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Research Article

Robust Stabilization and Disturbance Rejection of Positive Systems with Time-Varying Delays and Actuator Saturation

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This paper focuses on the problems of robust stabilization and disturbance rejection for a class of positive systems with time-varying delays and actuator saturation. First, a convex hull representation is used to describe the saturation characteristics. By constructing an appropriate copositive type Lyapunov functional, we give sufficient conditions for the existence of a state feedback controller such that the closed-loop system is positive and asymptotically stable at the origin of the state space with a domain of attraction. Then, the disturbance rejection performance analysis in the presence of actuator saturation is developed via L_1-gain. The design method is also extended to investigate the problem of L_1-gain analysis for uncertain positive systems with time-varying delays and actuator saturation. Finally, three examples are provided to demonstrate the effectiveness of the proposed method.

1. Introduction

Positive systems, whose states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative, are of fundamental importance to multitudinous applications in areas such as economics, biology, sociology, and communications [1–4]. Recently, positive systems have been investigated by many researchers [5–10]. The linear copositive Lyapunov functional approach has been used to study the stability of positive systems due to the fact that it is less conservative than the traditional quadratic Lyapunov functional method [11]. It is well known that, in real engineering, time-delays are involved in many subjects and fields, such as mechanics, medicine, chemistry, physics, engineering, and control theory [12]. The existence of time-delay may lead to the deterioration of system performance and instability. Many results have been reported for time-delay systems [13–18], and a few results on positive systems with time-delay have appeared in [19–21].

On the other hand, in practice, the reaction to exogenous signals is not instantaneous, and the outputs will be inevitably affected. Because of the peculiar nonnegative property of positive systems, it is natural to evaluate the size of such systems via the L_1-gain in terms of the ratio of input and output signals [19]. Some results on L_1-gain analysis and control of positive systems have been reported in the literature [19, 22].

Recently, several works on positive systems have been done [12, 19, 23–28]. It should be pointed out that, in almost all available results on positive systems, it has been assumed that the actuator provides unlimited amplitude signal. However, actuator saturation is commonly unavoidable in almost all practical control systems because of the existence of physical, technological, or even safety constraints [29, 30]. Actuator saturation can lead to performance degradation of the closed-loop system; even more, it will make the additional stable closed-loop system unstable for large perturbations. Thus, more and more attention has been focused on the analysis and control synthesis for dynamic systems with actuator saturation for a long time and many methods have been developed to deal with actuator saturation [31–42]. To the best of our knowledge, few results on positive systems with actuator saturation have been proposed [43, 44]. In addition, because of the phenomena of actuator saturation nonlinearities and the peculiar nonnegative property of positive systems, the research of positive systems with actuator

saturation becomes more difficult for both analysis and synthesis tasks.

In this paper, we focus our attention on the investigation of robust stabilization and disturbance rejection for a class of positive systems with time-varying delays and actuator saturation. The main contributions of this paper lie in three aspects. First, a convex hull representation is used to describe the saturation behavior, and a domain of attraction, which is different from the ellipsoid, is for the first time proposed for positive systems. Secondly, by constructing a copositive type Lyapunov functional, a state-feedback controller design scheme is developed to guarantee the stability with L_1-gain performance of the resulting closed-loop systems. Thirdly, the proposed controller design method is further extended to the case of uncertain positive systems.

The remainder of this paper is organized as follows. In Section 2, the necessary definitions and lemmas are reviewed. In Section 3, sufficient conditions for the existence of L_1-gain controller are presented. An extension of the obtained results to uncertain positive systems with time-varying delays and actuator saturation is given in Section 4. Three examples are provided to illustrate the feasibility of the proposed method in Section 5. Concluding remarks are given in Section 6.

Notation. In this paper, $A \succeq 0 (\preceq 0)$ means that all the entries of matrix A are nonnegative (nonpositive); $A \succ 0 (\prec 0)$ means that all the entries of A are positive (negative); $A \succ B (A \succeq B)$ means that $A - B \succ 0 (A - B \succeq 0)$; A^T means the transpose of matrix A; $R(R^+)$ is the set of all real (positive real) numbers; $R^n(R^n_+)$ is the n-dimensional real (positive) vector space; $R^{n\times k}$ is the set of all real matrices of dimension $(n\times k)$; Z_+ refers to the set of all positive integers. For the vector $x \in R^n$, 1-norm is denoted by $\|x\| = \sum_{l=1}^{n} |x_l|$, where x_l is the lth element of x; $1_n \in R^n$ denotes a column vector with n rows containing only 1 entry; $L_1[t_0, \infty)$ is the space of absolute integrable vector-valued functions on $[t_0, \infty)$; that is, we say $z : [t_0, \infty) \to R^k$ is in $L_1[t_0, \infty)$ if $\int_{t_0}^{\infty} \|z(t)\| dt < \infty$; $H_\kappa, \kappa = 1, 2, \ldots, m$, stands for the κth column of H_κ. For a vector $v \in R^n_+$ and a positive scalar ξ, $K(v, \xi) := \{x \in R^n_+, x^T v \le \xi\}$ denotes a domain of attractive regions, and for a matrix $H \in R^{m\times n}$, $L(H) := \{x \in R^n : |H_\kappa x| \le 1, \kappa = 1, 2, \ldots, m\}$ denotes a linear region.

