Research Article

The Stability of Solutions for the Generalized Degasperis-Procesi Equation with Variable Coefficients

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A generalized Degasperis-Procesi equation with variable coefficients is investigated. The $L^1(R)$ stability of the strong solution for the equation is established under certain assumptions.

1. Introduction

The Degasperis-Procesi (DP) equation

$$v_t - v_{txx} + 4vv_x = 3v_x v_{xx} + vv_{xxx}, \quad t > 0, \quad x \in R, \quad (1)$$

was discovered by Degasperis and Procesi [1] in a search for integrable equations similar to the Camassa-Holm equation. Degasperis and Procesi [1] studied a family of third order dispersive nonlinear equations

$$v_t + v_x + \gamma v_{xxx} - \alpha^2 v_{txx} = (c_1 v^2 + c_2 v_x^2 + c_3 v v_{xx})_x, \quad (2)$$

where $\alpha, \gamma, c_0, c_1, c_2, c_3 \in R$. It is found in [1] that there are only three equations that satisfy asymptotic integrability conditions within this family. By rescaling and applying a Galilean transformation, the three equations are Korteweg-de Vries equation

$$v_t + v_{xxx} + vv_x = 0, \quad t > 0, \quad x \in R, \quad (3)$$

the Camassa-Holm equation

$$v_t - v_{txx} + 3vv_x = 2v_x v_{xx} + vv_{xxx}, \quad t > 0, \quad x \in R, \quad (4)$$

and the Degasperis-Procesi equation (1). Degasperis et al. [2] proved the formal integrability of (1) and the existence of the nonsmooth solutions by constructing a Lax pair.

In recent years, (1) which plays a similar role in water wave theory as the Camassa-Holm equation has caused extensive concern of many scholars (see [1–11]). For example,

Coclite and Karlsen [3] established the well-posedness of $L^1 \cap BV$ weak solutions for (1). They proved uniqueness within a class of discontinuous solutions to (1) in [4]. Escher et al. [5] established the precise blow-up rate and proved the existence and uniqueness of global weak solutions to (1) in which the initial data satisfied appropriate conditions. Lai and Wu [7] investigated the local well-posedness of solutions to a generalization of both (1) and (4) in the Sobolev space $H^s(R)$ with $s > 3/2$. Lenells [8] classified all weak traveling wave solutions of the Degasperis-Procesi equation (1). Ai and Gui [9] proved global existence of solutions for the viscous Degasperis-Procesi equation and showed that the blow-up phenomena occurs in finite time. Fu et al. [11] studied the orbital stability of the peakons for the Degasperis-Procesi equation with a strong dispersive term on the line and proved that the shapes of these peakons were stable under small perturbations.

As we know, their coefficients play an important role to study the fundamental dynamical properties of the Degasperis-Procesi models. It prompts us to study the following generalized Degasperis-Procesi equation:

$$\begin{aligned} v_t - v_{txx} + m(t, x) f'(v) v_x \\ = b(t, x) (f'(v) v_{xxx} + 3f''(v) v_x v_{xx} + f'''(v) v_x^3) \\ + 2b_x(t, x) (f'(v) v_{xx} + f''(v) v_x^2) \\ + b_{xx}(t, x) f'(v) v_x, \end{aligned} \quad (5)$$

where $m(t, x) \in C_0(R)$, $b(t, x) \in C_0^3(R)$, and function $f(\cdot)$ is a polynomial of order n ($n \geq 2$). Letting $m = 4$, $b = 1$, $f(v) = v^2/2$, (5) reduces to the Degasperis-Procesi equation (1). We consider the Cauchy problem of (5) with an initial condition $v_0(x)$. Namely,

$$v(0, x) = v_0(x), \quad x \in R. \quad (6)$$

Assume that (5) possesses a bounded strong solution in its maximum existence time interval $[0, T)$ and v_0 lies in $L^1(R) \cap H^s(R)$ ($s > 3/2$). We use the approaches of Kružkov doubling the variables presented in [12] to prove the L^1 stability of the solution for the variable coefficients equation (5). From our knowledge, it has not been acquired in the literature.

