Research Article

Global Asymptotic Stabilization Control for a Class of Nonlinear Systems with Dynamic Uncertainties

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This paper is concerned with the global asymptotic stabilization control problem for a class of nonlinear systems with input-to-state stable (ISS) dynamic uncertainties and uncertain time-varying control coefficients. Unlike the existing works, the ISS dynamic uncertainty is characterized by the uncertain supply rates. By using the backstepping control approach, a systematic controller design procedure is developed. The designed control law can guarantee that the system states are asymptotically regulated to the origin from any initial conditions and the other signals are bounded in closed-loop systems. Moreover, it is shown that, under some additional conditions, a linear control law can be designed by the proposed methodology. The simulation example demonstrates its effectiveness.

1. Introduction

The nonlinear control theory is an active research direction in the control field because of its widespread applications in the real world. During the past two decades, various novel methodologies have been generated for the nonlinear feedback control; see the recent survey [1] and references therein for an interesting introduction to this area. One of the influential notions is the input-to-state stability (ISS) and its several variants. Since they are introduced by Sontag in [2, 3], the notion of ISS as well as its integral variant—integral ISS (iISS)—has become a foundational concept upon which much of modern nonlinear feedback analysis and design rest. As noted in [4], ISS provides a nonlinear generalization of finite gains with respect to supremum norms and also of finite L^2 gains, and it plays a central role in recursive design, coprime factorizations, controllers for nonminimum phase systems, and many other areas. Based on the series of works on ISS, the nonlinear small-gain theorem was proposed in the state-space setting and is widely used in the stability analysis and control design for complex interconnected systems in [5]. The stochastic results can be found in [6, 7] and the references therein.

It is noted that a unifying framework is presented in [8] for the global output feedback regulation control problem

from ISS to iISS. The framework established in [8] extends many known classes of output feedback form systems. However, the system uncertainties investigated there depend only on the system output and the inverse system state. With unmeasured states dependent growth, in [9, 10], the problem of global stabilization by output/state feedback is investigated for a class of nonlinear systems with uncertain control coefficients. However, there is no dynamic uncertainty for the system under consideration. In [11], this work is further studied for a larger class of nonlinear uncertain systems, in which the observer gain is governed by a Riccati differential equation. Moreover, the output regulation problem is also considered in [12] for this class of nonlinear systems with iISS inverse dynamics. Later, in [13, 14], this work is further investigated for the nonlinear systems with uncertain nonlinearities dependent on all unmeasured states. However, the control coefficients in above results are required to be known a priori or unknown nonzero constants. In [15], the global set-point tracking control is investigated for a class of cascaded nonlinear systems with unknown control coefficients. However, a restrictive condition is that the control coefficients are required to have the same signs.

In this paper, we will further investigate this problem for a class of nonlinear systems with more general nonlinear

uncertainties. Unlike the existing works such as in [9, 12, 15–17], the studied system is with the uncertain control coefficients, which could be unknown time-varying functions. Another feature of this work is that the dynamic uncertainties are characterized by the uncertain ISS supply rates. This is different from the existing results reported in literatures where the ISS dynamic uncertainty is investigated under the hypothesis that supply rates are known a priori such as in [8, 11, 13, 15]. With the help of the backstepping approach [18], we design a robust adaptive controller which could achieve the system states convergent to the origin while the other signals are bounded. Moreover, it is of interest to note that a linear control law can be designed using the developed scheme if some stronger conditions are imposed on the nonlinear system.

The rest of the paper is organized as follows. In Section 2, we provide some mathematical preliminaries and state the problem. The controller design procedure is developed in Section 3, and the main result is presented in Section 4. Section 5 illustrates the obtained results by a numerical example. Section 6 concludes this paper.

Notation. Let $\mathbf{R}(\mathbf{R}_+)$ denote the set of all (positive) real numbers and let $\mathbf{R}^n$ denote the real n -dimensional space. For a given vector or matrix X , X^T denotes its transpose. For any column vector $x = (x_1, \dots, x_n)^T \in \mathbf{R}^n$, $\bar{x}_i$ denotes the column vector consisting of the first i elements of x in the original order; that is, $\bar{x}_i = (x_1, \dots, x_i)^T$. Specifically, for $x = (x_1, \dots, x_n)^T$, $x_1 = \bar{x}_1$, $x = \bar{x}_n$. A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class- K if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class- K_∞ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$. The notation $\tilde{\beta}(s) = O(\beta(s))$ means that there exist two positive constants k and c such that $\tilde{\beta}(s) \leq k\beta(s)$, $\forall |s| < c$.

2. Problem Formulation

In this paper, we consider the following class of cascaded nonlinear systems with dynamic uncertainties:

$$\begin{aligned} \dot{\eta} &= q(t, \eta, y), \\ \dot{x}_1 &= d_1(t) x_2 + g_1(t, \eta, x), \\ &\vdots \\ \dot{x}_n &= d_n(t) u + g_n(t, \eta, x), \\ y &= x_1, \end{aligned} \quad (1)$$

where $u \in \mathbf{R}$ is the control input, $y \in \mathbf{R}$ is the system output, $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ are the system states, and $\eta \in \mathbf{R}^r$ is referred to as dynamic uncertainties, which is unmeasured and hence is not available for feedback design. The continuous functions $d_i(t)$ ($i = 1, \dots, n$) called the control coefficients are assumed to be unknown; particularly, $d_i(t) \neq 0$; the unmodeled (or uncertain) dynamics $q(\cdot)$ and $g_i(\cdot)$ ($i = 1, \dots, n$) are locally Lipschitz.

The control objective in this paper is to find a smooth, dynamic, partial-state feedback law of the form

$$\begin{aligned} \dot{\xi} &= \chi_\xi(\xi, x), \\ u &= \chi_u(\xi, x), \end{aligned} \quad (2)$$

where χ_ξ and χ_u are smooth functions such that all solutions $(\eta(t), x(t), \xi(t))$ in closed-loop system are bounded on $[0, \infty)$ and specially the system states $(\eta(t), x(t))$ asymptotically converge to the origin. Toward this end, throughout the paper, we make the following assumptions on system (1).

Assumption 1. The η -subsystem is input-to-state stable (ISS) with state η and input y ; that is, there exists a positive-definite and proper ISS-Lyapunov function V_0 , such that

$$\begin{aligned} \underline{\alpha}(|\eta|) \leq V_0(\eta) \leq \bar{\alpha}(|\eta|), \\ \frac{\partial V_0}{\partial \eta}(\eta) q(t, \eta, y) \leq -\alpha_0(|\eta|) + \delta_0 \gamma_0(|y|), \end{aligned} \quad (3)$$

where $\underline{\alpha}(\cdot), \bar{\alpha}(\cdot), \alpha_0(\cdot), \gamma_0(\cdot) \in K_\infty$ and $\delta_0 > 0$ is an unknown constant.

Remark 2. According to [19], one knows that η -subsystem satisfying (3) is ISS, and the function pair $(\alpha_0, \delta_0 \gamma_0)$ is viewed as the supply rates. Since δ_0 in (3) is unknown, the dynamic uncertainty has uncertain ISS supply rates. This is different from the existing results reported in literatures, where the ISS dynamic uncertainty is investigated with the supply rates assumed to be known a priori, such as [8, 11, 13, 15].

Assumption 3. For the uncertain nonlinearities $g_i(t, \eta, x)$ ($i = 1, \dots, n$), there exist unknown positive constants p_{ij} ($i = 1, \dots, n; j = 1, 2$) such that

$$|g_i(t, \eta, x)| \leq p_{i1} \phi_{i1}(|\eta|) + p_{i2} \phi_{i2}(|x_1, \dots, x_i|), \quad (4)$$

$i = 1, \dots, n,$

where $\phi_{ij}(\cdot)$ are known smooth functions and $\phi_{ij}(0) = 0$ ($i = 1, \dots, n; j = 1, 2$).

Assumption 4. There exist known positive constants $\underline{d}_i$ and $\bar{d}_i$ ($i = 1, \dots, n$), such that

$$0 < \underline{d}_i \leq d_i(t) \leq \bar{d}_i, \quad i = 1, \dots, n. \quad (5)$$

To deal with the unmeasured state η , we have the following lemma, which plays an important role in the coming feedback design and stability analysis.

