Research Article

Multivariate Spectral Gradient Algorithm for Nonsmooth Convex Optimization Problems

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Received 20 April 2015; Revised 4 July 2015; Accepted 5 July 2015

Academic Editor: Dapeng P. Du

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We propose an extended multivariate spectral gradient algorithm to solve the nonsmooth convex optimization problem. First, by using Moreau-Yosida regularization, we convert the original objective function to a continuously differentiable function; then we use approximate function and gradient values of the Moreau-Yosida regularization to substitute the corresponding exact values in the algorithm. The global convergence is proved under suitable assumptions. Numerical experiments are presented to show the effectiveness of this algorithm.

1. Introduction

Consider the unconstrained minimization problem

$$\min_{x \in \mathcal{R}^n} f(x), \quad (1)$$

where $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is a nonsmooth convex function. The Moreau-Yosida regularization [1] of f at $x \in \mathcal{R}^n$ associated with $z \in \mathcal{R}^n$ is defined by

$$F(x) = \min_{z \in \mathcal{R}^n} \left\{ f(z) + \frac{1}{2\lambda} \|z - x\|^2 \right\}, \quad (2)$$

where $\|\cdot\|$ is the Euclidean norm and λ is a positive parameter. The function minimized on the right-hand side is strongly convex and differentiable, so it has a unique minimizer for every $z \in \mathcal{R}^n$. Under some reasonable conditions, the gradient function of $F(x)$ can be proved to be semismooth [2, 3], though generally $F(x)$ is not twice differentiable. It is widely known that the problem

$$\min_{x \in \mathcal{R}^n} F(x) \quad (3)$$

and the original problem (1) are equivalent in the sense that the two corresponding solution sets coincidentally are the same. The following proposition shows some properties of the Moreau-Yosida regularization function $F(x)$.

Proposition 1 (see Chapter XV, Theorem 4.1.4, [1]). *The Moreau-Yosida regularization function F is convex, finite-valued, and differentiable everywhere with gradient*

$$g(x) \equiv \nabla F(x) = \frac{1}{\lambda} (x - p(x)), \quad (4)$$

where

$$p(x) = \arg \min_{z \in \mathcal{R}^n} \left\{ f(z) + \frac{1}{2\lambda} \|z - x\|^2 \right\} \quad (5)$$

is the unique minimizer in (2). Moreover, for all $x, y \in \mathcal{R}^n$, one has

$$\|g(x) - g(y)\| \leq \frac{1}{\lambda} \|x - y\|. \quad (6)$$

This proposition shows that the gradient function $g : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is Lipschitz continuous with modulus $1/\lambda$. In this case, the gradient function g is differentiable almost everywhere by the Rademacher theorem; then the B-subdifferential [4] of g at $x \in \mathcal{R}^n$ is defined by

$$\partial_B g(x) = \left\{ V \in \mathcal{R}^{n \times n} : V = \lim_{x_k \rightarrow x} \nabla g(x_k), x_k \in D_g \right\}, \quad (7)$$

where $D_g = \{x \in \mathcal{R}^n : g \text{ is differentiable at } x\}$, and the next property of BD-regularity holds [4–6].

Proposition 2. If g is BD-regular at x , then

- (i) all matrices $V \in \partial_B g(x)$ are nonsingular;
- (ii) there exists a neighborhood $\mathcal{N}$ of $x \in \mathfrak{R}^n$, $\kappa_1 > 0$, and $\kappa_2 > 0$; for all $y \in \mathcal{N}$, one has

$$d^T V d \geq \kappa_1 \|d\|^2, \quad \|V^{-1}\| \leq \kappa_2, \quad (8)$$

$$\forall d \in \mathfrak{R}^n, V \in \partial_B g(y).$$

Instead of the corresponding exact values, we often use the approximate value of function $F(x)$ and gradient $g(x)$ in the practical computation, because $p(x)$ is difficult and sometimes impossible to be solved precisely. Suppose that, for any $\varepsilon > 0$ and for each $x \in \mathfrak{R}^n$, there exists an approximate vector $p^a(x, \varepsilon) \in \mathfrak{R}^n$ of the unique minimizer $p(x)$ in (2) such that

$$f(p^a(x, \varepsilon)) + \frac{1}{2\lambda} \|p^a(x, \varepsilon) - x\|^2 \leq F(x) + \varepsilon. \quad (9)$$