This paper is organized as follows. Section 2 gives several lemmas. The proof of local solution stability is presented in Section 3.

2. Preliminaries

Applying the operator $\Lambda^{-2} = (1 - \partial_x^2)^{-1}$ to (5), we obtain its equivalent form

$$v_t + b(t, x) f'(v) v_x + \Lambda^{-2} (m(t, x) - b(t, x)) \partial_x f(v) = 0, \quad (7)$$

where $\Lambda^{-2} p(t, x) = (1 - \partial_x^2)^{-1} p(t, x) = (1/2) \int_R e^{-|x-y|} p(t, y) dy$.

Equations (5) and (6) are equivalent to the problem

$$\begin{aligned} v_t + \partial_x P(t, x, v) + \Psi(t, x, v) &= 0, \\ v(0, x) &= v_0(x), \end{aligned} \quad (8)$$

where $P(t, x, v) = b(t, x) f(v)$ and $\Psi(t, x, v) = \Lambda^{-2} (m(t, x) - b(t, x)) \partial_x f(v)$. Notice that $\partial_x P = P_x + P_v v_x$.

Remark. According to the statements presented in [7] or [12], we know that problem (8) has a unique local solution in the space $C([0, T], H^s(R))$ if we assume $v_0(x) \in H^s(R)$ ($s > 3/2$).

Assume that $v_1(t, x)$ and $v_2(t, x)$ are solutions of problem (8) in the domain $[0, T) \times R$ with initial functions $v_{10}(x)$ and $v_{20}(x) \in L^1(R) \cap H^s(R)$ ($s > 3/2$), where T is the maximum existence time of solutions. For simplicity, we denote by c any positive constants. Now we give several lemmas.

Lemma 1. Let $v(t, x)$ be the solution of problem (8) and $\|v\|_{L^\infty(R)} \leq M$. Then

$$\|\Psi(t, x, v)\|_{L^\infty(R)} \leq cM^n, \quad (9)$$

where positive constant c depends on $\|v_0\|_{L^\infty}$ and T .

Proof. We have

$$\begin{aligned} |\Psi(t, x, v)| &= \left| \Lambda^{-2} (m(t, x) - b(t, x)) \partial_x f(v) - b_x(t, x) f(v) \right| \\ &\leq \left| \frac{1}{2} \int_R e^{-|x-y|} m(t, y) \partial_y f(v) dy \right| \\ &\quad + \left| \frac{1}{2} \int_R e^{-|x-y|} b(t, y) \partial_y f(v) dy \right| + |b_x f(v)| \\ &\leq c \left| \int_R e^{-|x-y|} \text{sign}(y-x) f(v) dy \right| + c |f(v)| \\ &\leq cM^n, \end{aligned} \quad (10)$$

in which we have used $\int_R e^{-|x-y|} dx = 2$ to complete the proof. $\square$

Lemma 2. Assume that $v_1(t, x)$ and $v_2(t, x)$ are solutions of problem (8) in the domain $[0, T) \times R$, $\|v_1\|_{L^\infty(R)} \leq M$, and $\|v_2\|_{L^\infty(R)} \leq M$. Then

$$\begin{aligned} \int_{-\infty}^{+\infty} |\Psi(t, x, v_1) - \Psi(t, x, v_2)| dx \\ \leq c \int_{-\infty}^{+\infty} |v_1 - v_2| dx, \end{aligned} \quad (11)$$

where $c > 0$ depends on $\|v_{10}\|_{L^\infty(R)}$, $\|v_{20}\|_{L^\infty(R)}$ and T .