2. Problem Statements and Preliminaries

Consider the following system with time-varying delays:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d(t)) + Ew(t), \\ x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $z(t) \in R^q$ is the controlled output vector and $w(t) \in R^p$ is the disturbance input which belongs to $L_1[t_0, \infty)$, $\phi(\theta)$ is the initial condition on $[-\tau, 0]$, $\tau > 0$, t_0 is the initial time, A, A_d, B, C, and E are constant matrices of appropriate dimensions, and $d(t)$ denotes the time-varying delay satisfying

$$0 \le d(t) \le \tau, \qquad \dot{d}(t) \le d, \tag{2}$$

where τ and d are known constants.

Definition 1 (see [24]). System (1) is said to be positive if for any initial condition $\phi(\theta) \succeq 0$, $\theta \in [-\tau, 0]$, and any inputs $w(t) \succeq 0$, it satisfies $x(t) \succeq 0$ and $z(t) \succeq 0$, $t \ge t_0$.

Definition 2 (see [45]). A is called a Metzler matrix if its off-diagonal entries are nonnegative.

Lemma 3 (see [24]). *System (1) is positive if and only if A is a Metzler matrix, and $A_d \succeq 0$, $E \succeq 0$, and $C \succeq 0$.*

Definition 4 (see [19]). For a given positive scalar γ, system (1) is said to have an L_1-gain performance level γ if the following conditions hold.

(a) System (1) is asymptotically stable when $w(t) = 0$.

(b) Under the zero initial condition, that is, $\phi(\theta) = 0$, $\theta \in [-\tau, 0]$, system (1) satisfies

$$\int_{t_0}^{\infty} \|z(t)\| dt \le \gamma \int_{t_0}^{\infty} \|w(t)\| dt, \quad w(t) \ne 0. \tag{3}$$

Now let us consider the following system subject to actuator saturation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d(t)) + B\,\mathrm{sat}(u(t)) + Ew(t), \\ x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) &= Cx(t), \end{aligned} \tag{4}$$

where $u(t) \in R^m$ is the control input vector.

The function $\mathrm{sat}(\cdot) : R^m \to R^m$ is the saturation function which is defined as

$$\mathrm{sat}(u) = \left[\mathrm{sat}(u_1) \;\; \mathrm{sat}(u_2) \;\; \cdots \;\; \mathrm{sat}(u_m)\right]^T, \tag{5}$$

where

$$\mathrm{sat}(u_j) = \mathrm{sgn}(u_j) \min\{1, |u_j|\}, \quad \forall j = 1, \ldots, m. \tag{6}$$

Let Ω be the set of all diagonal matrices in $R^{m\times m}$ with diagonal elements that are either 1 or 0; then there are 2^m elements D_i in Ω, and for each $i = 1, 2, \ldots, 2^m$, $D_i^- = I - D_i$ is also an element in Ω.

Lemma 5 (see [46]). *Given F and H in $R^{m\times n}$, then*

$$\mathrm{sat}(Fx) \in \mathrm{co}\{D_i Fx + D_i^- Hx : i = 1, 2, \ldots, 2^m\}, \tag{7}$$

for all $x \in R^n$ *satisfying* $|H_\kappa x| \le 1$, $\kappa = 1, 2, \ldots, m$, *and* co$\{\cdot\}$ *represents the convex hell. Consequently,* sat(Fx) *can be expressed as*

$$\operatorname{sat}(Fx) = \sum_{i=1}^{2^m} \eta_i \left(D_i F + D_i^- H\right) x, \tag{8}$$

where $\sum_{i=1}^{2^m} \eta_i = 1$ *with* $0 \le \eta_i \le 1$.

Consider the following state-feedback control law:

$$u(t) = Fx(t), \tag{9}$$

where F *is a gain matrix to be determined.*

From Lemma 5, it is clear that sat$(Fx(t))$ satisfies (8) when $x \in L(H)$. Applying controller (9) to system (4) yields the closed-loop system:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2^m} \eta_i \left[\left(A + BD_i F + BD_i^- H\right) x(t) \right. \\ &\quad \left. + A_d x(t - d(t)) + Ew(t)\right], \\ x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) &= Cx(t). \end{aligned} \tag{10}$$

By Lemma 3, $\sum_{i=1}^{2^m} \eta_i (A + BD_i F + BD_i^- H)$ should be Metzler matrices to ensure the positivity of system (10).

The aim of the paper is to determine the controller gain matrix F such that the resulting closed-loop system (10) is positive and asymptotically stable with an L_1-gain performance.

3. Main Results

3.1. Stability Analysis. In this section, we firstly consider the stability of the closed-loop system (10) with $w(t) = 0$.