Lemma 5. Consider the η -subsystem satisfying Assumption 1. Suppose

$$\gamma_0(s) = O(s^2), \quad (6)$$

and then we can choose a positive continuous function $\rho(\cdot)$, such that the function

$$\bar{V}_0(\eta) = \int_0^{V_0(\eta)} \rho(s) ds \quad (7)$$

is another candidate ISS-Lyapunov function satisfying

$$\frac{\partial \bar{V}_0}{\partial \eta}(\eta) q(t, \eta, y) \leq -(1 - \epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) + \delta \gamma(|y|), \quad (8)$$

where ϵ ($0 < \epsilon < 1$) is a small design constant, $\delta > 0$ is an unknown constant, and γ is a K_∞ -function with $\gamma(s) = O(s^2)$.

Remark 6. If $\gamma(s) = O(s^2)$, according to Lemma 2 in [19], there exists a smooth nondecreasing function $\bar{\gamma}(\cdot)$ satisfying

$$\gamma(s) \leq \bar{\gamma}(s) s^2. \quad (9)$$

Lemma 7. For any C^1 function $f(x_1, \dots, x_n)$, there exist continuous functions $f_i(x_1, \dots, x_n)$ ($1 \leq i \leq n$), such that

$$f(x_1, \dots, x_n) = f(0, \dots, 0) + \sum_{i=1}^n x_i f_i(x_1, \dots, x_n). \quad (10)$$

Remark 8. According to Lemma 7, from $\phi_{i2}(0) = 0$ ($i = 1, \dots, n$) in Assumption 3, it is known that, for each $\phi_{i2}(\cdot)$, there exist smooth functions $\phi_{i2j}(\cdot)$ ($1 \leq j \leq i$) satisfying

$$\phi_{i2}(x_1, \dots, x_i) = \sum_{j=1}^i x_j \phi_{i2j}(x_1, \dots, x_i), \quad i = 1, \dots, n. \quad (11)$$

3. Controller Design

In this section, we give the controller design procedure using the backstepping design method.

Step 1. Starting with the x_1 -subsystem $\dot{x}_1 = d_1(t)x_2 + g_1(t, \eta, x)$. We consider the variable x_2 as the virtual control input. Let $z_1 = x_1$ and $z_2 = x_2 - \vartheta_1$ where ϑ_1 is the intermediate control input. Considering Lemma 5 and Remark 6, along solutions of (1), the time derivative of the function

$$V_1 = \frac{1}{2} z_1^2 + \bar{V}_0(\eta) \quad (12)$$

satisfies

$$\begin{aligned} \dot{V}_1 &\leq z_1 (d_1(t) \vartheta_1 + g_1(t, \eta, x)) + d_1(t) z_1 z_2 \\ &\quad - (1 - \epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) + \delta \bar{\gamma}(|x_1|) x_1^2. \end{aligned} \quad (13)$$

According to Assumption 3 and the completion of squares, we have

$$\begin{aligned} z_1 g_1(t, \eta, x) &\leq |z_1| (p_{11} \phi_{11}(|\eta|) + p_{12} |z_1| \phi_{121}(x_1)) \\ &\leq \phi_{11}^2(|\eta|) + \frac{1}{4} z_1^2 p_{11}^2 \\ &\quad + z_1^2 p_{12} |\phi_{121}(x_1)|. \end{aligned} \quad (14)$$

Define $p^* = \max\{\delta, p_{11}, p_{12}, p_{11}^2, p_{12}^2 \mid i = 1, \dots, n\}$, and we get

$$\begin{aligned} z_1 g_1(t, \eta, x) &+ \delta \bar{\gamma}(|x_1|) x_1^2 \\ &\leq \phi_{11}^2(|\eta|) + \frac{1}{4} z_1^2 p_{11}^2 + z_1^2 p_{12} |\phi_{121}(x_1)| \\ &\quad + \delta \bar{\gamma}(|x_1|) x_1^2 \\ &\leq \phi_{11}^2(|\eta|) + p^* z_1^2 \left(\frac{1}{4} + |\phi_{121}(x_1)| + \bar{\gamma}(|x_1|) \right) \\ &\leq \phi_{11}^2(|\eta|) + z_1^2 \hat{\phi}_1(x_1) p^* \end{aligned} \quad (15)$$

with a new smooth function $\hat{\phi}_1(x_1) \geq 1/4 + |\phi_{121}(x_1)| + \bar{\gamma}(|x_1|) > 0$. As a result, there holds

$$\begin{aligned} \dot{V}_1 &\leq -(1 - \epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) + d_1(t) z_1 \vartheta_1 \\ &\quad + d_1(t) z_1 z_2 + \phi_{11}^2(|\eta|) + z_1^2 \hat{\phi}_1(x_1) p^*. \end{aligned} \quad (16)$$

Considering the unknown constant p^* in (16), we use an adaptive signal $\hat{p}$ to estimate p^* . Consequently, we augment V_1 with the parameter estimation error $\tilde{p} = p^* - \hat{p}$, such as

$$\bar{V}_1 = V_1 + \frac{1}{2Y} \tilde{p}^2, \quad (17)$$

where $Y > 0$ is the design parameter. In view of (16) and $p^* = \hat{p} + \tilde{p}$, a direct substitution leads to

$$\begin{aligned} \dot{\bar{V}}_1 &\leq -(1 - \epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) + d_1(t) z_1 \vartheta_1 \\ &\quad + z_1^2 \hat{\phi}_1(x_1) \hat{p} + d_1(t) z_1 z_2 + \phi_{11}^2(|\eta|) \\ &\quad + \frac{1}{Y} \tilde{p} (Y \hat{\phi}_1(x_1) z_1^2 - \dot{\hat{p}}). \end{aligned} \quad (18)$$

Considering Assumption 4, we take the virtual control

$$\vartheta_1 = -\frac{1}{\underline{d}_1} (\nu_1 + \hat{\phi}_1(x_1) \hat{p}) z_1, \quad (19)$$

where $\nu_1 > 0$ is a design constant to be determined later. Let

$$\tau_1 = Y \hat{\phi}_1(x_1) z_1^2, \quad (20)$$

and then we get

$$\begin{aligned} \dot{\bar{V}}_1 &\leq -(1 - \epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - \nu_1 z_1^2 + \phi_{11}^2(|\eta|) \\ &\quad + d_1(t) z_1 z_2 + \frac{1}{Y} \tilde{p} (\tau_1 - \dot{\hat{p}}). \end{aligned} \quad (21)$$

Remark 9. It is noted that, in (19), we assume that $\hat{p}(t) \geq 0$. In fact, from the updating law of $\hat{p}$ given later, this property can be guaranteed by choosing the initial condition $\hat{p}(0) \geq 0$. Alternatively, using the idea in [20], we also can apply the $\hat{p} \leq \sqrt{1 + \hat{p}^2}$ or $\hat{p} \leq (1 + \hat{p}^2)/2$ instead of $\hat{p}$.

Step 2. Let $z_3 = x_3 - \vartheta_2$, where ϑ_2 is the virtual control law. We consider the Lyapunov function

$$V_2 = \bar{V}_1 + \frac{1}{2} z_2^2. \quad (22)$$

In view of (21), we have

$$\begin{aligned} \dot{V}_2 \leq & -(1 - \epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - \nu_1 z_1^2 + \phi_{11}^2(|\eta|) \\ & + d_2(t) z_2 z_3 + \frac{1}{Y} \tilde{p}(\tau_1 - \hat{p}) + z_2 \left(d_2(t) \vartheta_2 \right. \\ & + g_2(t, \eta, x) - \frac{\partial \vartheta_1}{\partial x_1} g_1(t, \eta, x) + d_1(t) z_1 \\ & \left. - \frac{\partial \vartheta_1}{\partial x_1} d_1(t) x_2 - \frac{\partial \vartheta_1}{\partial \hat{p}} \dot{\hat{p}} \right). \end{aligned} \quad (23)$$

From (4) and (11), the following calculations hold:

$$\begin{aligned} z_2 g_2(t, \eta, x) \leq & |z_2| p_{21} \phi_{21}(|\eta|) \\ & + |z_2| p_{22} |x_1| \phi_{121}(\bar{x}_2) \\ & + |z_2| p_{22} (|z_2 + \vartheta_1|) \phi_{122}(\bar{x}_2). \end{aligned} \quad (24)$$