The implementable algorithms to find such approximate vector $p^a(x, \varepsilon) \in \mathfrak{R}^n$ can be found, for example, in [7, 8]. The existence theorem of the approximate vector $p^a(x, \varepsilon)$ is presented as follows.

Proposition 3 (see Lemma 2.1 in [7]). Let $\{x_k\}$ be generated according to the formula

$$x_{k+1} = x_k - \alpha_k v_k, \quad \text{for } k = 1, 2, \dots, \quad (10)$$

where $\alpha_k > 0$ is a stepsize and v_k is an approximate subgradient at x_k ; that is,

$$v_k \in \partial_{\varepsilon_k} f(x_k) = \{v \mid f(v) - \langle v, x_k \rangle \leq f(x_k) + \varepsilon_k\}, \quad (11)$$

$$\text{for } k = 1, 2, \dots$$

(i) If v_k satisfies

$$v_k \in \partial f(x_{k+1}), \quad \text{for } k = 1, 2, \dots, \quad (12)$$

then (11) holds with

$$\varepsilon_k = f(x_k) - f(x_{k+1}) - \alpha_k \|v_k\|^2 \geq 0. \quad (13)$$

(ii) Conversely, if (11) holds with ε_k given by (13), then (12) holds: $x_{k+1} = p^a(x_k, \varepsilon_k)$.

We use the approximate vector $p^a(x, \varepsilon)$ to define approximate function and gradient values of the Moreau-Yosida regularization, respectively, by

$$F^a(x, \varepsilon) = f(p^a(x, \varepsilon)) + \frac{1}{2\lambda} \|p^a(x, \varepsilon) - x\|^2, \quad (14)$$

$$g^a(x, \varepsilon) = \frac{(x - p^a(x, \varepsilon))}{\lambda}. \quad (15)$$

The following proposition is crucial in the convergence analysis. The proof of this proposition can be found in [2].

Proposition 4. Let ε be arbitrary positive number and let $p^a(x, \varepsilon)$ be a vector satisfying (9). Then, one gets

$$F(x) \leq F^a(x, \varepsilon) \leq F(x) + \varepsilon, \quad (16)$$

$$\|g^a(x, \varepsilon) - g(x)\| \leq \sqrt{\frac{2\varepsilon}{\lambda}}, \quad (17)$$

$$\|p^a(x, \varepsilon) - p(x)\| \leq \sqrt{2\lambda\varepsilon}. \quad (18)$$

Algorithms which combine the proximal techniques with Moreau-Yosida regularization for solving the nonsmooth problem (1) have been proved to be effective [7, 9, 10], and also some trust region algorithms for solving (1) have been proposed in [5, 11, 12], and so forth. Recently, Yuan et al. [13, 14] and Li [15] have extended the spectral gradient method and conjugate gradient-type method to solve (1), respectively.

Multivariate spectral gradient (MSG) method was first proposed by Han et al. [16] for optimization problems. This method has a nice property that it converges quadratically for objective function with positive definite diagonal Hessian matrix [16]. Further studies on such method for nonlinear equations and bound constrained optimization can be found, for instance, in [17, 18]. By using nonmonotone technique, some effective spectral gradient methods are presented in [13, 16, 17, 19]. In this paper, we extend the multivariate spectral gradient method by combining with a nonmonotone line search technique as well as the Moreau-Yosida regularization function to solve the nonsmooth problem (1) and do some numerical experiments to test its efficiency.