Theorem 6. *Given a matrix* $H \in R^{m \times n}$, *if there exist vectors* $v, \upsilon \in R_+^n$, *and a matrix* $F \in R^{m \times n}$, *such that, for* $i = 1, 2, \ldots, 2^m$,

$$A + BD_i F + BD_i^- H \text{ are Metzler matrices}, \tag{11}$$

$$\left(A + BD_i F + BD_i^- H\right)^T v + \upsilon \prec 0, \tag{12}$$

$$A_d^T v - (1 - d)\upsilon \prec 0, \tag{13}$$

$$v \succeq \left|H_\kappa\right|^T, \quad \kappa = 1, 2, \ldots, m, \tag{14}$$

then the closed-loop system (10) with $w(t) = 0$ *is positive and asymptotically stable for any initial states satisfying*

$$\begin{aligned} \Gamma(\phi) = (1 + \tau)\left(\max_{r \in \underline{n}} v_r + \max_{r \in \underline{n}} \upsilon_r\right) \sup_{\theta \in [-\tau, 0]} \left|\phi(\theta)\right| \le 1, \\ \underline{n} = \{1, 2, \ldots, n\}, \end{aligned} \tag{15}$$

where $v_r(\upsilon_r)$ *is the* r*th element of* $v(\upsilon)$.

Proof. For any $x(t) \in K(v, 1)$, that is, for any $x(t) \in R_n^+$ satisfying $x^T(t) v \le 1$, we have from (14) that $|H_\kappa x(t)| \le 1$, $\kappa = 1, 2, \ldots, m$, that is, $x \in L(H)$; therefore, $K(v, 1) \subset L(H)$. When $w(t) = 0$, system (10) can be written by the following representation:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2^m} \eta_i \left[\left(A + BD_i F + BD_i^- H\right) x(t) \right. \\ &\quad \left. + A_d x(t - d(t))\right], \\ x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) &= Cx(t). \end{aligned} \tag{16}$$

By Lemma 3, it is easy to get from (11) that system (10) is positive. Choose the following copositive type Lyapunov functional candidate for system (10):

$$V(t) = V_1(t) + V_2(t), \tag{17}$$

where

$$\begin{aligned} V_1(t) &= x^T(t) v, \\ V_2(t) &= \int_{t-d(t)}^{t} x^T(s) \upsilon \, ds, \end{aligned} \tag{18}$$

and v, υ are positive vectors to be determined.

When $w(t) = 0$, along the trajectory of system (10), we have, $\forall x(t) \in K(v, 1)$,

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{2^m} \eta_i \left[x^T(t) \left(A + BD_i F + BD_i^- H\right)^T v \right. \\ &\quad \left. + x^T(t - d(t)) A_d^T v\right], \\ \dot{V}_2(t) &= x^T(t) \upsilon - \left(1 - \dot{d}(t)\right) x^T(t - d(t)) \upsilon \\ &\le x^T(t) \upsilon - (1 - d) x^T(t - d(t)) \upsilon. \end{aligned} \tag{19}$$

It follows that

$$\dot{V}(t) \le \sum_{i=1}^{2^m} \eta_i \left\{ x^T(t) \left[\left(A + BD_iF + BD_i^- H\right)^T v + v \right] + x^T(t-d(t)) \left(A_d^T v - (1-d) v\right) \right\}, \quad \forall x(t) \in K(v,1). \tag{20}$$

Combining (12)–(14), we obtain

$$\dot{V}(t) < 0, \quad \forall x(t) \in K(v,1). \tag{21}$$

Therefore, system (10) with $w(t) = 0$ is locally asymptotically stable.

Furthermore, it can be obtained from (17) that

$$\begin{aligned} x^T(t) v &\le V(t), \\ V(t_0) &= x^T(t_0) v + \int_{t_0-d(t_0)}^{t_0} x^T(s) v \, ds \\ &\le x^T(t_0) v + \tau \sup_{\theta\in[-\tau,0]} \phi^T(\theta) v \\ &\le (1+\tau)\left(\max_{r\in\underline{n}} v_r + \max_{r\in\underline{n}} v_r\right) \sup_{\theta\in[-\tau,0]} |\phi(\theta)| \\ &= \Gamma(\phi). \end{aligned} \tag{22}$$

Therefore, all the trajectories of $x(t)$ that start from $\Gamma(\phi) \le 1$ will remain inside of $K(v,1)$.

The proof is completed. □

3.2. L_1-Gain Analysis. The following theorem establishes a condition under which the closed-loop system (10) possesses positivity and has an L_1-gain performance.

Theorem 7. *Given positive constants γ, w_m, and a matrix $H \in R^{m\times n}$, if there exist $v, \upsilon \in R_+^n$, and a matrix $F \in R^{m\times n}$, such that, for $i = 1, 2, \ldots, 2^m$, (11), (13), and the following conditions hold,*

$$\left(A + BD_iF + BD_i^- H\right)^T v + \upsilon + C^T 1_q \prec 0, \tag{23}$$

$$E^T v - \gamma I_p \prec 0, \tag{24}$$

$$v \succeq (1+\gamma w_m) \left|H_\kappa\right|^T, \quad \kappa = 1, 2, \ldots, m, \tag{25}$$

then the closed-loop system (10) is positive and asymptotically stable with an L_1-gain performance level γ for any initial states satisfying (15).