Like the calculations in (14), by completing the squares, we have

$$\begin{aligned} & -z_2 \frac{\partial \vartheta_1}{\partial x_1} g_1(t, \eta, x) \\ & \leq \phi_{11}^2(|\eta|) + z_1^2 \\ & + p^* z_2^2 \left(\frac{1}{4} \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 + \frac{1}{4} \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 \phi_{121}^2(x_1) \right). \end{aligned} \quad (25)$$

Define $\hat{\phi}_{21}(x_1, \hat{p}) = (1/4)(\partial \vartheta_1 / \partial x_1)^2 (1 + \phi_{121}^2(x_1))$, and then we have

$$\begin{aligned} -z_2 \frac{\partial \vartheta_1}{\partial x_1} g_1(t, \eta, x) \leq & \phi_{11}^2(|\eta|) + z_1^2 \\ & + z_2^2 \hat{\phi}_{21}(x_1, \hat{p}) p^*. \end{aligned} \quad (26)$$

In the same manner, using the completion of squares again, it can be verified that

$$\begin{aligned} |z_2| p_{21} \phi_{21}(|\eta|) & \leq \phi_{21}^2(|\eta|) + \frac{1}{4} p_{21}^2 z_2^2, \\ |z_2| p_{22} |z_1| \phi_{121}(\bar{x}_2) & \leq z_1^2 + \frac{1}{4} p_{22}^2 z_2^2 \phi_{121}^2(\bar{x}_2), \\ |z_2| p_{22} (|z_2 + \vartheta_1|) \phi_{122}(\bar{x}_2) \\ & \leq z_2^2 p_{22} \phi_{122}(\bar{x}_2) + |z_2| p_{22} |\vartheta_1| \phi_{122}(\bar{x}_2) \\ & \leq z_2^2 p_{22} \phi_{122}(\bar{x}_2) \\ & + |z_2| p_{22} \left| -\frac{1}{\underline{d}_1} (\nu_1 + \hat{\phi}_1(x_1) \hat{p}) z_1 \right| \phi_{122}(\bar{x}_2) \end{aligned}$$

$$\begin{aligned} & \leq z_2^2 p_{22} \phi_{122}(\bar{x}_2) + z_1^2 \\ & + \frac{1}{4} p_{22}^2 z_2^2 \frac{1}{\underline{d}_1^2} (\nu_1 + \hat{\phi}_1(x_1) \hat{p})^2 \phi_{122}^2(\bar{x}_2). \end{aligned} \quad (27)$$

As a result,

$$\begin{aligned} z_2 g_2(t, \eta, x) \leq & \phi_{21}^2(|\eta|) + 2z_1^2 + z_2^2 p^* \left(\frac{1}{4} \right. \\ & + \frac{1}{4} \phi_{121}^2(\bar{x}_2) + \phi_{122}(\bar{x}_2) \\ & \left. + \frac{1}{4} \frac{1}{\underline{d}_1^2} (\nu_1 + \hat{\phi}_1(x_1) \hat{p})^2 \phi_{122}^2(\bar{x}_2) \right). \end{aligned} \quad (28)$$

Define $\hat{\phi}_{22}(\bar{x}_2, \hat{p}) = 1/4 + (1/4)\phi_{121}^2(\bar{x}_2) + \phi_{122}(\bar{x}_2) + (1/4)(1/\underline{d}_1^2)(\nu_1 + \hat{\phi}_1(x_1) \hat{p})^2 \phi_{122}^2(\bar{x}_2)$, and then

$$z_2 g_2(t, \eta, x) \leq \phi_{21}^2(|\eta|) + 2z_1^2 + z_2^2 p^* \hat{\phi}_{22}(\bar{x}_2, \hat{p}). \quad (29)$$

Let $\hat{\phi}_2(\bar{x}_2, \hat{p}) = \hat{\phi}_{21}(x_1, \hat{p}) + \hat{\phi}_{22}(\bar{x}_2, \hat{p}) > 0$, and then we get

$$\begin{aligned} & z_2 \left(g_2(t, \eta, x) - \frac{\partial \vartheta_1}{\partial x_1} g_1(t, \eta, x) \right) \\ & \leq \sum_{j=1}^2 \phi_{j1}^2(|\eta|) + 3z_1^2 + z_2^2 \hat{\phi}_2(\bar{x}_2, \hat{p}) p^*. \end{aligned} \quad (30)$$

From Assumption 4, it is deduced that

$$\begin{aligned} & z_2 \left(d_1(t) z_1 - \frac{\partial \vartheta_1}{\partial x_1} d_1(t) x_2 \right) \\ & \leq |z_2| |d_1(t)| |z_1| + |z_2| \left| \frac{\partial \vartheta_1}{\partial x_1} \right| |d_1(t)| (|z_2| + |\vartheta_1|) \\ & \leq z_1^2 + \frac{\bar{d}_1^2}{4} z_2^2 + z_2^2 \frac{\bar{d}_1}{2} \left(1 + \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 \right) + z_1^2 \\ & + \frac{1}{4} z_2^2 \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 \frac{\bar{d}_1^2}{\underline{d}_1^2} (\nu_1 + \hat{\phi}_1(x_1) \hat{p})^2. \end{aligned} \quad (31)$$

Define $\bar{\phi}_{21}(\bar{x}_2, \hat{p}) = \bar{d}_1^2/4 + (1/4)(\bar{d}_1^2/\underline{d}_1^2)(\partial \vartheta_1 / \partial x_1)^2 (\nu_1 + \hat{\phi}_1(x_1) \hat{p})^2 + (\bar{d}_1/2)(1 + (\partial \vartheta_1 / \partial x_1)^2)$, and there holds

$$z_2 \left(d_1(t) z_1 - \frac{\partial \vartheta_1}{\partial x_1} d_1(t) x_2 \right) \leq 2z_1^2 + z_2^2 \bar{\phi}_{21}(\bar{x}_2, \hat{p}). \quad (32)$$

Take the following notation:

$$\tau_2 = \tau_1 + Y z_2^2 \hat{\phi}_2(\bar{x}_2, \hat{p}), \quad (33)$$

and furthermore, in view of (30) and (32), we obtain

$$\begin{aligned} \dot{V}_2 \leq & -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - (\nu_1 - 5) z_1^2 \\ & + 2\phi_{11}^2(|\eta|) + \phi_{21}^2(|\eta|) + d_2(t) z_2 z_3 + z_2 \left(d_2(t) \vartheta_2 \right. \\ & + z_2 \hat{\phi}_2(\bar{x}_2, \hat{p}) \hat{p} + z_2 \bar{\phi}_{21}(\bar{x}_2, \hat{p}) - \frac{\partial \vartheta_1}{\partial \hat{p}} \dot{\hat{p}} \left. \right) + \frac{1}{Y} \\ & \cdot \tilde{p}(\tau_2 - \dot{\hat{p}}). \end{aligned} \quad (34)$$

Considering $d_2(t)$ is unknown, the term of $-(\partial \vartheta_1 / \partial \hat{p}) \dot{\hat{p}}$ can not be canceled. In fact, we express (34) as

$$\begin{aligned} \dot{V}_2 \leq & -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - (\nu_1 - 5) z_1^2 \\ & + 2\phi_{11}^2(|\eta|) + \phi_{21}^2(|\eta|) + d_2(t) z_2 z_3 + z_2 \left(d_2(t) \vartheta_2 \right. \\ & + z_2 \hat{\phi}_2(\bar{x}_2, \hat{p}) \hat{p} + z_2 \bar{\phi}_{21}(\bar{x}_2, \hat{p}) - \frac{\partial \vartheta_1}{\partial \hat{p}} \tau_2 \left. \right) + \left(\frac{1}{Y} \tilde{p} \right. \\ & + z_2 \frac{\partial \vartheta_1}{\partial \hat{p}} \left. \right) (\tau_2 - \dot{\hat{p}}). \end{aligned} \quad (35)$$