Proof. Using the property of the operator Λ^{-2} , we get

$$\begin{aligned} \int_{-\infty}^{+\infty} |\Psi(t, x, v_1) - \Psi(t, x, v_2)| dx \\ \leq \int_{-\infty}^{+\infty} \left| \Lambda^{-2} (m - b) \partial_x (f(v_1) - f(v_2)) \right| dx \\ + \int_{-\infty}^{+\infty} |(f(v_1) - f(v_2))| dx \leq c \int_{-\infty}^{+\infty} dx \\ \cdot \int_{-\infty}^{+\infty} |e^{-|x-y|} \text{sign}(y-x) (f(v_1) - f(v_2))| dy \\ + c \int_{-\infty}^{+\infty} |v_1 - v_2| dx \leq c \int_{-\infty}^{+\infty} |f(v_1) - f(v_2)| dy \\ \cdot \int_{-\infty}^{+\infty} e^{-|x-y|} dx + c \int_{-\infty}^{+\infty} |v_1 - v_2| dx \\ \leq c \int_{-\infty}^{+\infty} |v_1 - v_2| dx, \end{aligned} \quad (12)$$

in which we apply the Tonelli Theorem to complete the proof. $\square$

Let $\delta(\sigma) \geq 0$, $\delta(\sigma) \equiv 0$, for $|\sigma| \geq 1$; $\int_{-\infty}^{+\infty} \delta(\sigma) d\sigma = 1$ and $\delta(\sigma)$ is infinitely differential on $(-\infty, +\infty)$. Set

$\delta_\varepsilon(\sigma) = \delta(\varepsilon^{-1}\sigma)/\varepsilon$, where ε is an arbitrary positive constant. It is found that $\delta_\varepsilon(\sigma) \in C_0^\infty(-\infty, +\infty)$ and

$$\begin{aligned} \delta_\varepsilon(\sigma) &\geq 0, \\ \delta_\varepsilon(\sigma) &= 0 \end{aligned} \quad \text{for } |\sigma| \geq \varepsilon, \quad (13)$$

$$|\delta_\varepsilon(\sigma)| \leq \frac{c}{\varepsilon},$$

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(\sigma) d\sigma = 1.$$

Let the function $\phi(x)$ be defined and locally integrable on $(-\infty, +\infty)$. Set $\phi^\varepsilon(x)$; denote the approximation function of $\phi(x)$ as

$$\phi^\varepsilon(x) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \delta\left(\frac{x-y}{\varepsilon}\right) \phi(y) dy. \quad (14)$$

We call x_0 a Lebesgue point of the function $\phi(x)$ if

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} |\phi(x) - \phi(x_0)| dx = 0. \quad (15)$$

At any Lebesgue point x_0 , we get

$$\lim_{\varepsilon \rightarrow 0} \phi^\varepsilon(x_0) = \phi(x_0). \quad (16)$$

Since the set of points which are not Lebesgue points of $\phi(x)$ has measure zero, we have $\phi^\varepsilon(x) \rightarrow \phi(x)$ as $\varepsilon \rightarrow 0$ almost everywhere.

For any $T > 0$, we denote the band $\{(t, x) : [0, T] \times R\}$ by ω_T . Let $K_r = \{x : |x| \leq r\}$ and

$$\begin{aligned} \Pi = \left\{ (t, x, \tau, y) : \left| \frac{t-\tau}{2} \right| \leq \varepsilon, \rho \leq \frac{t+\tau}{2} \leq T \right. \\ \left. -\rho, \left| \frac{x-y}{2} \right| \leq \varepsilon, \left| \frac{x+y}{2} \right| \leq r-\rho \right\}, \end{aligned} \quad (17)$$

where $r > 0, \rho > 0$.

We state the concept of a characteristic cone. Let $\|v\|_{L^\infty(R)} \leq M$, for any $R_1 > 0$; we define

$$N > \max_{(t,x) \in [0,T] \times K_{R_1}} |f'(v)|. \quad (18)$$

Let Ω represent the cone $\{(t, x) : |x| \leq R_1 - Nt, 0 \leq t \leq T_1 = \min(T, R_1 N^{-1})\}$ and let S_τ designate the cross section of the cone Ω by the plane $t = \tau, \tau \in [0, T_1]$.