Proof. By Lemma 3, it is easy to get from (11) that system (10) is positive. Choose the Lyapunov functional candidate (17). By Theorem 6, the stability of system (10) with $w(t) = 0$ is ensured if (13), (23)–(25) hold. To establish the L_1-gain performance, we define

$$J_N = \int_{t_0}^{\infty} \left(\|z(t)\| - \gamma \|w(t)\|\right) dt. \tag{26}$$

Under zero initial condition, we have

$$\begin{aligned} J_N &= \int_{t_0}^{\infty} \left(\|z(t)\| - \gamma \|w(t)\| + \dot{V}(t)\right) dt \\ &\quad + V(t)|_{t=t_0} - V(t)|_{t=\infty} \\ &\le \int_{t_0}^{\infty} \left(\|z(t)\| - \gamma \|w(t)\| + \dot{V}(t)\right) dt. \end{aligned} \tag{27}$$

From (13), (23)–(25), we obtain

$$\|z(t)\| - \gamma \|w(t)\| + \dot{V}(t) < 0. \tag{28}$$

Under zero initial condition, it gives rise to

$$\int_{t_0}^{\infty} \|z(t)\| \, dt < \gamma \int_{t_0}^{\infty} \|w(t)\| \, dt. \tag{29}$$

Considering that $w(t) \in L_1[t_0, \infty)$, there exists a constant $w_m < \infty$ satisfying

$$\int_{t_0}^{\infty} \|w(t)\| \, dt \le w_m. \tag{30}$$

Then from (28) and (30) we can get

$$x^T(t) v \le V(t) \le V(t_0) + \gamma w_m. \tag{31}$$

It means that if $\Gamma(\phi) \le 1$, that is, $V(t_0) \le 1$, then $V(t) \le 1 + \gamma w_m$ and hence $x(t) \in K(v, 1+\gamma w_m)$.

Therefore, system (10) has an L_1-gain performance level γ, and all trajectories will remain inside of $K(v, 1+\gamma w_m)$.

The proof is completed. □

In what follows, we will give a method for the controller design based on Theorem 7.

From the definition of the Metzler matrix, (11) can be converted into

$$e_g \left(A + BD_iF + BD_i^- H\right) e_l^T \ge 0, \quad \forall g, l = 1, 2, \ldots n, \; g \ne l, \tag{32}$$

where

$$e_g = \left[\underbrace{0\cdots0}_{g-1} \;\; 1 \;\; \underbrace{0\cdots0}_{n-g}\right], \qquad e_l = \left[\underbrace{0\cdots0}_{l-1} \;\; 1 \;\; \underbrace{0\cdots0}_{n-l}\right]. \tag{33}$$

Remark 8. It should be noted that both F and v are variables to be determined in Theorem 7, and we cannot directly compute F by using the LMI (linear matrix inequality) method. In order to obtain F, we introduce vectors g_i satisfying $g_i \succeq F^T D_i^T B^T v$, $i = 1, 2, \ldots, 2^m$; then (23) holds if the following inequality is satisfied:

$$\left(A + BD_i^- H\right)^T v + g_i + \upsilon + C^T 1_q \prec 0. \tag{34}$$

Thus, we can firstly obtain v, υ, and g_i by solving (13), (24)-(25), and (34). Then from (32) and $g_i \succeq F^T D_i^T B^T v$, $i = 1, 2, \ldots, 2^m$, we can get F.

Remark 9. There are several results on the stabilization of positive systems with time-varying delays [19, 22, 26]; however, the controllers proposed in these papers may fail to work when the actuator is subject to saturation. In this paper, the actuator saturation, which brings difficulties for the controller design, is taken into account, and the convex hull technique is used to deal with it. The controller proposed in Theorem 7 can guarantee the positivity and the L_1-gain performance of the closed-loop system despite the existence of actuator saturation.

4. Extension to Uncertain Case

In this section, we will extend the results proposed in previous section to the following uncertain positive system:

$$\begin{gathered}\dot{x}(t) = \widehat{A}x(t) + \widehat{A}_d x(t - d(t)) + Ew(t), \\ x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) = Cx(t),\end{gathered} \tag{35}$$

where $x(t) \in R_+^n$ is the state vector, $z(t) \in R_+^q$ is the controlled output vector, $w(t) \in R_+^p$ is the disturbance input which belongs to $L_1[0, \infty)$, $\phi(\theta)$ is the initial condition on $[-\tau, 0]$, $\tau > 0$, and $d(t)$ denotes the time-varying delay satisfying (2).