For the term of $-z_2(\partial \vartheta_1 / \partial \hat{p}) \tau_2$, according to (33), it can be dealt with as follows:

$$\begin{aligned} -z_2 \frac{\partial \vartheta_1}{\partial \hat{p}} \tau_2 &= -z_2 \frac{\partial \vartheta_1}{\partial \hat{p}} Y (z_1^2 \hat{\phi}_1(x_1) + z_2^2 \hat{\phi}_2(\bar{x}_2, \hat{p})) \\ &\leq z_1^2 + \frac{1}{4} Y^2 z_2^2 \left(\frac{\partial \vartheta_1}{\partial \hat{p}} \right)^2 z_1^2 \hat{\phi}_1^2(x_1) + z_2^2 \frac{1}{4} Y (1 + z_2^2) \\ &\quad \cdot \hat{\phi}_2(\bar{x}_2, \hat{p}) \left(1 + \left(\frac{\partial \vartheta_1}{\partial \hat{p}} \right)^2 \right) = z_1^2 \\ &\quad + z_2^2 \left(\frac{1}{4} Y^2 \left(\frac{\partial \vartheta_1}{\partial \hat{p}} \right)^2 z_1^2 \hat{\phi}_1^2(x_1) \right. \\ &\quad + \left. \frac{1}{4} Y (1 + z_2^2) \hat{\phi}_2(\bar{x}_2, \hat{p}) \left(1 + \left(\frac{\partial \vartheta_1}{\partial \hat{p}} \right)^2 \right) \right). \end{aligned} \quad (36)$$

Define the following smooth function: $\bar{\phi}_{22}(\bar{x}_2, \hat{p}) = (1/4)Y^2 z_1^2 (\partial \vartheta_1 / \partial \hat{p})^2 \hat{\phi}_1^2(x_1) + (1/4)Y(1 + z_2^2) \hat{\phi}_2(\bar{x}_2, \hat{p}) (1 + (\partial \vartheta_1 / \partial \hat{p})^2)$, and we get

$$-z_2 \frac{\partial \vartheta_1}{\partial \hat{p}} \tau_2 \leq z_1^2 + z_2^2 \bar{\phi}_{22}(\bar{x}_2, \hat{p}). \quad (37)$$

Denote $\bar{\phi}_2(\bar{x}_2, \hat{p}) = \bar{\phi}_{21}(\bar{x}_2, \hat{p}) + \bar{\phi}_{22}(\bar{x}_2, \hat{p}) > 0$, and a direct substitution leads to

$$\begin{aligned} \dot{V}_2 \leq & -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - (\nu_1 - 6) z_1^2 \\ & + 2\phi_{11}^2(|\eta|) + \phi_{21}^2(|\eta|) + d_2(t) z_2 z_3 \\ & + z_2 (d_2(t) \vartheta_2 + z_2 \hat{\phi}_2(\bar{x}_2, \hat{p}) \hat{p} + z_2 \bar{\phi}_2(\bar{x}_2, \hat{p})) \\ & + \left(\frac{1}{Y} \tilde{p} + z_2 \frac{\partial \vartheta_1}{\partial \hat{p}} \right) (\tau_2 - \dot{\hat{p}}). \end{aligned} \quad (38)$$

Take the virtual control

$$\vartheta_2 = -\frac{1}{d_2} \left(\nu_2 + \hat{\phi}_2(\bar{x}_2, \hat{p}) \hat{p} + \bar{\phi}_2(\bar{x}_2, \hat{p}) \right) z_2, \quad (39)$$

which is such that

$$\begin{aligned} \dot{V}_2 \leq & -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - (\nu_1 - 6) z_1^2 - \nu_2 z_2^2 \\ & + 2\phi_{11}^2(|\eta|) + \phi_{21}^2(|\eta|) + d_2(t) z_2 z_3 \\ & + \left(\frac{1}{Y} \tilde{p} + z_2 \frac{\partial \vartheta_1}{\partial \hat{p}} \right) (\tau_2 - \dot{\hat{p}}). \end{aligned} \quad (40)$$

Step i ($3 \leq i \leq n$). Assume that, in Step $i-1$, we have designed the virtual control ϑ_j ($j = 1, \dots, i-1$) and the tuning function τ_j ($j = 1, \dots, i-1$), such that the Lyapunov function

$$V_{i-1} = \sum_{j=1}^{i-1} \frac{1}{2} z_j^2 + \frac{1}{2Y} \tilde{p}^2 \quad (41)$$

satisfies

$$\begin{aligned} \dot{V}_{i-1} \leq & -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) \\ & - \left(\nu_1 - 6 - \sum_{j=3}^{i-1} (2j+1) \right) z_1^2 \\ & - \left(\nu_2 - 7 - \sum_{j=4}^{i-1} (2j) \right) z_2^2 - \dots \\ & - (\nu_{i-2} - 7) z_{i-2}^2 - \nu_{i-1} z_{i-1}^2 \\ & + \left(\frac{1}{Y} \tilde{p} + \sum_{j=2}^{i-1} z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \right) (\tau_{i-1} - \dot{\hat{p}}) \\ & + d_{i-1}(t) z_{i-1} z_i + \sum_{j=1}^{i-1} (i-j) \phi_{j1}^2(|\eta|). \end{aligned} \quad (42)$$

In what follows, it will be shown that the property (42) also holds in Step i .

Let ϑ_i be the virtual control and $z_{i+1} = x_{i+1} - \vartheta_i$. Consider the Lyapunov function

$$V_i = V_{i-1} + \frac{1}{2} z_i^2. \quad (43)$$

To begin with, the dynamics of z_i can be expressed as

$$\begin{aligned} \dot{z}_i &= d_i(t) \vartheta_i + g_i(t, \eta, x) - \sum_{j=1}^{i-1} \frac{\partial \vartheta_{i-1}}{\partial x_j} g_j(t, \eta, x) \\ &\quad - \sum_{j=1}^{i-1} \frac{\partial \vartheta_{i-1}}{\partial x_j} d_j(t) x_{j+1} - \frac{\partial \vartheta_{i-1}}{\partial \hat{p}} \dot{\hat{p}} + d_i(t) z_{i+1}. \end{aligned} \quad (44)$$

For notational convenience, denote $\vartheta_0 = 0$. From $x_k = z_k + \vartheta_{k-1} = z_k - (1/d_{k-1})(\nu_{k-1} + \hat{\phi}_{k-1}(x_{[k-1]}, \hat{p}) \hat{p} +$

$\bar{\phi}_i(x_{[k-1]}, \hat{p})z_{k-1}$ ($k = 1, \dots, i-1$), like the calculations in (26), it can be verified that there exist smooth functions $\hat{\phi}_{ij}(\cdot)$ ($j = 1, \dots, i$) > 0 , such that, for $j = 1, \dots, i-1$,

$$\begin{aligned} -z_i \frac{\partial \vartheta_{i-1}}{\partial x_j} g_j(t, \eta, x) &\leq \phi_{j1}^2(|\eta|) + \sum_{k=1}^{j-1} (2z_k^2) + z_j^2 \\ &\quad + z_i^2 \hat{\phi}_{ij}(\bar{x}_j, \hat{p}) p^*, \\ z_i g_i(t, \eta, x) &\leq \phi_{i1}^2(|\eta|) \\ &\quad + \sum_{k=1}^{i-1} (2z_k^2) + z_i^2 \hat{\phi}_{ii}(\bar{x}_i, \hat{p}) p^*. \end{aligned} \quad (45)$$

Define $\hat{\phi}_i(\bar{x}_i, \hat{p}) = \sum_{j=1}^i \hat{\phi}_{ij}(\bar{x}_j, \hat{p})$; from (45), it follows that

$$\begin{aligned} z_i \left(g_i(t, \eta, x) - \sum_{j=1}^{i-1} \frac{\partial \vartheta_{i-1}}{\partial x_j} g_j(t, \eta, x) \right) \\ \leq \sum_{j=1}^i \phi_{j1}^2(|\eta|) \\ + \sum_{j=1}^{i-1} (1 + 2(i-j)) z_j^2 + z_i^2 \hat{\phi}_i(\bar{x}_i, \hat{p}) p^*. \end{aligned} \quad (46)$$

Similar to (31) and (32), there exists a smooth function $\bar{\phi}_{i1}(\bar{x}_i, \hat{p})$ such that

$$\begin{aligned} z_i \left(d_{i-1}(t) z_{i-1} - \sum_{j=1}^{i-1} \frac{\partial \vartheta_{i-1}}{\partial x_j} d_j(t) x_{j+1} \right) \\ \leq z_1^2 + \sum_{k=2}^{i-2} (2z_k^2) + 3z_{i-1}^2 + z_i^2 \bar{\phi}_{i1}(\bar{x}_i, \hat{p}). \end{aligned} \quad (47)$$