The rest of this paper is organized as follows. In Section 2, we propose multivariate spectral gradient algorithm to solve (1). In Section 3, we prove the global convergence of the proposed algorithm; then some numerical results are presented in Section 4. Finally, we have a conclusion section.

2. Algorithm

In this section, we present the multivariate spectral gradient algorithm to solve the nonsmooth convex unconstrained optimization problem (1). Our approach is using the tool of the Moreau-Yosida regularization to smoothen the nonsmooth function and then make use of the approximate values of function F and gradient g in multivariate spectral gradient algorithm.

We first recall the multivariate spectral gradient algorithm [16] for smooth optimization problem:

$$\min \{f(x) \mid x \in \mathfrak{R}^n\}, \quad (19)$$

where $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$ is continuously differentiable and its gradient is denoted by g . Let x_k be the current iteration; multivariate spectral gradient algorithm is defined by

$$x_{k+1} = x_k - \text{diag} \left\{ \frac{1}{\lambda_k^1}, \frac{1}{\lambda_k^2}, \dots, \frac{1}{\lambda_k^n} \right\} g_k, \quad (20)$$

where g_k is the gradient vector of f at x_k and $\text{diag}\{\lambda_k^1, \lambda_k^2, \dots, \lambda_k^n\}$ is solved by minimizing

$$\|\text{diag} \{\lambda^1, \lambda^2, \dots, \lambda^n\} s_{k-1} - u_{k-1}\|_2 \quad (21)$$

with respect to $\{\lambda^i\}_{i=1}^n$, where $s_{k-1} = x_k - x_{k-1}$, $u_{k-1} = g_k - g_{k-1}$.

Denote the i th element of s_k and y_k by s_k^i and y_k^i , respectively. We present the following multivariate spectral gradient (MSG) algorithm.

Algorithm 5. Set $x_0 \in \mathfrak{R}^n$, $\sigma \in (0, 1)$, $\beta > 0$, $\lambda > 0$, $\gamma \geq 0$, $\delta > 0$, $\rho \in [0, 1]$, $\epsilon \in (0, 1)$, $E_0 = 1$, and $\tau_0 \in (0, 1]$; $\{\tau_k\}$ is a strictly decreasing sequence with $\lim_{k \rightarrow 0} \tau_k = 0$, $k := 0$.

Step 1. Set $\epsilon_0 = \tau_0$. Calculate $F^a(x_0, \epsilon_0)$ by (14) as well as $g^a(x_0, \epsilon_0)$ by (15). Let $J_0 = F^a(x_0, \epsilon_0)$, $d_0 = -g^a(x_0, \epsilon_0)$.

Step 2. Stop if $\|g^a(x_k, \epsilon_k)\| = 0$. Otherwise, go to Step 3.

Step 3. Choose ϵ_{k+1} satisfying $0 < \epsilon_{k+1} \leq \min\{\tau_k, \tau_k \|g^a(x_k, \epsilon_k)\|^2\}$; find α_k which satisfies

$$F^a(x_k + \alpha_k d_k, \epsilon_{k+1}) - J_k \leq \sigma \alpha_k g^a(x_k, \epsilon_k)^T d_k, \quad (22)$$

where $\alpha_k = \beta 2^{-i_k}$ and i_k is the smallest nonnegative integer such that (22) holds.

Step 4. Let $x_{k+1} = x_k + \alpha_k d_k$. Stop if $\|g^a(x_{k+1}, \epsilon_{k+1})\| = 0$.