$\widehat{A}$ and $\widehat{A}_d$ are uncertain matrices satisfying

$$\underline{A} \preceq \widehat{A} \preceq \overline{A}, \qquad \underline{A}_d \preceq \widehat{A}_d \preceq \overline{A}_d, \tag{36}$$

where $\underline{A}$, $\overline{A}$, $\underline{A}_d$, and $\overline{A}_d$ are known constant matrices.

Lemma 10. *If $\underline{A}$ is a Metzler matrix and $\underline{A}_d \succeq 0$, $C \succeq 0$, and $E \succeq 0$, then system (35) is positive.*

Proof. Because $\underline{A}$ is a Metzler matrix and $\underline{A}_d \succeq 0$, $C \succeq 0$, and $E \succeq 0$, it is easy to obtain that $\widehat{A}$ is a Metzler matrix and $\widehat{A}_d \succeq 0$; then system (35) is positive.

The proof is completed. □

Consider the following system with actuator saturation:

$$\begin{gathered}\dot{x}(t) = \widehat{A}x(t) + \widehat{A}_d x(t - d(t)) + B\,\mathrm{sat}(u(t)) + Ew(t), \\ x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) = Cx(t),\end{gathered} \tag{37}$$

where $u(t) \in R_+^m$ is the control input vector.

Similarly, the system (37) can be rewritten as the following closed-loop system:

$$\begin{gathered}\dot{x}(t) = \sum_{i=1}^{2^m} \eta_i \left[\left(\widehat{A} + BD_iF + BD_i^- H\right) x(t) \right. \\ \left. + \widehat{A}_d x(t - d(t)) + Ew(t)\right], \\ x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \\ z(t) = Cx(t).\end{gathered} \tag{38}$$

By Lemma 10, $\sum_{i=1}^{2^m} \eta_i(\underline{A} + BD_iF + BD_i^- H)$ should be Metzler matrices to ensure the positivity of system (38).

The following theorem gives sufficient conditions which ensure the positivity and L_1-gain property of the closed-loop system (38).

Theorem 11. *Given positive constants γ, w_m, and a matrix $H \in R^{m\times n}$, if there exist $v, \upsilon \in R_+^n$, and a matrix $F \in R^{m\times n}$, such that, for $i = 1, 2, \ldots, 2^m$, (24)-(25) and the following conditions hold,*

$$\underline{A} + BD_iF + BD_i^- H \text{ are Metzler matrices}, \tag{39}$$

$$\left(\overline{A} + BD_iF + BD_i^- H\right)^T v + \upsilon + C^T 1_q \prec 0, \tag{40}$$

$$\overline{A}_d^T v - (1 - d)\upsilon \prec 0, \tag{41}$$

then the closed-loop system (38) is positive and asymptotically stable with an L_1-gain performance level γ for any initial states satisfying (15).

Proof. By Theorem 7, it is easy to get from (14) and (39) that $K(v, 1 + \gamma w_m) \subset L(H)$ and system (38) is positive.

Choose the Lyapunov functional candidate (17). Along the trajectory of system (38), we have

$$\begin{gathered}\dot{V}(t) \le \sum_{i=1}^{2^m} \eta_i \left\{x^T(t)\left[\left(\overline{A} + BD_iF + BD_i^- H\right)^T v + \upsilon + C^T 1_q\right]\right. \\ + x^T(t - d(t))\left(\overline{A}_d^T v - (1 - d)\upsilon\right) \\ \left. + w^T(t)\left(E^T v - \gamma I_p\right)\right\}, \\ \forall x(t) \in K(v, 1 + \gamma w_m).\end{gathered} \tag{42}$$

When $w(t) = 0$, we obtain

$$\dot{V}(t) < 0, \quad \forall x(t) \in K(v, 1 + \gamma w_m). \tag{43}$$

Therefore, system (38) with $w(t) = 0$ is locally asymptotically stable.

When $w(t) \neq 0$, similar to the proof line of Theorem 7, the L_1-gain performance can be obtained.

The proof is completed. □

From the definition of the Metzler matrix, (39) can be converted into

$$e_g\left(\underline{A} + BD_iF + BD_i^- H\right)e_l^T \geq 0, \quad \forall g, l = 1, 2, \ldots n,\ g \neq l, \tag{44}$$

where

$$e_g = \left[\overbrace{0\cdots0}^{g-1}\ \ 1\ \ \overbrace{0\cdots0}^{n-g}\right], \qquad e_l = \left[\overbrace{0\cdots0}^{l-1}\ \ 1\ \ \overbrace{0\cdots0}^{n-l}\right]. \tag{45}$$

Remark 12. It should be noted that condition (40) is not expressed in the form of LMI. We can adopt the method proposed in Remark 8 to find the gain matrix F; that is to say, we can firstly get v, υ, and g_i by solving (13), (24)-(25), and

$$\left(\overline{A} + BD_i^- H\right)^T v + g_i + \upsilon + C^T 1_q \prec 0. \tag{46}$$

Then from (44) and $g_i \succeq F^T D_i^T B^T v$, $i = 1, 2, \ldots, 2^m$, we can obtain the gain matrix F.