Let

$$\tau_i = \tau_{i-1} + \Upsilon z_i^2 \hat{\phi}_i(\bar{x}_i, \hat{p}), \quad (48)$$

and then, from (47) and (48), there holds

$$\begin{aligned} \dot{V}_i &\leq -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - \left(\nu_1 - 6 \right. \\ &\quad - \sum_{j=3}^{i-1} (2j+1) - 2i \left. \right) z_1^2 - \left(\nu_2 - 7 \right. \\ &\quad - \sum_{j=4}^{i-1} (2j) - 1 - 2(i-1) \left. \right) z_2^2 - \dots - (\nu_{i-2} - 7 - 7) \\ &\quad \cdot z_{i-2}^2 - (\nu_{i-1} - 6) z_{i-1}^2 + \sum_{j=1}^i (i-j+1) \phi_{j1}^2(|\eta|) \\ &\quad + d_i(t) z_i z_{i+1} + z_i \left(d_i(t) \vartheta_i + z_i \hat{\phi}_i(\bar{x}_i, \hat{p}) \hat{p} \right. \\ &\quad \left. + z_i \bar{\phi}_{i1}(\bar{x}_i, \hat{p}) \right) \end{aligned}$$

$$\begin{aligned} - \sum_{j=2}^{i-1} z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \cdot \Upsilon z_i \hat{\phi}_i(\bar{x}_i, \hat{p}) - \frac{\partial \vartheta_{i-1}}{\partial \hat{p}} \tau_i \left. \right) + \left(\frac{1}{\Upsilon} \tilde{p} \right. \\ \left. + \sum_{j=2}^i z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \right) (\tau_i - \dot{\hat{p}}). \end{aligned} \quad (49)$$

Remark 10. In (49), we subtract two terms $-\sum_{j=2}^{i-1} z_j (\partial \vartheta_{j-1} / \partial \hat{p}) \cdot \Upsilon z_i \hat{\phi}_i(\bar{x}_i, \hat{p})$ and $-(\partial \vartheta_{i-1} / \partial \hat{p}) \tau_i$ in the brackets to generate the term $((1/\Upsilon) \tilde{p} + \sum_{j=2}^i z_j (\partial \vartheta_{j-1} / \partial \hat{p})) (\tau_i - \dot{\hat{p}})$.

However, from (49), it can be seen that, due to the unknown control coefficient $d_i(t)$, the terms $-\sum_{j=2}^{i-1} z_j (\partial \vartheta_{j-1} / \partial \hat{p}) \cdot z_i \hat{\phi}_i(\bar{x}_i, \hat{p})$ and $-(\partial \vartheta_{i-1} / \partial \hat{p}) \tau_i$ could not be directly canceled by the coming virtual control ϑ_i . We get around this burden by the following estimates:

$$\begin{aligned} -z_i \sum_{j=2}^{i-1} z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \cdot \Upsilon z_i \hat{\phi}_i(\bar{x}_i, \hat{p}) \\ \leq z_i^2 \frac{1}{2} \left(1 + \Upsilon^2 \left(\sum_{j=2}^{i-1} z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \right)^2 \hat{\phi}_i^2(\bar{x}_i, \hat{p}) \right), \\ -z_i \frac{\partial \vartheta_{i-1}}{\partial \hat{p}} \tau_i = -z_i \frac{\partial \vartheta_{i-1}}{\partial \hat{p}} \Upsilon (z_1^2 \hat{\phi}_1 + \dots + z_i^2 \hat{\phi}_i) \\ \leq \sum_{j=1}^{i-1} z_j^2 + z_i^2 \frac{1}{4} \Upsilon^2 \left(\frac{\partial \vartheta_{i-1}}{\partial \hat{p}} \right)^2 \sum_{j=1}^i z_j^2 \hat{\phi}_j^2(\bar{x}_j, \hat{p}). \end{aligned} \quad (50)$$

Let $\bar{\phi}_i(\bar{x}_i, \hat{p}) = (1/4) \Upsilon^2 (\partial \vartheta_{i-1} / \partial \hat{p})^2 \sum_{j=1}^i z_j^2 \hat{\phi}_j^2(\bar{x}_j, \hat{p}) + (1/2)(1 + \Upsilon^2 (\sum_{j=2}^{i-1} z_j (\partial \vartheta_{j-1} / \partial \hat{p}))^2 \hat{\phi}_i^2(\bar{x}_i, \hat{p})) + \bar{\phi}_{i1}(\bar{x}_i, \hat{p}) > 0$, and then we get

$$\begin{aligned} z_i \left(z_i \bar{\phi}_{i1}(\bar{x}_i, \hat{p}) - \sum_{j=2}^{i-1} z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \cdot z_i \hat{\phi}_i(\bar{x}_i, \hat{p}) \right. \\ \left. - \frac{\partial \vartheta_{i-1}}{\partial \hat{p}} \tau_i \right) \leq \sum_{j=1}^{i-1} z_j^2 + z_i^2 \bar{\phi}_i(\bar{x}_i, \hat{p}). \end{aligned} \quad (51)$$

By substituting (51) into (49), it follows that

$$\begin{aligned} \dot{V}_i &\leq -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) \\ &\quad - \left(\nu_1 - 6 - \sum_{j=3}^i (2j+1) \right) z_1^2 \\ &\quad - \left(\nu_2 - 7 - \sum_{j=4}^i (2j) \right) z_2^2 - \dots \\ &\quad - (\nu_{i-2} - 7 - 8) z_{i-2}^2 - (\nu_{i-1} - 7) z_{i-1}^2 \\ &\quad + z_i (d_i(t) \vartheta_i + z_i \hat{\phi}_i(\bar{x}_i, \hat{p}) \hat{p} + z_i \bar{\phi}_i(\bar{x}_i, \hat{p})) \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{Y} \tilde{p} + \sum_{j=2}^i z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \right) (\tau_i - \dot{\hat{p}}) \\
& + \sum_{j=1}^i (i-j+1) \phi_{j1}^2(|\eta|) + d_i(t) z_i z_{i+1}.
\end{aligned} \quad (52)$$

Take the virtual control

$$\vartheta_i = -\frac{1}{\underline{d}_i} \left(v_i + \hat{\phi}_i(\bar{x}_i, \hat{p}) \hat{p} + \bar{\phi}_i(\bar{x}_i, \hat{p}) \right) z_i, \quad (53)$$

and then the following holds:

$$\begin{aligned}
\dot{V}_i & \leq -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) \\
& - \left(v_1 - 6 - \sum_{j=3}^i (2j+1) \right) z_1^2 \\
& - \left(v_2 - 7 - \sum_{j=4}^i (2j) \right) z_2^2 - \cdots - (v_{i-1} - 7) z_{i-1}^2 \\
& - v_i z_i^2 + \sum_{j=1}^i (i-j+1) \phi_{j1}^2(|\eta|) \\
& + \left(\frac{1}{Y} \tilde{p} + \sum_{j=2}^i z_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}} \right) (\tau_i - \dot{\hat{p}}) + d_i(t) z_i z_{i+1}.
\end{aligned} \quad (54)$$

In particular, when $i = n$, the actual control u appears, and we choose the controller u and updating law τ_n for $\hat{p}(t)$ as follows:

$$u = -\frac{1}{\underline{d}_n} \left(v_n + \hat{\phi}_n(x, \hat{p}) \hat{p} + \bar{\phi}_n(x, \hat{p}) \right) z_n, \quad (55)$$

$$\dot{\hat{p}} = \tau_n = \sum_{j=1}^n Y z_j^2 \hat{\phi}_j(\bar{x}_j, \hat{p}), \quad (56)$$

such that the Lyapunov function

$$V_n = \sum_{j=1}^n \frac{1}{2} z_j^2 + \frac{1}{2Y} \tilde{p}^2 \quad (57)$$

satisfies

$$\begin{aligned}
\dot{V}_n & \leq -(1-\epsilon) \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) \\
& - \left(v_1 - 6 - \sum_{j=3}^n (2j+1) \right) z_1^2 \\
& - \left(v_2 - 7 - \sum_{j=4}^n (2j) \right) z_2^2 - \cdots - (v_{n-1} - 7) z_{n-1}^2 \\
& - v_n z_n^2 + \sum_{j=1}^n (n-j+1) \phi_{j1}^2(|\eta|).
\end{aligned} \quad (58)$$

This completes the controller design procedure.

4. Main Results

After the above controller design procedure, we are now ready to state the main results.