Lemma 3 (see [12]). *Let the function $\phi(t, x)$ be bounded and measurable in cylinder $[0, T_1] \times K_r$. For any $\rho \in (0, \min[r, T_1])$ and any $\varepsilon \in (0, \rho)$, the function*

$$V_\varepsilon = \frac{1}{\varepsilon^2} \iiint_{\Pi} |\phi(t, x) - \phi(\tau, y)| dt dx d\tau dy \quad (19)$$

satisfies

$$\lim_{\varepsilon \rightarrow 0} V_\varepsilon = 0. \quad (20)$$

Lemma 4 (see [12]). *If $\partial F(\phi)/\partial \phi$ is bounded, the function $H(\phi, \psi) = \text{sign}(\phi - \psi)(F(\phi) - F(\psi))$ satisfies the Lipschitz condition in ϕ and ψ .*

Lemma 5. *If $v(t, x)$ is the solution of problem (11) on ω_T , $\eta(t, x) \in C_0^\infty(\omega_T)$, it holds that*

$$\begin{aligned} \iint_{\omega_T} \{ |v-k| \eta_t \\ + \text{sign}(v-k) [P(t, x, v) - P(t, x, k)] \eta_x \\ - \text{sign}(v-k) (P_x(t, x, k) + \Psi(t, x, v)) \eta \} dt dx \\ = 0, \end{aligned} \quad (21)$$

where k is an arbitrary constant.

Proof. Suppose that $\Phi(v)$ is a twice differential function. Multiplying the first equation of problem (8) by $\Phi'(v)\eta(t, x)$ and integrating over ω_T , we get

$$\begin{aligned} \iint_{\omega_T} \{ \Phi'(v) \eta v_t + \Phi'(v) \eta P_x + \Phi'(v) \eta P_v v_x \\ + \Phi'(v) \eta \Psi(t, x, v) \} dt dx = 0. \end{aligned} \quad (22)$$

Using the method of integration by parts, we get

$$\iint_{\omega_T} \Phi'(v) \eta v_t dt dx = - \iint_{\omega_T} \Phi(v) \eta_t dt dx. \quad (23)$$

Notice that

$$\begin{aligned} \left(\int_k^v \Phi'(z) P_z(t, x, z) dz \right)'_x \\ = \Phi'(v) P_v(v) v_x \\ + \int_k^v (\Phi''(z) P_z z_x + \Phi'(z) P_{zx}) dz. \end{aligned} \quad (24)$$

So

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(\int_k^v \Phi'(z) P_z dz \right) \eta_x dx \\ = - \int_{-\infty}^{+\infty} \Phi'(v) P_v v_x \eta dx \\ - \int_{-\infty}^{+\infty} \left[\int_k^v (\Phi''(z) P_z z_x + \Phi'(z) P_{zx}) dz \right] \eta dx. \end{aligned} \quad (25)$$

Then we have

$$\begin{aligned} \iint_{\omega_T} \Phi'(v) P_v v_x \eta dt dx = - \iint_{\omega_T} \left(\int_k^v v'(z) P_z dz \right) \\ \cdot \eta_x dt dx \\ + \iint_{\omega_T} \left[\int_k^v (\Phi''(z) P_z z_x + \Phi'(z) P_{zx}) dz \right] \eta dt dx. \end{aligned} \quad (26)$$

Substituting (23) and (26) into (22), we get

$$\begin{aligned} & \iint_{\omega_T} \left\{ \Phi'(\nu) \eta_t + \left(\int_k^\nu \Phi'(z) P_z dz \right) \eta_x \right. \\ & \quad + \left(\int_k^\nu \Phi''(z) P_z z_x dz \right) \eta + \left(\int_k^\nu \Phi'(z) P_{zx} dz \right) \eta \\ & \quad \left. - \Phi'(\nu) \eta P_x - \Phi'(\nu) \eta \Psi \right\} dt dx = 0. \end{aligned} \quad (27)$$