Step 5. Update J_{k+1} by the following formula:

$$\begin{aligned} E_{k+1} &= \rho E_k + 1, \\ J_{k+1} &= \frac{\rho E_k J_k + F^a(x_k + \alpha_k d_k, \epsilon_{k+1})}{E_{k+1}}. \end{aligned} \quad (23)$$

Step 6. Compute the search direction d_{k+1} by the following:

- (a) If $y_k^i/s_k^i > 0$, then set $\lambda_{k+1}^i = y_k^i/s_k^i$; otherwise set $\lambda_{k+1}^i = s_k^T y_k / s_k^T s_k$ for $i = 1, 2, \dots, n$, where $y_k = g^a(x_{k+1}, \epsilon_{k+1}) - g^a(x_k, \epsilon_k) + \gamma s_k$, $s_k = x_{k+1} - x_k$.
- (b) If $\lambda_{k+1}^i \leq \epsilon$ or $\lambda_{k+1}^i \geq 1/\epsilon$, then set $\lambda_{k+1}^i = \delta$ for $i = 1, 2, \dots, n$.

Let $d_{k+1} = -\text{diag}\{1/\lambda_{k+1}^1, 1/\lambda_{k+1}^2, \dots, 1/\lambda_{k+1}^n\} g^a(x_{k+1}, \epsilon_{k+1})$.

Step 7. Set $k := k + 1$; go back to Step 2.

Remarks. (i) The definition of $\epsilon_{k+1} = o(\|g^a(x_k, \epsilon_k)\|^2)$ in Algorithm 5, together with (15) and Proposition 3, deduces that

$$\begin{aligned} \epsilon_{k+1} &= o(\|x_k - p^a(x_k, \epsilon_k)\|^2) = o(\|x_k - x_{k+1}\|^2) \\ &= o(\alpha_k^2 \|d_k\|^2); \end{aligned} \quad (24)$$

then, with the decreasing property of ϵ_{k+1} , the assumed condition $\epsilon_k = o(\alpha_k^2 \|d_k\|^2)$ in Lemma 7 holds.

(ii) From the nonmonotone line search technique (22), we can see that J_{k+1} is a convex combination of the function value $F^a(x_{k+1}, \epsilon_{k+1})$ and J_k . Also J_k is a convex combination of the function values $F^a(x_k, \epsilon_k)$, $\dots$, $F^a(x_1, \epsilon_1)$, $F^a(x_0, \epsilon_0)$ as $J_0 = F^a(x_0, \epsilon_0)$. ρ is a positive value that plays an important

role in manipulating the degree of nonmonotonicity in the nonmonotone line search technique, with $\rho = 0$ yielding a strictly monotone scheme and with $\rho = 1$ yielding $J_k = C_k$, where

$$C_k = \frac{1}{k+1} \sum_{i=0}^k F^a(x_i, \epsilon_i) \quad (25)$$

is the average function value.

(iii) From Step 6, we can obtain that

$$\min \left\{ \epsilon, \frac{1}{\delta} \right\} \|g^a(x_k, \epsilon_k)\| \leq \|d_k\| \quad (26)$$

$$\leq \max \left\{ \frac{1}{\epsilon}, \frac{1}{\delta} \right\} \|g^a(x_k, \epsilon_k)\|;$$

then there is a positive constant μ such that, for all k ,

$$g^a(x_k, \epsilon_k)^T d_k \leq -\mu \|g^a(x_k, \epsilon_k)\|^2, \quad (27)$$

which shows that the proposed multivariate spectral gradient algorithm possesses the sufficient descent property.

3. Global Convergence

In this section, we provide a global convergence analysis for the multivariate spectral gradient algorithm. To begin with, we make the following assumptions which have been given in [5, 12–14].

Assumption A. (i) F is bounded from below.

(ii) The sequence $\{V_k\}$, $V_k \in \partial_B g(x_k)$, is bounded; that is, there exists a constant $M > 0$ such that, for all k ,

$$\|V_k\| \leq M. \quad (28)$$

The following two lemmas play crucial roles in establishing the convergence theorem for the proposed algorithm. By using (26) and (27) and Assumption A, similar to Lemma 1.1 in [20], we can get the next lemma which shows that Algorithm 5 is well defined. The proof ideas of this lemma and Lemma 1.1 in [20] are similar, hence omitted.