Theorem 11. Suppose the investigated system (1) satisfies Assumptions 1, 3, and 4 together with the local conditions

$$\phi_{i1}^2(s) = O(\alpha_0(s)), \quad i = 1, \dots, n. \quad (59)$$

Then all the signals of the closed-loop system (1) with the controller (55) and updating law (56) are bounded on $[0, +\infty)$. Specifically, the following convergent property holds:

$$\lim_{t \rightarrow \infty} (|\eta(t)| + |x(t)| + |u(t)|) = 0. \quad (60)$$

Proof. From the local conditions (59), one can choose the smooth function $\rho(\cdot)$ such that

$$\sum_{j=1}^n (n-j+1) \phi_{j1}^2(|\eta|) \leq \frac{1-\epsilon}{2} \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|). \quad (61)$$

One can choose positive constants v_j ($j = 1, \dots, n$) satisfying

$$\begin{aligned}
v_1 - 6 - \sum_{j=3}^n (2j+1) & \geq 1, \\
v_2 - 7 - \sum_{j=4}^n (2j) & \geq 1, \\
& \vdots \\
v_{n-1} - 7 & \geq 1, \\
v_n & \geq 1.
\end{aligned} \quad (62)$$

Then, from (58), (61), and (62), it follows that

$$\dot{V}_n \leq -\frac{1-\epsilon}{2} \rho \circ \underline{\alpha}(|\eta|) \alpha_0(|\eta|) - \sum_{i=1}^n z_i^2. \quad (63)$$

Now, assume that the maximal interval of existence of the solution of the closed-loop system starting from any given initial conditions is $[0, t_f)$ for some $t_f > 0$. In view of $0 < \epsilon < 1$, from (63), it can be concluded that V_n and hence the variables $(\eta(t), z_1(t), \dots, z_n(t), \tilde{p}(t))$ are bounded on $[0, t_f)$. In terms of $\tilde{p}(t) = p^* - \hat{p}(t)$, we obtain the boundedness of $\hat{p}(t)$. Considering (53), it can be derived that ϑ_i ($i = 1, \dots, n$) are bounded. In view of $z_i = x_i - \vartheta_{i-1}$ ($\vartheta_0 = 0$), we further obtain that the states x_i ($i = 1, \dots, n$) are bounded on $[0, t_f)$.

So far all the closed-loop system signals are bounded on $[0, t_f)$. This guarantees that the finite time escape will not happen. Therefore, it is natural that t_f can be maximized to $+\infty$ by means of Theorem 3.3 in [21]. Next we will prove the convergence property of (60).

Again, according to (63), considering $\underline{\alpha}$ and α_0 are K_∞ -functions, a direct application of LaSalle's invariance

principle in [21] guarantees the convergence property of $(\eta(t), z_1(t), \dots, z_n(t))$; that is,

$$\begin{aligned} \lim_{t \rightarrow \infty} \eta(t) &= 0, \\ \lim_{t \rightarrow \infty} z_i(t) &= 0 \quad (i = 1, \dots, n). \end{aligned} \quad (64)$$

As a consequence, from (53) and (55), the following holds:

$$\lim_{t \rightarrow \infty} \vartheta_i(t) = 0 \quad (i = 1, \dots, n). \quad (65)$$

Particularly,

$$\lim_{t \rightarrow \infty} u(t) = 0. \quad (66)$$

In terms of (64), (65), and $x_i = z_i + \vartheta_{i-1}$ ($i = 1, \dots, n$), we can obtain

$$\lim_{t \rightarrow \infty} x_i(t) = 0 \quad (i = 1, \dots, n). \quad (67)$$

This completes the proof. $\square$

It is noted that, under some stronger conditions, the designed control law can be a linear controller. In fact, we have the following statement.

Theorem 12. Suppose that the conditions for Theorem 11 are satisfied with δ_0 known a priori and the following additional assumptions hold:

(i) The uncertain functions $\phi_{i1}(\cdot)$ ($i = 1, \dots, n$) satisfy

$$\limsup_{s \rightarrow +\infty} \frac{\phi_{i1}^2(s)}{\alpha_0(s)} < +\infty. \quad (68)$$

(ii) There exist known constants $p_{ij} > 0$ ($i = 1, \dots, n; j = 1, 2$), such that

$$|g_i(t, \eta, x)| \leq p_{i1}\phi_{i1}(|\eta|) + p_{i2}(|x_1| + \dots + |x_i|), \quad (69)$$

$i = 1, \dots, n.$

(iii) $\gamma_0(|s|) = ks^2$ where k is a positive constant.

Then, the proposed design method can result in a linear control law

$$\begin{aligned} u &= -\frac{1}{\underline{d}_n} \nu_n z_n \\ &= -\frac{1}{\underline{d}_n} \nu_n (x_n + \nu_{n-1} (x_{n-1} + \dots + \nu_1 x_1)), \end{aligned} \quad (70)$$

where ν_i ($i = 1, \dots, n$) are some sufficiently large positive constants.

Proof. Under the above hypotheses (i)–(iii), it is known that the constant p^* is known, and hence the estimation $\hat{p}$ for p^* is no longer needed. Moreover, since conditions (59) and (68) are satisfied, the function $\rho(\cdot)$ in (7) can be chosen as

a constant $\rho > 0$. For $i = 1, \dots, n$, we consider the following function:

$$\bar{V}_{0i}(\eta) = \int_0^{V_0(\eta)} \rho_i ds, \quad (71)$$

where ρ_i ($i = 1, \dots, n$) are design constants. In view of (3), we can get

$$\begin{aligned} \dot{\bar{V}}_{0i}(\eta) &\leq -\rho_i \alpha_0(|\eta|) + \rho_i \delta_0 \gamma_0(|y|) \\ &= -\phi_{i1}^2(|\eta|) + \tilde{\rho}_i \delta_0 \gamma_0(|y|) + \phi_{i1}^2(|\eta|) \\ &\quad - \rho_i \alpha_0(|\eta|) - \tilde{\rho}_i \delta_0 \gamma_0(|y|) + \rho_i \delta_0 \gamma_0(|y|) \end{aligned} \quad (72)$$

with some positive constants $\tilde{\rho}_i$ ($i = 1, \dots, n$). We will prove that if the constants ρ_i and $\tilde{\rho}_i$ are chosen suitably, the following inequality holds:

$$\begin{aligned} \phi_{i1}^2(|\eta|) - \rho_i \alpha_0(|\eta|) - \tilde{\rho}_i \delta_0 \gamma_0(|y|) + \rho_i \delta_0 \gamma_0(|y|) &\leq 0, \\ i &= 1, \dots, n. \end{aligned} \quad (73)$$

In fact, because of $\phi_{i1}^2(s) = O(\alpha_0(s))$, there exist positive constants $s_i > 0$, $c_i > 0$ satisfying $\phi_{i1}^2(s) \leq c_i \alpha_0(s)$, for $s \in [0, s_i]$. Take $\rho_i \geq c_i + 1$, $\tilde{\rho}_i = \rho_i + 1$, and then $\phi_{i1}^2(s) - \rho_i \alpha_0(s) \leq 0$ and $(\rho_i - \tilde{\rho}_i) \delta_0 \gamma_0(|y|) \leq 0$. In view of $\limsup_{s \rightarrow \infty} (\phi_{i1}^2(s)/\alpha_0(s)) < \infty$, there exist positive constants $s'_i > 0$, $c'_i > 0$ satisfying $\phi_{i1}^2(s) \leq c'_i \alpha_0(s)$ if $s \in [s'_i, \infty]$. Similarly, if we take $\rho_i \geq c'_i + 1$ and $\tilde{\rho}_i = \rho_i + 1$, then $\phi_{i1}^2(s) - \rho_i \alpha_0(s) \leq 0$, $(\rho_i - \tilde{\rho}_i) \delta_0 \gamma_0(|y|) \leq 0$. In the finite closed interval $[s_i, s'_i]$, let $\alpha_0(\underline{s})$ be the minimum value of $\alpha_0(s)$ and let $\phi_{i1}^2(\bar{s})$ be the maximum value of $\phi_{i1}^2(s)$, respectively, and if we take $\rho_i \geq \phi_{i1}^2(\bar{s})/\alpha_0(\underline{s})$, $\tilde{\rho}_i = \rho_i + 1$, then $\phi_{i1}^2(s) - \rho_i \alpha_0(s) \leq 0$, $(\rho_i - \tilde{\rho}_i) \delta_0 \gamma_0(|y|) \leq 0$. According to the previous analysis, we choose $\rho_i = c_i + 1 + c'_i + 1 + \phi_{i1}^2(\bar{s})/\alpha_0(\underline{s})$ and $\tilde{\rho}_i = \rho_i + 1$, and then (72) holds. Therefore we get

$$\dot{\bar{V}}_{0i}(\eta) \leq -\phi_{i1}^2(|\eta|) + \tilde{\rho}_i \delta_0 \gamma_0(|y|), \quad i = 1, \dots, n. \quad (74)$$