Let $\Phi^\varepsilon(\nu)$ be an approximation of the function $|\nu - k|$. When $\varepsilon \rightarrow 0$, $\Phi^\varepsilon(\nu) \rightarrow \Phi(\nu)$. Setting $\Phi(\nu) = |\nu - k|$, then $\Phi'(\nu) = \text{sign}(\nu - k)$, $\Phi''(\nu) = 0$. Hence,

$$\begin{aligned} & \int_k^\nu \Phi'(z) P_z dz \\ & = \text{sign}(\nu - k) (P(t, x, \nu) - P(t, x, k)), \end{aligned} \quad (28)$$

$$\begin{aligned} & \int_k^\nu \Phi'(z) P_{zx} dz \\ & = \text{sign}(\nu - k) (P_x(t, x, \nu) - P_x(t, x, k)), \end{aligned} \quad (29)$$

combining with (27), we complete the proof. $\square$

3. Main Result

Set function $\eta(t, x) \in C_0^\infty(\omega_T)$, $\eta(t, x) \equiv 0$, outside the cylinder $\{(t, x)\} = [\rho, T - 2\rho] \times K_{r-2\rho}$, where $K_{r-2\rho} = \{|x| : |x| \leq r - 2\rho\}$, $r > 0$, $0 < 2\rho < \min(T, r)$. Now we give the main result of this work.

Theorem 6. Assume that $v_1(t, x)$ and $v_2(t, x)$ are two strong solutions of problem (8) with initial data $v_{10}(x), v_{20}(x) \in L^1(R) \cap H^s(R)$ ($s > 3/2$). Let $T > 0$ be the maximum existence time of $v_1(t, x)$ and $v_2(t, x)$. If $\|v_1(t, x)\|_{L^\infty} \leq M$ and $\|v_2(t, x)\|_{L^\infty} \leq M$, for any $t \in [0, T]$, it holds that

$$\begin{aligned} & \|v_1(t, x) - v_2(t, x)\|_{L^1} \\ & \leq e^{ct} \int_{-\infty}^{+\infty} |v_{10}(x) - v_{20}(x)| dx, \end{aligned} \quad (30)$$

where c is a positive constant depending on $\|v_{10}\|_{L^\infty(R)}$ and $\|v_{20}\|_{L^\infty(R)}$.

Proof. Set $k = \psi(\tau, y)$, $\eta(t, x) = h(t, x, \tau, y)$. Using the Kružkov device of doubling the variables in [12] and Lemma 5, we get

$$\begin{aligned} & \iiint_{\omega_T \times \omega_T} \{ |v_1(t, x) - v_2(\tau, y)| h_t \\ & \quad + \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \quad \cdot [P(t, x, v_1(t, x)) - P(\tau, y, v_2(\tau, y))] h_x \\ & \quad - \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \quad \cdot [P_x(t, x, v_2(\tau, y)) + \Psi(t, x, v_1(t, x))] h \} dt dx d\tau dy \\ & = 0. \end{aligned} \quad (31)$$

Similarly, we have

$$\begin{aligned} & \iiint_{\omega_T \times \omega_T} \{ |v_2(\tau, y) - v_1(t, x)| h_\tau \\ & \quad + \text{sign}(v_2(\tau, y) - v_1(t, x)) \\ & \quad \cdot [P(\tau, y, v_2(\tau, y)) - P(t, x, v_1(t, x))] h_y \\ & \quad - \text{sign}(v_2(\tau, y) - v_1(t, x)) \\ & \quad \cdot [P_y(\tau, y, v_1(t, x)) + \Psi(\tau, y, v_2(\tau, y))] h \} dt dx d\tau dy \\ & = 0. \end{aligned} \quad (32)$$