Lemma 6. Let $\{F^a(x_k, \epsilon_k)\}$ be the sequence generated by Algorithm 5. Suppose that Assumption A holds and C_k is defined by (25). Then one has $F^a(x_k, \epsilon_k) \leq J_k \leq C_k$ for all k . Also, there exists a stepsize α_k satisfying the nonmonotone line search condition.

Lemma 7. Let $\{(x_k, \epsilon_k)\}$ be the sequence generated by Algorithm 5. Suppose that Assumption A and $\epsilon_k = o(\alpha_k^2 \|d_k\|^2)$ hold. Then, for all k , one has

$$\alpha_k \geq m_0, \quad (29)$$

where $m_0 > 0$ is a constant.

Proof (Proof by Contradiction). Let α_k satisfy the nonmonotone Armijo-type line search (22). Assume on the contrary

that $\liminf_{k \rightarrow \infty} \alpha_k = 0$ does hold; then there exists a subsequence $\{\alpha_k\}_{k'}$ such that $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$. From the nonmonotone line search rule (22), $\alpha'_k = \alpha_k/2$ satisfies

$$F^a(x_k + \alpha'_k d_k, \varepsilon_{k+1}) - J_k > \sigma \alpha'_k g^a(x_k, \varepsilon_k)^T d_k; \quad (30)$$

together with $F^a(x_k, \varepsilon_k) \leq J_k \leq C_k$ in Lemma 6, we have

$$\begin{aligned} F^a(x_k + \alpha'_k d_k, \varepsilon_{k+1}) - F^a(x_k, \varepsilon_k) \\ \geq F^a(x_k + \alpha'_k d_k, \varepsilon_{k+1}) - J_k > \sigma \alpha'_k g^a(x_k, \varepsilon_k)^T d_k. \end{aligned} \quad (31)$$

By (28) and (31) and Proposition 4 and using Taylor's formula, there is

$$\begin{aligned} \frac{\alpha_k}{2} = \alpha'_k &> \left[\frac{(g^a(x_k, \varepsilon_k) - g(x_k))^T d_k - (1 - \sigma) g^a(x_k, \varepsilon_k)^T d_k - \varepsilon_{k+1} \alpha'_k}{\|d_k\|^2} \right] \frac{2}{M} \\ &\geq \left[\frac{\mu(1 - \sigma) \|g^a(x_k, \varepsilon_k)\|^2 - \sqrt{2\varepsilon_k/\lambda} \|d_k\| - \varepsilon_k/\alpha'_k}{\|d_k\|^2} \right] \frac{2}{M} = \left[\frac{\mu(1 - \sigma) \|g^a(x_k, \varepsilon_k)\|^2}{\|d_k\|^2} - \frac{o(\alpha_k)}{\sqrt{\lambda}} - o(\alpha_k) \right] \frac{2}{M} \\ &\geq \left[\frac{\mu(1 - \sigma)}{(\max\{1/\varepsilon, 1/\delta\})^2} - \frac{o(\alpha_k)}{\sqrt{\lambda}} - o(\alpha_k) \right] \frac{2}{M}, \end{aligned} \quad (33)$$

where the second inequality follows from (26), Part 3 in Proposition 4, and $\varepsilon_{k+1} \leq \varepsilon_k$, the equality follows from $\varepsilon_k = o(\alpha_k^2 \|d_k\|^2)$, and the last inequality follows from (27). Dividing each side by α_k and letting $k \rightarrow \infty$ in the above inequality, we can deduce that

$$\frac{1}{2} \geq \lim_{k \rightarrow \infty} \left(\frac{2\mu(1 - \sigma)}{(\max\{1/\varepsilon, 1/\delta\})^2 M} \right) \frac{1}{\alpha_k} = +\infty, \quad (34)$$

which is impossible, so the conclusion is obtained. $\square$

By using the above lemmas, we are now ready to prove the global convergence of Algorithm 5.

