrix} \right. \\
& \quad \left. \begin{pmatrix} \dots & \frac{n + (r-1)m}{n + rm} \\ \ddots & \vdots \\ \ddots & \frac{n + (r-1)m}{n + rm} \\ \frac{n + (r-1)m}{n + rm} & \frac{(r-1)n^2 + [(r^2 - r + 1)m + 1]n + rm^2 + (r-1)m}{n + rm} \end{pmatrix}_{n \times n} \right)^{r-1} \\
& \times (rm + n)((r-1)m + n)^{m-1} \\
& \times \det \begin{pmatrix} \frac{(r-1)n^2 + [(r^2 - r + 1)m + 1]n + rm^2 + (-r^2 + 3r - 1)m}{n + rm} & \frac{n + (-r^2 + 3r - 1)m}{n + rm} \\ \frac{n + (-r^2 + 3r - 1)m}{n + rm} & \ddots \\ \vdots & \ddots \\ \frac{n + (-r^2 + 3r - 1)m}{n + rm} & \dots \end{pmatrix} \\
& \quad \begin{pmatrix} \dots & \frac{n + (-r^2 + 3r - 1)m}{n + rm} \\ \ddots & \vdots \\ \ddots & \frac{n + (-r^2 + 3r - 1)m}{n + rm} \\ \frac{n + (-r^2 + 3r - 1)m}{n + rm} & \frac{(r-1)n^2 + [(r^2 - r + 1)m + 1]n + rm^2 + (-r^2 + 3r - 1)m}{n + rm} \end{pmatrix}_{n \times n} \\
& = \frac{(rm + n)^{r-1}((r-1)m + n)^{(r-1)(m-1)}}{r^2(n + m)^2} \times \left(\frac{n + (r-1)m}{n + rm} \right)^{n(r-1)}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\det \begin{pmatrix} \frac{(r-1)n^2 + [(r^2 - r + 1)m + 1]n + rm^2 + (r-1)m}{n + (r-1)m} & 1 & \cdots & 1 \\ & 1 & \ddots & \vdots \\ & \vdots & \ddots & 1 \\ & 1 & \cdots & 1 \end{pmatrix} \right)^{r-1} \\
& \times (rm + n)((r-1)m + n)^{m-1} \times \left(\frac{n + (-r^2 + 3r - 1)m}{n + rm} \right)^n \\
& \times \det \begin{pmatrix} \frac{(r-1)n^2 + [(r^2 - r + 1)m + 1]n + rm^2 + (-r^2 + 3r - 1)m}{n + (-r^2 + 3r - 1)m} & 1 & \cdots & 1 \\ & 1 & \ddots & \vdots \\ & \vdots & \ddots & 1 \\ & 1 & \cdots & 1 \end{pmatrix}^{n \times n} \quad (32)
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
& \tau(K_r \oplus K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} \times (rm + n)^{r-1} \times ((r-1)m + n)^{(r-1)(m-1)} \\
& \times \left(\frac{n + (r-1)m}{n + rm} \right)^{(r-1)n} \times \frac{1}{(n + (r-1)m)^{(r-1)n}} \\
& \times (rn^2 + rm^2 + r^2nm)^{r-1} \\
& \times (rm^2 + (r^2 - r + 1)nm + (r-1)n^2)^{(r-1)(n-1)} \\
& \times (rm + n) \times ((r-1)m + n)^{m-1} \\
& \times \left(\frac{n + (-r^2 + 3r - 1)m}{n + rm} \right)^n \\
& \times \frac{1}{(n + (-r^2 + 3r - 1)m)^n} \times (rn^2 + rm^2 + 2rmn) \\
& \times (rm^2 + (r^2 - r + 1)nm + (r-1)n^2)^{n-1} \\
& = r^{r-2} \times ((r-1)m + n)^{r(m-1)} \\
& \times ((r-1)n + m)^{r(n-1)} \times (m^2 + n^2 + rmn)^{r-1}. \quad (33)
\end{aligned}$$

Specially,

$$\tau(K_r \oplus K_{n,n}) = r^{2rn-r-2} \times n^{2rn-2} \times (r+2)^{r-1}; \quad n \geq 1. \quad (34)$$

□

7. Number of Spanning Trees of Strong Sum of Graphs

The strong sum, $G_1 * G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 * G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 * G_2$ if and only if u_2 is adjacent to v_2 in G_2 and either u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ [24].

Theorem 10. For $m, n \geq 1$ and $r \geq 2$, we have

$$\tau(K_r * K_{m,n}) = r^{(m+n)r-2} \times m^{rn-1} \times n^{rm-1}. \quad (35)$$

$$\tau(K_r * K_{m,n})$$

$$= \frac{1}{r^2(m+n)^2}$$

(36)

Using Lemma 3, we obtain

$$\begin{aligned}
 \tau(K_r * K_{m,n}) &= \frac{1}{r^2(m+n)^2} \det \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix} \\
 &= \frac{1}{(r(m+n))^2} [\det(A-B)]^{r-1} [\det(A+(r-1)B)] \\
 &= \frac{1}{r^2(m+n)^2} \det \left(\det \begin{pmatrix} rn & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & rn & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & rm & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & rm \end{pmatrix} \right)^{r-1} \\
 &\quad \times \det \begin{pmatrix} r(n+1) & r & \cdots & r & 0 & \cdots & \cdots & 0 \\ r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & r(n+1) & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & r(m+1) & r & \cdots & r \\ \vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\ 0 & \cdots & \cdots & 0 & r & \cdots & r & r(m+1) \end{pmatrix} \\
 &= \frac{1}{r^2(m+n)^2} [(rn)^m (rm)^n]^{r-1} \times r^m \det \begin{pmatrix} n+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & n+1 \end{pmatrix}_{m \times m} \\
 &\quad \times r^n \det \begin{pmatrix} m+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & m+1 \end{pmatrix}_{n \times n}.
 \end{aligned}$$

Using Lemma 2, we get

$$\begin{aligned}\tau(K_r * K_{m,n}) &= \frac{1}{r^2(m+n)^2} \times r^{(m+n)(r-1)} \\ &\quad \times n^{m(r-1)} \times m^{n(r-1)} \times r^{(m+n)} \\ &\quad \times (n+1+m-1)(n+1-1)^{m-1} \\ &\quad \times (m+1+n-1)(m+1-1)^{n-1} \\ &= r^{(m+n)r-2} \times m^{nr-1} \times n^{mr-1}.\end{aligned}\quad (38)$$

Specially,

$$\tau(K_r * K_{n,n}) = (nr)^{2(rn-1)}; \quad n \geq 1. \quad (39)$$

□

8. Conclusion

Driving formulas for different types of graphs can prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequence related to network reliability. Some computationally hard problems, such as the Steiner tree problem and the traveling salesperson problem, can be solved approximately by using spanning trees [25]. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and lemmas and their proofs.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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