Adding (31) and (32), we obtain

$$\begin{aligned} & \iiint_{\omega_T \times \omega_T} \{ |v_1(t, x) - v_2(\tau, y)| (h_t + h_\tau) \\ & \quad + \text{sign}(v_1(t, x) - v_2(\tau, y)) [P(t, x, v_1(t, x)) \\ & \quad - P(t, x, v_2(\tau, y))] (h_x + h_y) \\ & \quad + \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \quad \cdot ([P(\tau, y, v_2(\tau, y)) - P(t, x, v_2(\tau, y))] h_x \\ & \quad + [P(\tau, y, v_1(t, x)) - P(t, x, v_1(t, x))] h_y \\ & \quad + [P_y(\tau, y, v_1(t, x)) - P_x(t, x, v_2(\tau, y))] h \\ & \quad - \text{sign}(v_1(t, x) - v_2(\tau, y)) [\Psi(t, x, v_1(t, x)) \\ & \quad - \Psi(\tau, y, v_2(\tau, y))] h \} dt dx d\tau dy \\ & = \iiint_{\omega_T \times \omega_T} (A_1 + A_2 + A_3 + A_4) dt dx d\tau dy \\ & = 0. \end{aligned} \quad (33)$$

Set function

$$\begin{aligned} & h(t, x, \tau, y) \\ & = \eta\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \delta_\varepsilon\left(\frac{t-\tau}{2}\right) \delta_\varepsilon\left(\frac{x-y}{2}\right) \\ & = \eta(\cdots) \lambda_\varepsilon\left(\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}\right), \end{aligned} \quad (34)$$

in which $(\cdots) = ((t+\tau)/2, (x+y)/2)$ and $\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = ((t-\tau)/2, (x-y)/2)$. Thus, we obtain

$$\begin{aligned} & h_t + h_\tau = \eta_t(\cdots) \lambda_\varepsilon\left(\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}\right), \\ & h_x + h_y = \eta_x(\cdots) \lambda_\varepsilon\left(\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}\right). \end{aligned} \quad (35)$$

We will prove that the form A_3 in (33) approaches zero as $\varepsilon \rightarrow 0$. In fact, the coefficients of η_x and η_y in A_3 vanish for

$|t - \tau| + |x - y| = 0$. Thus the integrals of A_3 can be rewritten as the following concrete form:

$$\begin{aligned} & \iiint_{\omega_T \times \omega_T} \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \cdot \{ [P(\tau, y, v_2(\tau, y)) - P(t, x, v_2(\tau, y))] (\lambda_\varepsilon)_x \\ & - P_x(t, x, v_2(\tau, y)) \lambda_\varepsilon \\ & + [P(\tau, y, v_1(t, x)) - P(t, x, v_1(t, x))] (\lambda_\varepsilon)_y \\ & + P_y(\tau, y, v_1(t, x)) \lambda_\varepsilon \} \eta(\cdot \cdot) dt dx d\tau dy = A_\varepsilon. \end{aligned} \quad (36)$$

In the following computations, we omit the index ε of function λ . Applying the Taylor formula, we have the relations

$$\begin{aligned} & [P(\tau, y, v_2(\tau, y)) - P(t, x, v_2(\tau, y))] \lambda_x \\ & - P_x(t, x, v_2(\tau, y)) \lambda \\ & = ([P(\tau, y, v_2(\tau, y)) - P(t, x, v_2(\tau, y))] \lambda)_x \\ & = P_\tau(\tau, y, v_2(\tau, y)) (\tau - t) \lambda_x \\ & + P_y(\tau, y, v_2(\tau, y)) (y - x) \lambda_x \\ & - P_y(\tau, y, v_2(\tau, y)) \lambda + \alpha_1 \lambda_x + \alpha_0 \lambda \\ & = P_\tau(\tau, y, v_2(\tau, y)) ((\tau - t) \lambda)_x \\ & + P_y(\tau, y, v_2(\tau, y)) ((y - x) \lambda)_x + \alpha_1 \lambda_x + \alpha_0 \lambda. \end{aligned} \quad (37)$$