Theorem 8. *Let $\{x_k\}$ be generated by Algorithm 5 and suppose that the conditions of Lemma 7 hold. Then one has*

$$\lim_{k \rightarrow \infty} \|g(x_k)\| = 0; \quad (35)$$

sequence $\{x_k\}_{k=0}^\infty$ has accumulation point, and every accumulation point of $\{x_k\}_{k=0}^\infty$ is optimal solution of problem (1).

Proof. Suppose that there exist $\varepsilon_0 > 0$ and $k_0 > 0$ such that

$$\|g^a(x_k, \varepsilon_k)\| \geq \varepsilon_0, \quad \forall k > k_0. \quad (36)$$

From (22), (26), and (29), we get

$$\begin{aligned} F^a(x_k + \alpha_k d_k, \varepsilon_{k+1}) - J_k &\leq \sigma \alpha_k g^a(x_k, \varepsilon_k)^T d_k \\ &\leq -\sigma \alpha_k \min \left\{ \varepsilon, \frac{1}{\delta} \right\} \|g^a(x_k, \varepsilon_k)\|^2 \\ &\leq -\sigma m_0 \varepsilon_0 \min \left\{ \varepsilon, \frac{1}{\delta} \right\}, \quad \forall k > k_0. \end{aligned} \quad (37)$$

$$\begin{aligned} \sigma \alpha'_k g^a(x_k, \varepsilon_k)^T d_k &< F^a(x_k + \alpha'_k d_k, \varepsilon_{k+1}) \\ &\quad - F^a(x_k, \varepsilon_k) \\ &\leq F(x_k + \alpha'_k d_k) - F(x_k) + \varepsilon_{k+1} \\ &= \alpha'_k d_k^T g(x_k) \\ &\quad + \frac{1}{2} (\alpha'_k)^2 d_k^T V(u_k) d_k + \varepsilon_{k+1} \\ &\leq \alpha'_k d_k^T g(x_k) + \frac{M}{2} (\alpha'_k)^2 \|d_k\|^2 \\ &\quad + \varepsilon_{k+1}, \end{aligned} \quad (32)$$

where $u_k \in (x_k, x_{k+1})$. From (32) and Proposition 4, we have

Therefore, it follows from the definition of J_{k+1} and (23) that

$$\begin{aligned} J_{k+1} &= \frac{\rho E_k J_k + F^a(x_k + \alpha_k d_k, \varepsilon_{k+1})}{E_{k+1}} \\ &\leq \frac{\rho E_k J_k + J_k - \sigma m_0 \varepsilon_0 \min\{\varepsilon, 1/\delta\}}{E_{k+1}} \\ &\leq J_k - \frac{\sigma m_0 \varepsilon_0 \min\{\varepsilon, 1/\delta\}}{E_{k+1}}. \end{aligned} \quad (38)$$

By Assumption A, F is bounded from below. Further by Proposition 4, $F(x_k) \leq F^a(x_k, \varepsilon_k)$ for all k , we see that $F^a(x_k, \varepsilon_k)$ is bounded from below. Together with $F^a(x_k, \varepsilon_k) \leq J_k$ for all k from Lemma 6, it shows that J_k is also bounded from below. By (38), we obtain

$$\sum_{k=k_0}^{\infty} \frac{\sigma m_0 \varepsilon_0 \min\{\varepsilon, 1/\delta\}}{E_{k+1}} < \infty. \quad (39)$$

On the other hand, the definition of E_{k+1} implies that $E_{k+1} \leq k + 2$, and it follows that

$$\begin{aligned} \sum_{k=k_0}^{\infty} \frac{\sigma m_0 \varepsilon_0 \min\{\varepsilon, 1/\delta\}}{E_{k+1}} &\geq \sum_{k=k_0}^{\infty} \frac{\sigma m_0 \varepsilon_0 \min\{\varepsilon, 1/\delta\}}{k + 2} \\ &= +\infty. \end{aligned} \quad (40)$$