5. Examples

In this section, three examples are presented to check the validity of the proposed results.

Example 1. Consider system (4) with the following parameters:

$$A = \begin{bmatrix} -1.5 & 2 \\ 2.5 & -2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.02 & 0.012 \\ 0 & 0.03 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.31 & 0.42 \\ 0.22 & 0.1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.33 & 0.12 \\ 0.13 & 0.2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.35 \\ 0.4 \end{bmatrix}. \tag{47}$$

Let $\tau = 0.1$, $d = 0.2$, $\gamma = 1.9$, $w_m = 0.1$, and

$$H = \begin{bmatrix} -1.4 & -0.5 \\ -0.9 & -1.2 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{48}$$

Solving the matrix inequalities (13), (24)-(25), and (34) in Theorem 7 gives rise to

$$v = \begin{bmatrix} 2.2811 \\ 1.9938 \end{bmatrix}, \qquad \upsilon = \begin{bmatrix} 0.8324 \\ 0.6800 \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 0.4790 \\ 0.1886 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} -1.9882 \\ -1.7587 \end{bmatrix}, \tag{49}$$

$$g_3 = \begin{bmatrix} -1.3932 \\ -0.9427 \end{bmatrix}, \qquad g_4 = \begin{bmatrix} -3.5923 \\ -2.3316 \end{bmatrix}.$$

Then from (32) and $g_i \succeq F^T D_i^T B^T v$, $i = 1, 2, \ldots, 2^m$, we can get

$$F = \begin{bmatrix} -2.2708 & -1.6108 \\ -2.7555 & -2.1416 \end{bmatrix}, \tag{50}$$

and $A + BD_iF + BD_i^- H$, $i = 1, 2, \ldots, 2^m$, are Metzler matrices.

The simulation results are shown in Figures 1–3, where the initial condition of the systems is $x(0) = [0.19\ \ 0.28]^T$, $x(\theta) = [0\ \ 0]^T$, $\theta = [-0.1, 0)$, and the disturbance input is $w(t) = 0.05e^{-0.5t}$. Figure 1 shows the domain of attraction. Figure 2 plots the state responses of the closed-loop system. Figure 3 shows the control signals $u(t)$ and $\mathrm{sat}(u(t))$. It is not hard to find that the feedback controller can guarantee the positivity and the asymptotical stability of the closed-loop system.

Example 2. Consider system (37) with the following parameters:

$$\underline{A} = \begin{bmatrix} -2 & 3 \\ 3 & -3 \end{bmatrix}, \qquad \overline{A} = \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix},$$

$$\underline{A}_d = \begin{bmatrix} 0.01 & 0.005 \\ 0 & 0.02 \end{bmatrix}, \qquad \overline{A}_d = \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.03 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.31 & 0.42 \\ 0.22 & 0.1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}. \tag{51}$$

Let $\tau = 0.1$, $d = 0.2$, $\gamma = 2.87$, $w_m = 0.1$, and

$$H = \begin{bmatrix} -1.2 & -0.4 \\ -0.8 & -1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{52}$$

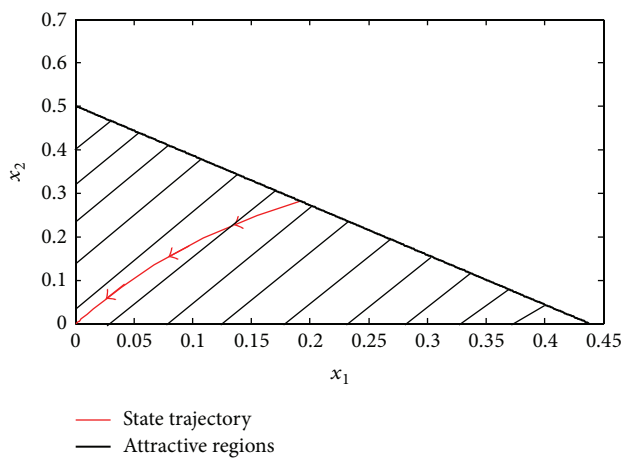

FIGURE 1: Attractive regions and state trajectory in Example 1.

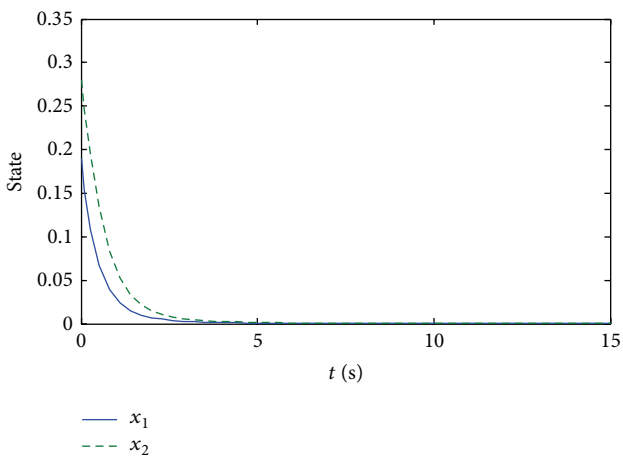

FIGURE 2: State responses of the closed-loop system in Example 1.