In view of $\gamma_0(s) = ks^2$, a direct substitution in (74) results in

$$\dot{\bar{V}}_{0i}(\eta) \leq -\phi_{i1}^2(|\eta|) + \tilde{\rho}_i \delta_0 k y^2, \quad i = 1, \dots, n. \quad (75)$$

To deal with the unmeasured dynamics η in this case, we can choose the candidate Lyapunov function as follows:

$$V_n = \sum_{i=1}^n \frac{1}{2} z_i^2 + \sum_{i=1}^n (n-i+2) \bar{V}_{0i}(\eta). \quad (76)$$

Consequently, a modified version of the design procedure in Section 3 leads to the linear control law

$$\begin{aligned} u &= -\frac{1}{\underline{d}_n} \nu_n z_n \\ &= -\frac{1}{\underline{d}_n} \nu_n (x_n + \nu_{n-1} (x_{n-1} + \dots + \nu_1 x_1)) \end{aligned} \quad (77)$$

with some sufficiently large positive constants ν_i ($i = 1, \dots, n$). $\square$

5. Simulation Example

In this section, we provide a simulation example to illustrate the proposed method in the paper. Consider the following nonlinear systems:

$$\begin{aligned}\dot{\eta} &= -\eta + \delta(t) y^2, \\ \dot{x}_1 &= d_1(t) x_2 + \theta_1 x_1 + \theta_2 \frac{x_1 x_2}{1 + x_2^2}, \\ \dot{x}_2 &= d_2(t) u + \theta_3 \sin(tx_1 x_2) \eta, \\ y &= x_1\end{aligned}\quad (78)$$

with $q(t, \eta, y) = -\eta + \delta(t) y^2$, $\delta(t) = 1 + e^{-t}$, $g_1(t, \eta, x) = \theta_1 x_1 + \theta_2 (x_1 x_2 / (1 + x_2^2))$, and $g_2(t, \eta, x) = \theta_3 \sin(tx_1 x_2) \eta$. The inverse system $\dot{\eta} = -\eta + \delta(t) y^2$ is ISS, and $V_0(\eta) = (1/2)\eta^2$ is a candidate ISS-Lyapunov function with the supply pair $\alpha_0(|\eta|) = (1/2)\eta^2$, $\delta(t)\gamma_0(|x_1|) = (1 + e^{-t})(1/4)x_1^4$.

Next, we use the proposed algorithm in Section 3 to design the partial-state feedback controller.

Step 1. We consider the function $V_1 = (1/2)z_1^2 + (1/2)\eta^2$, whose time derivative satisfies

$$\begin{aligned}\dot{V}_1 &= z_1 \left(d_1(t) x_2 + \theta_1 x_1 + \theta_2 \frac{x_1 x_2}{1 + x_2^2} \right) \\ &\quad + \eta (-\eta + \delta(t) y^2).\end{aligned}\quad (79)$$

Like the calculations in (14), we have

$$\begin{aligned}z_1 \theta_1 x_1 &\leq |\theta_1| z_1^2, \\ z_1 \theta_2 \frac{x_1 x_2}{1 + x_2^2} &\leq \frac{1}{2} |\theta_2| z_1^2, \\ \eta \delta(t) y^2 &\leq \frac{1}{4} \eta^2 + \delta^2(t) z_1^4 \leq \frac{1}{4} \eta^2 + \delta_0 z_1^4,\end{aligned}\quad (80)$$

where $\delta_0 > 0$ satisfying $\delta^2(t) \leq \delta_0$.

Define $p^* = \max\{\delta_0, |\theta_i|, \theta_i^2 \mid i = 1, 2, 3\}$, and we get

$$\begin{aligned}z_1 \theta_1 x_1 + z_1 \theta_2 \frac{x_1 x_2}{1 + x_2^2} + \eta \delta(t) y^2 \\ \leq \frac{1}{4} \eta^2 + p^* z_1^2 \left(1 + \frac{1}{2} + z_1^2 \right) \leq \frac{1}{4} \eta^2 + z_1^2 \hat{\phi}_1(x_1) p^*\end{aligned}\quad (81)$$

with a new smooth function $\hat{\phi}_1(x_1) \geq 1 + 1/2 + z_1^2 > 0$.

Similar to (17), we augment V_1 as follows:

$$\bar{V}_1 = V_1 + \frac{1}{2Y} \bar{p}^2, \quad (82)$$

where $Y > 0$ is the design parameter. In view of (79) and $p^* = \hat{p} + \bar{p}$, a direct substitution leads to

$$\begin{aligned}\dot{\bar{V}}_1 &\leq -\left(1 - \frac{1}{4}\right) \eta^2 + d_1(t) z_1 \vartheta_1 + d_1(t) z_1 z_2 \\ &\quad + z_1^2 \hat{\phi}_1(x_1) \bar{p} + \frac{1}{Y} \bar{p} (Y \hat{\phi}_1(x_1) z_1^2 - \dot{\bar{p}}).\end{aligned}\quad (83)$$

We take the virtual control and the tuning function

$$\begin{aligned}\vartheta_1 &= -\frac{1}{d_1} (\nu_1 + \hat{\phi}_1(x_1) \bar{p}) z_1, \quad \nu_1 > 0, \\ \tau_1 &= Y \hat{\phi}_1(x_1) z_1^2,\end{aligned}\quad (84)$$

and then we get

$$\begin{aligned}\dot{\bar{V}}_1 &\leq -\left(1 - \frac{1}{4}\right) \eta^2 - \nu_1 z_1^2 + d_1(t) z_1 z_2 \\ &\quad + \frac{1}{Y} \bar{p} (\tau_1 - \dot{\bar{p}}).\end{aligned}\quad (85)$$

Step 2. To find the actual control law u , we consider the Lyapunov function

$$V_2 = \bar{V}_1 + \frac{1}{2} z_2^2. \quad (86)$$

In view of (78) and (85), we have

$$\begin{aligned}\dot{V}_2 &\leq -\left(1 - \frac{1}{4}\right) \eta^2 - \nu_1 z_1^2 + d_1(t) z_1 z_2 + \frac{1}{Y} \bar{p} (\tau_1 \\ &\quad - \dot{\bar{p}}) + z_2 \left(d_2(t) u + \theta_3 \sin(tx_1 x_2) \eta \right. \\ &\quad \left. - \frac{\partial \vartheta_1}{\partial x_1} \left(d_1(t) x_2 + \theta_1 x_1 + \theta_2 \frac{x_1 x_2}{1 + x_2^2} \right) - \frac{\partial \vartheta_1}{\partial \bar{p}} \dot{\bar{p}} \right).\end{aligned}\quad (87)$$

As in (80), we have

$$\begin{aligned}d_1(t) z_1 z_2 &\leq z_1^2 + \frac{\bar{d}_1^2}{4} z_2^2, \\ z_2 \theta_3 \sin(tx_1 x_2) \eta &\leq |z_2 \theta_3 \sin(tx_1 x_2) \eta| \\ &\leq \frac{1}{4} \eta^2 + \theta_3^2 z_2^2, \\ -z_2 \frac{\partial \vartheta_1}{\partial x_1} \theta_1 x_1 &\leq z_1^2 + \frac{1}{4} \theta_1^2 z_2^2 \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2, \\ -z_2 \frac{\partial \vartheta_1}{\partial x_1} \theta_2 \frac{x_1 x_2}{1 + x_2^2} &\leq z_1^2 + \frac{1}{4} \theta_2^2 z_2^2 \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 \frac{1}{4}, \\ -z_2 \frac{\partial \vartheta_1}{\partial x_1} d_1(t) x_2 &\leq z_2^2 \frac{\bar{d}_1}{2} \left(1 + \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 \right) + z_1^2 \\ &\quad + \frac{1}{4} z_2^2 \left(\frac{\partial \vartheta_1}{\partial x_1} \right)^2 \frac{\bar{d}_1^2}{d_1^2} (\nu_1 + \hat{\phi}_1(x_1) \bar{p})^2.\end{aligned}\quad (88)$$