It is seen that the identity $\lambda_y = -\lambda_x$. In a similar way, we obtain

$$\begin{aligned} & [P(\tau, y, v_1(t, x)) - P(t, x, v_1(t, x))] \lambda_y \\ & + P_y(\tau, y, v_1(t, x)) \lambda \\ & = P_\tau(\tau, y, v_1(t, x)) (\tau - t) \lambda_y \\ & + P_y(\tau, y, v_1(t, x)) (y - x) \lambda_y \\ & + P_y(\tau, y, v_1(t, x)) \lambda + \beta_1 \lambda_y \\ & = P_\tau(\tau, y, v_1(t, x)) ((t - \tau) \lambda)_x \\ & + P_y(\tau, y, v_1(t, x)) ((x - y) \lambda)_x + \beta_1 \lambda_y. \end{aligned} \quad (38)$$

The functions α_0 , α_1 , and β_1 in (37) and (38) satisfy

$$|\alpha_0| + |\alpha_1| + |\beta_1| \leq d\varepsilon(d), \quad (39)$$

where $d = |t - \tau| + |x - y|$ and $\varepsilon(d) \rightarrow 0$ as $d \rightarrow 0$. There are $\lambda = \lambda_\varepsilon \equiv 0$ for $|t - \tau| \geq 2\varepsilon$ or $|x - y| \geq 2\varepsilon$ and

$$\begin{aligned} & |\lambda_x| + |\lambda_y| \leq \frac{c}{\varepsilon^3}, \\ & |\eta(\cdot \cdot) - \eta(\tau, y)| \leq c(|t - \tau| + |x - y|). \end{aligned} \quad (40)$$

Hence, we get

$$\begin{aligned} A_\varepsilon = & \iiint_{\omega_T \times \omega_T} \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \cdot \{ [P_\tau(\tau, y, v_2(\tau, y)) - P_\tau(t, x, v_2(\tau, y))] \\ & \cdot ((\tau - t) \lambda)_x \\ & + [P_y(\tau, y, v_2(t, x)) - P_y(\tau, y, v_1(t, x))] \\ & \cdot ((y - x) \lambda)_x \} \eta(\tau, y) dt dx d\tau dy + I(\varepsilon), \end{aligned} \quad (41)$$

where $I(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

We denote the integrand in (41) as

$$\begin{aligned} B_\varepsilon = & K_1(\tau, y, v_1(t, x), v_2(\tau, y)) ((\tau - t) \eta(\tau, y) \lambda)_x \\ & + K_2(\tau, y, v_1(t, x), v_2(\tau, y)) ((y - x) \eta(\tau, y) \lambda)_x \\ = & \{ K_1(\tau, y, v_1(t, x), v_2(\tau, y)) \\ & - K_1(\tau, y, v_1(\tau, y), v_2(\tau, y)) \} ((\tau - t) \eta(\tau, y) \lambda)_x \\ & + \{ K_2(\tau, y, v_1(t, x), v_2(\tau, y)) \\ & - K_2(\tau, y, v_1(\tau, y), v_2(\tau, y)) \} \\ & \cdot ((y - x) \eta(\tau, y) \lambda)_x \\ & + K_1(\tau, y, v_1(\tau, y), v_2(\tau, y)) ((\tau - t) \eta(\tau, y) \lambda)_x \\ & + K_2(\tau, y, v_1(\tau, y), v_2(\tau, y)) ((y - x) \eta(\tau, y) \lambda)_x, \end{aligned} \quad (42)$$

where K_1 and K_2 satisfy the Lipschitz condition in v . Applying the property of the function $\eta(\tau, y) \lambda_\varepsilon$ and the method of integration by parts, we have

$$\begin{aligned} & \iiint_{\omega_T \times \omega_T} \{ K_1(\tau, y, v_1(\tau, y), v_2(\tau, y)) \\ & \cdot ((\tau - t) \eta(\tau, y) \lambda)_x \\ & + K_2(\tau, y, v_1(\tau, y), v_2(\tau, y)) \\ & \cdot ((y - x) \eta(\tau, y) \lambda)_x \} dt dx d\tau dy = 0. \end{aligned} \quad (43)$$