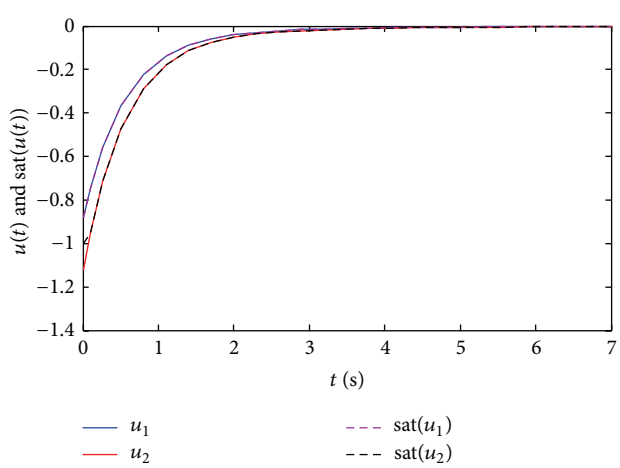

FIGURE 3: $u(t)$ and sat$(u(t))$ of the closed-loop system in Example 1.

Solving the matrix inequalities (13), (24)-(25), and (46) in Theorem 11 gives rise to

$$v = \begin{bmatrix} 3.7490 \\ 3.0111 \end{bmatrix}, \qquad \upsilon = \begin{bmatrix} 0.5552 \\ 0.5930 \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 0.1308 \\ 0.1184 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} -6.1343 \\ -6.5343 \end{bmatrix},$$

$$g_3 = \begin{bmatrix} -6.8233 \\ -5.3885 \end{bmatrix}, \qquad g_4 = \begin{bmatrix} -8.3239 \\ -7.2642 \end{bmatrix}. \tag{53}$$

Then from (44) and $g_i \geq F^T D_i^T B^T v$, $i = 1, 2, \ldots, 2^m$, we can get

$$F = \begin{bmatrix} -4.0921 & -3.3218 \\ -3.7483 & -3.6654 \end{bmatrix}, \tag{54}$$

and $A + BD_iF + BD_i^- H$, $i = 1, 2, \ldots, 2^m$, are Metzler matrices.

The simulation results are shown in Figures 4–6, where the initial condition of the systems is $x(0) = [0.25 \;\; 0.01]^T$, $x(\theta) = [0 \;\; 0]^T$, $\theta = [-0.1, 0)$ and the disturbance input is $w(t) = 0.05e^{-0.5t}$. Figure 4 plots the domain of attraction. Figure 5 shows the state responses of the closed-loop system. Figure 6 shows the control signals $u(t)$ and sat$(u(t))$. It can be seen from Figures 4–6 that the closed-loop system is positive and asymptotically stable. This demonstrates the effectiveness of the proposed approach.

Example 3. Consider a model of virus treatment which can be described as system (4) [6]. The system parameters are as follows:

$$A = \begin{bmatrix} -1.6 & 1 \\ 1.59 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.02 & 0.012 \\ 0 & 0.03 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 & 0.42 \\ 0.21 & 0.1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.33 & 0.12 \\ 0.13 & 0.2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.35 \\ 0.4 \end{bmatrix}, \tag{55}$$

where $x_1(t)$ and $x_2(t)$ are virus populations of two different viral genotypes.

Let $\tau = 0.1$, $d = 0.2$, $\gamma = 1.2$, $w_m = 0.1$, and

$$H = \begin{bmatrix} -1.4 & -0.5 \\ -0.9 & -1.2 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{56}$$

Solving the matrix inequalities (13), (24)-(25), and (32)-(34) in Theorem 7 gives rise to

$$F = \begin{bmatrix} -2.6601 & -2.0812 \\ -1.5620 & -1.8498 \end{bmatrix}. \tag{57}$$

Choosing the initial condition $x(0) = [0.2 \;\; 0.4]^T$, $x(\theta) = [0 \;\; 0]^T$, $\theta = [-0.1, 0)$ and the disturbance input $w(t) = 0.05e^{-0.5t}$, the simulation results are shown in Figures 7 and 8.

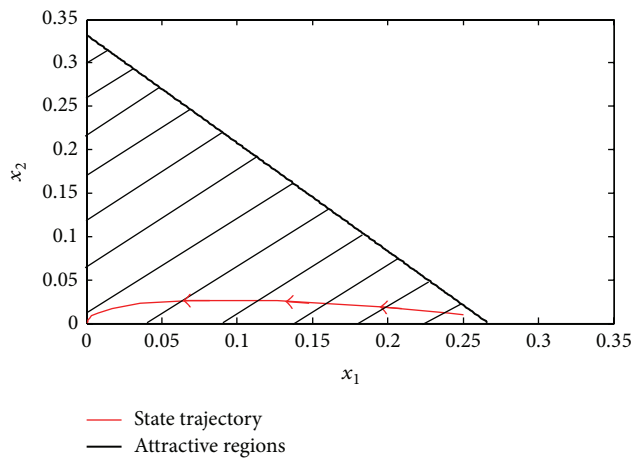

FIGURE 4: Attractive regions and state trajectory in Example 2.