This is a contradiction. Therefore, we should have

$$\lim_{k \rightarrow \infty} \|g^a(x_k, \varepsilon_k)\| = 0. \quad (41)$$

From (17) in Proposition 4 together with ε_k as $k \rightarrow \infty$, which comes from the definition of ε_k and $\lim_{k \rightarrow 0} \tau_k = 0$ in Algorithm 5, we obtain

$$\lim_{k \rightarrow \infty} \|g(x_k)\| = 0. \quad (42)$$

Set x^* as an accumulation point of sequence $\{x_k\}_{k=0}^\infty$; there is a convergent subsequence $\{x_{k_l}\}_{l=0}^\infty$ such that

$$\lim_{l \rightarrow \infty} x_{k_l} = x^*. \quad (43)$$

From (4) we know that $g(x_k) = (x_k - p(x_k))/\lambda$. Consequently, (42) and (43) show that $x^* = p(x^*)$. Hence, x^* is an optimal solution of problem (1). $\square$

4. Numerical Results

This section presents some numerical results from experiments using our multivariate spectral gradient algorithm for the given test nonsmooth problems which come from [21]. We also list the results of [14] (modified Polak-Ribière-Polyak gradient method, MPRP) and [22] (proximal bundle method, PBL) to make a comparison with the result of Algorithm 5. All codes were written in MATLAB R2010a and were implemented on a PC with 2.8 GHz CPU, 2 GB of memory, and Windows 8. We set $\beta = \lambda = 1$, $\sigma = 0.9$, $\varepsilon = 10^{-10}$, and $\gamma = 0.01$, and the parameter δ is chosen as

$$\delta = \begin{cases} 1 & \text{if } \|g^a(x_k, \varepsilon_k)\| > 1, \\ \|g^a(x_k, \varepsilon_k)\|^{-1} & \text{if } 10^{-5} \leq \|g^a(x_k, \varepsilon_k)\| \leq 1, \\ 10^{-5} & \text{if } \|g^a(x_k, \varepsilon_k)\| < 10^{-5}; \end{cases} \quad (44)$$

then we adopt the termination condition $\|g^a(x_k, \varepsilon_k)\| \leq 10^{-10}$. For subproblem (5), the classical PRP CG method (called subalgorithm) is used to solve it; the algorithm stops if $\|\partial f(x_k)\| \leq 10^{-4}$ or $f(x_{k+1}) - f(x_k) + \|\partial f(x_{k+1})\|^2 - \|\partial f(x_k)\|^2 \leq 10^{-3}$ holds, where $\partial f(x_k)$ is the subgradient of $f(x)$ at the point x_k . The subalgorithm will also stop if the iteration number is larger than fifteen. In its line search, the Armijo line search technique is used and the step length is accepted if the search number is larger than five. Table 1 contains problem names, problem dimensions, and the optimal values.

The summary of the test results is presented in Tables 2-3, where "Nr." denotes the name of the tested problem, "NF" denotes the number of function evaluations, "NI" denotes the number of iterations, and " $f(x)$ " denotes the function value at the final iteration.

TABLE 1: Test problems.

Nr.	Problems	Dim.	$f_{\text{ops}}(x)$
1	Rosenbrock	2	0
2	Crescent	2	0
3	CB2	2	1.9522245
4	CB3	2	2.0
5	DEM	2	-3
6	QL	2	7.20
7	LQ	2	-1.4142136
8	Mifflin 1	2	-1.0
9	Mifflin 2	2	-1.0
10	Wolfe	2	-8.0
11	Rosen-Suzuki	4	-44
12	Shor	5	22.600162

The value of ρ controls the nonmonotonicity of line search which may affect the performance of the MSG algorithm. Table 2 shows the results for different parameter ρ , as well as different values of the parameter τ_k ranging from $1/6(k+2)^6$ to $1/2k^2$ on problem Rosenbrock, respectively. We can conclude from the table that the proposed algorithm works reasonably well for all the test cases. This table also illustrates that the value of ρ can influence the performance of the algorithm significantly if the value of ε is within a certain range, and the choice $\rho = 0.75$ is better than $\rho = 0$.