Consequently, in view of the definition of p^* , the following holds:

$$\begin{aligned}z_2 \theta_3 \sin(tx_1 x_2) \eta - z_2 \frac{\partial \vartheta_1}{\partial x_1} \theta_1 x_1 - z_2 \frac{\partial \vartheta_1}{\partial x_1} \theta_2 \frac{x_1 x_2}{1 + x_2^2} \\ \leq \frac{1}{4} \eta^2 + 2z_1^2 + z_2^2 \hat{\phi}_2(x_1, \bar{p}) p^*,\end{aligned}\quad (89)$$

FIGURE 1: The response of the closed-loop system.

with $\hat{\phi}_2(x_1, \hat{p}) = 1 + (1/4)(\partial\vartheta_1/\partial x_1)^2 + (1/4)(\partial\vartheta_1/\partial \hat{p})^2(1/4)$. Define $\bar{\phi}_{21}(x_1, \hat{p}) = \bar{d}_1^2/4 + (\bar{d}_1/2)(1 + (\partial\vartheta_1/\partial x_1)^2) + (1/4)(\bar{d}_1^2/d_1^2)(\partial\vartheta_1/\partial x_1)^2(\nu_1 + \hat{\phi}_1(x_1)\hat{p})^2$, and one gets

$$z_2 \left(d_1(t) z_1 - \frac{\partial\vartheta_1}{\partial x_1} d_1(t) x_2 \right) \leq 2z_1^2 + z_2^2 \bar{\phi}_{21}(x_1, \hat{p}). \quad (90)$$

As a result, with $\tau_2 = \tau_1 + \Upsilon z_2^2 \hat{\phi}_2(x_1, \hat{p})$, we have

$$\begin{aligned} \dot{V}_2 \leq & -\left(1 - \frac{2}{4}\right) \eta^2 - (\nu_1 - 4) z_1^2 + z_2^2 \hat{\phi}_2(x_1, \hat{p}) \hat{p} \\ & + z_2^2 \bar{\phi}_{21}(x_1, \hat{p}) + \left(\frac{1}{\Upsilon} \hat{p} + z_2 \frac{\partial\vartheta_1}{\partial \hat{p}}\right) (\tau_2 - \dot{\hat{p}}) \\ & + z_2 d_2(t) u - z_2 \frac{\partial\vartheta_1}{\partial \hat{p}} \tau_2. \end{aligned} \quad (91)$$

For the term of $-z_2(\partial\vartheta_1/\partial \hat{p})\tau_2$, according to (36), it can be verified that

$$\begin{aligned} -z_2 \frac{\partial\vartheta_1}{\partial \hat{p}} \tau_2 \leq & z_1^2 + z_2^2 \left(\frac{1}{4} \Upsilon^2 \left(\frac{\partial\vartheta_1}{\partial \hat{p}} \right)^2 z_1^2 \hat{\phi}_1^2(x_1) \right. \\ & \left. + \frac{1}{4} \Upsilon (1 + z_2^2) \hat{\phi}_2(x_1, \hat{p}) \left(1 + \left(\frac{\partial\vartheta_1}{\partial \hat{p}} \right)^2 \right) \right). \end{aligned} \quad (92)$$

Define $\bar{\phi}_{22}(\bar{x}_2, \hat{p}) = (1/4)\Upsilon^2 z_1^2 (\partial\vartheta_1/\partial \hat{p})^2 \hat{\phi}_1^2(x_1) + (1/4)\Upsilon(1 + z_2^2) \hat{\phi}_2(x_1, \hat{p})(1 + (\partial\vartheta_1/\partial \hat{p})^2)$, and we get

$$-z_2 \frac{\partial\vartheta_1}{\partial \hat{p}} \tau_2 \leq z_1^2 + z_2^2 \bar{\phi}_{22}(\bar{x}_2, \hat{p}). \quad (93)$$

Denote $\bar{\phi}_2(\bar{x}_2, \hat{p}) = \bar{\phi}_{21}(x_1, \hat{p}) + \bar{\phi}_{22}(\bar{x}_2, \hat{p}) > 0$, and a direct substitution leads to

$$\begin{aligned} \dot{V}_2 \leq & -\left(1 - \frac{2}{4}\right)\eta^2 - (\nu_1 - 5)z_1^2 \\ & + z_2 \left(d_2(t)u + z_2 \hat{\phi}_2(\bar{x}_2, \hat{p}) \hat{p} + z_2 \bar{\phi}_2(\bar{x}_2, \hat{p}) \right) \\ & + \left(\frac{1}{Y} \tilde{p} + z_2 \frac{\partial \theta_1}{\partial \hat{p}} \right) (\tau_2 - \hat{p}). \end{aligned} \quad (94)$$

As in Step 1, we design the following partial-state feedback controller:

$$u = -\frac{1}{\underline{d}_2} \left(\nu_2 + \hat{\phi}_2(\bar{x}_2, \hat{p}) \hat{p} + \bar{\phi}_2(\bar{x}_2, \hat{p}) \right) z_2 \quad (95)$$

with the updating law for unknown parameter p^*

$$\dot{\hat{p}} = Y \hat{\phi}_1(x_1) z_1^2 + Y \hat{\phi}_2(x_1, x_2, \hat{p}) z_2^2, \quad (96)$$

which is such that

$$\dot{V}_2 \leq -\frac{1}{2}\eta^2 - (\nu_1 - 5)z_1^2 - \nu_2 z_2^2. \quad (97)$$

The Lyapunov function V_2 can be made $\dot{V}_2 \leq 0$ by choosing $\nu_1 > 5$, $\nu_2 > 0$, and the stability analysis can be done in the similar way to Theorem 11. The simulation plots shown in Figures 1 and 2 are performed by MATLAB with the following parameters: $d_1(t) = 2 - \sin(t)$, $d_2(t) = 2 + \sin(t)$, $\underline{d}_1 = 1$, $\bar{d}_1 = 3$, $\underline{d}_2 = 1$, $\bar{d}_2 = 3$, $\nu_1 = 6$, $\nu_2 = 1$, $Y = 1$, the derived functions: $\phi_{11}(|\eta|) = 0$, $\phi_{21}(|\eta|) = |\eta|$, $\phi_{12}(x_1) = |x_1| + (1/2)|x_1|$, $\phi_{22}(x_1, x_2) = 0$, $\hat{\phi}_1(x_1) = 1 + 1/2 + x_1^2$, $\hat{\phi}_2(x_1, x_2, \hat{p}) = 2 + (3/2)(\partial \theta_1 / \partial x_1)^2 (\nu_1 + \hat{p})^2$, and the initial conditions: $\eta(0) = 0.5$, $x_1(0) = 1$, $z_2(0) = 0.1$, $\hat{p}(0) = 1$.

According to our results reported in Theorem 11, the states (η, x_1, x_2) must asymptotically converge to the origin and the parameter estimate $\hat{p}$ is bounded on $[0, \infty)$. This fact can be verified from Figure 1, which plots the trajectories of these dynamic signals $(\eta(t), x_1(t), x_2(t), \hat{p}(t))$. It can be seen that, at about $t = 4.17$ s, $t = 0.68$ s, and $t = 0.59$ s, the states approach the origin, and at $t = 0.36$ s, the parameter estimate $\hat{p}(t)$ is bounded near $p^* = 1.1$. In addition, according to Theorem 11, the control input u is convergent to the origin. Figure 2 demonstrates this result, and it can be shown that, at about $t = 0.51$ s, the input signal u approaches the origin. As can be seen from Figures 1 and 2, our control scheme provides a fairly good asymptotic stabilization performance.

6. Conclusion

The state feedback stabilization problem is investigated for a class of nonlinear systems with dynamic uncertainties and uncertain control coefficients in this paper. The dynamic uncertainty is characterized by the uncertain ISS supply rates. A global asymptotic stabilization control scheme is proposed using the backstepping design scheme. The tuning function technique is applied in this procedure, which avoids the disadvantage of overparameterization. It is shown that, under some more restrictive conditions, a linear state feedback controller can be designed by the presented algorithm. The simulation example demonstrates the effectiveness of the proposed method.

FIGURE 2: The control input of the closed-loop system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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