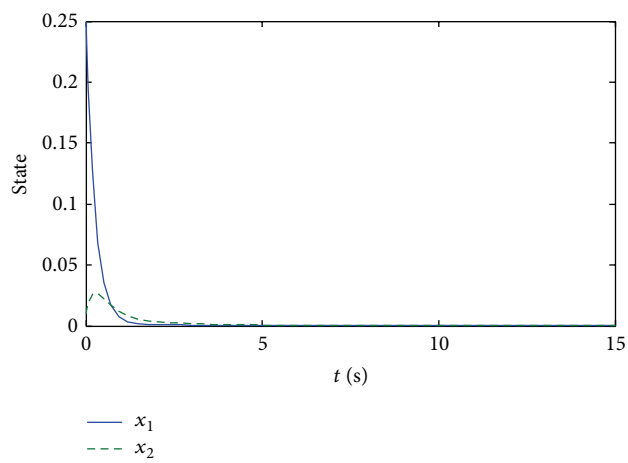

FIGURE 5: State responses of the closed-loop system in Example 2.

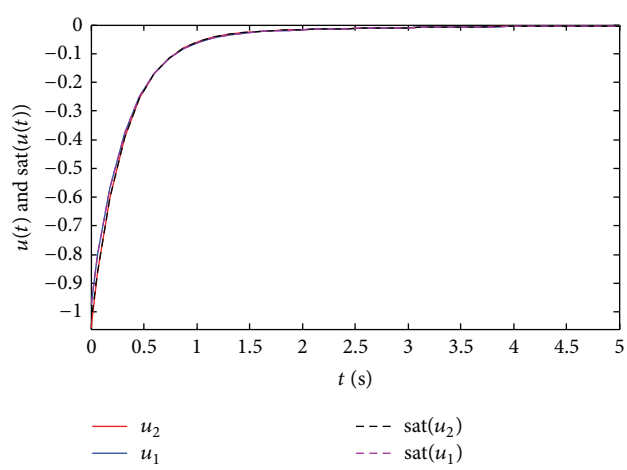

FIGURE 6: $u(t)$ and sat$(u(t))$ of the closed-loop system in Example 2.

6. Conclusions

In this paper, we have investigated the problems of robust stabilization and disturbance rejection for a class of positive systems with time-varying delays and actuator saturation. An appropriate copositive type Lyapunov functional is employed to ensure the stability and L_1-gain performance of the positive systems. Both the existence conditions and the explicit characterization of the desired controller are derived in terms of LMIs. Finally, three examples are provided to illustrate the validity of the theoretical results.

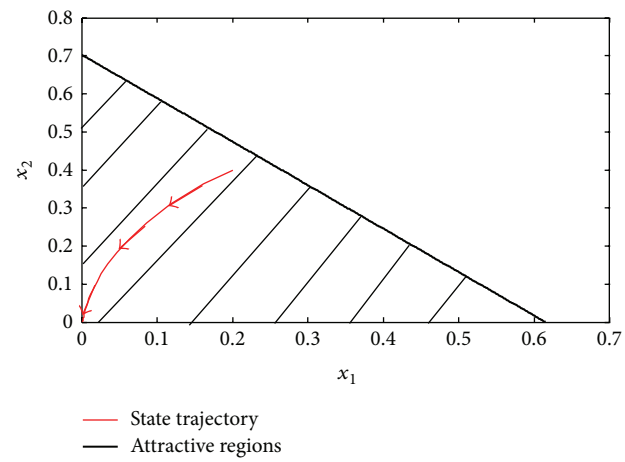

FIGURE 7: Attractive regions and state trajectory in Example 3.

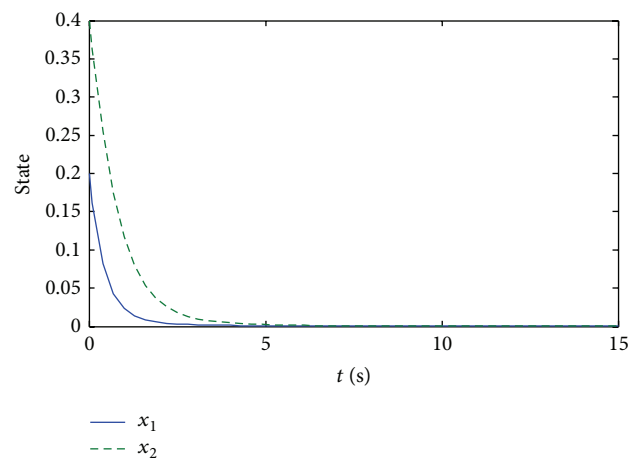

FIGURE 8: State responses of the closed-loop system in Example 3.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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