Then, we compare the performance of MSG to that of the algorithms MPRP and PBL. In this test, we fix $\tau_k = 1/2k^2$ and $\rho = 0.75$. To illustrate the performance of each algorithm more specifically, we present three comparison results in terms of number of iterations, number of function evaluations, and the final objective function value in Table 3.

The numerical results indicate that Algorithm 5 can successfully solve the test problems. From the number of iterations in Table 3, we see that Algorithm 5 performs best among these three methods, and the final function value obtained by Algorithm 5 is closer to the optimal function value than those obtained by MPRP and PBL. In a word, the numerical experiments show that the proposed algorithm provides an efficient approach to solve nonsmooth problems.

5. Conclusions

We extend the multivariate spectral gradient algorithm to solve nonsmooth convex optimization problems. The proposed algorithm combines a nonmonotone line search technique and the idea of Moreau-Yosida regularization. The algorithm satisfies the sufficient descent property and its global convergence can be established. Numerical results show the efficiency of the proposed algorithm.

TABLE 2: Results on Rosenbrock with different ρ and ε .

τ_k	$\rho = 0$ NI/NF/ $f(x)$	Time	$\rho = 0.75$ NI/NF/ $f(x)$	Time
$1/2k^2$	30/46/1.581752e - 9	1.794	29/30/7.778992e - 9	1.076
$1/3(k+2)^3$	28/38/5.207744e - 9	1.420	26/27/6.541087e - 9	1.023
$1/4(k+2)^4$	29/37/1.502034e - 9	1.388	27/28/5.112699e - 9	1.030
$1/5(k+2)^5$	27/37/1.903969e - 9	1.451	27/28/6.329141e - 9	1.092
$1/6(k+2)^6$	27/36/4.859901e - 9	1.376	27/28/6.073222e - 9	1.025

TABLE 3: Numerical results for MSG/MPRP/PBL on problems 1–12.

Nr.	MSG NI/NF/ $f(x)$	MPRP NI/NF/ $f(x)$	PBL NI/NF/ $f(x)$	$f_{\text{ops}}(x)$
1	29/30/7.778992e - 9	46/48/7.091824e - 7	42/45/0.381e - 6	0
2	9/10/1.450669e - 5	11/13/6.735123e - 5	18/20/0.679e - 6	0
3	9/10/1.9522245	12/14/1.952225	32/34/1.9522245	1.9522245
4	4/9/2.000009	2/6/2.000098	14/16/2.0000000	2.0
5	3/4/-2.999949	4/6/-2.999866	17/19/-3.0000000	-3
6	11/12/7.200000	10/12/7.200011	13/15/7.2000015	7.20
7	3/4/-1.4142136	2/3/-1.414214	11/12/-1.4142136	-1.4142136
8	9/10/-0.9999638	4/6/-0.9919815	66/68/-0.9999994	-1.0
9	12/13/-0.9999978	20/23/-0.9999925	13/15/-1.0000000	-1.0
10	5/6/-7.999999	—	43/46/-8.0000000	-8.0
11	6/7/-43.99797	28/58/-43.99986	43/45/-43.999999	-44
12	12/13/2.260017	33/91/22.60023	27/29/22.600162	22.600162

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments and suggestions which help a lot to improve the paper greatly. The author also thanks Professor Gong-lin Yuan for his kind offer of the source BB codes on nonsmooth problems. This work is supported by the National Natural Science Foundation of China (Grant no. 11161003).

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