Hence

$$\begin{aligned} |A_\varepsilon - I(\varepsilon)| = & \left| \iiint_{\omega_T \times \omega_T} B_\varepsilon dt dx d\tau dy \right| \leq c \\ & \cdot \iiint_{\omega_T \times \omega_T} |v_1(t, x) - v_2(\tau, y)| \\ & \cdot \{ (|t - \tau| + |y - x|) \eta(\tau, y) |\lambda_x| + \lambda_\varepsilon \} dt dx d\tau dy \\ & \leq \frac{c}{\varepsilon^2} \iiint_{\Pi} |v_1(t, x) - v_2(\tau, y)| dt dx d\tau dy. \end{aligned} \quad (44)$$

Using Lemma 3, $A_\varepsilon - I(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, we have

$$\lim_{\varepsilon \rightarrow 0} \iiint_{\omega_T \times \omega_T} A_3 dt dx d\tau dy = 0. \quad (45)$$

It follows from (33) to (45) that

$$\begin{aligned}
& \iiint_{\omega_T \times \omega_T} \{ |v_1(t, x) - v_2(\tau, y)| \eta_t \\
& + \text{sign}(v_1(t, x) - v_2(\tau, y)) \\
& \cdot [P(t, x, v_1(t, x)) - P(t, x, v_2(\tau, y))] \} \eta_x \} \\
& \cdot \lambda_\varepsilon dt dx d\tau dy \\
& + \left| \iiint_{\omega_T \times \omega_T} \text{sign}(v_1(t, x) - v_2(\tau, y)) \right. \\
& \cdot [\Psi(t, x, v_1(t, x)) - \Psi(\tau, y, v_2(\tau, y))] \\
& \cdot \eta \lambda_\varepsilon dt dx d\tau dy \Big| \geq 0.
\end{aligned} \quad (46)$$

We note that the first two terms of the integrand of (46) have the form

$$F_\varepsilon = F(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \lambda_\varepsilon \left(\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right), \quad (47)$$

where F satisfies the Lipschitz condition in all its variables. Then

$$\begin{aligned}
& \iiint_{\omega_T \times \omega_T} F_\varepsilon dt dx d\tau dy \\
& = \iiint_{\omega_T \times \omega_T} F(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \\
& \cdot \lambda_\varepsilon dt dx d\tau dy \\
& = \iiint_{\omega_T \times \omega_T} \{ F(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \\
& - F(t, x, t, x, v_1(t, x), v_2(t, x)) \} \lambda_\varepsilon dt dx d\tau dy \\
& + \iiint_{\omega_T \times \omega_T} F(t, x, t, x, v_1(t, x), v_2(t, x)) \\
& \cdot \lambda_\varepsilon dt dx d\tau dy = J_{11}(\varepsilon) + J_{12}.
\end{aligned} \quad (48)$$

Note that $F_\varepsilon = 0$ outside the region Π . Applying the estimate $\left| \lambda_\varepsilon \left(\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right) \right| \leq c/\varepsilon^2$ and Lemma 4, we get

$$\begin{aligned}
& |J_{11}(\varepsilon)| \\
& \leq c \left[\varepsilon + \frac{1}{\varepsilon^2} \iiint_{\Pi} |v_1(t, x) - v_2(\tau, y)| dt dx d\tau dy \right],
\end{aligned} \quad (49)$$

where c is a positive constant independent of ε . Using Lemma 3, we know $J_{11}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For the term J_{12} , we substitute $t = \alpha$, $(t - \tau)/2 = \beta$, $x = \zeta$, $(x - y)/2 = \gamma$. Combining with the identity

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{+\infty} \lambda_\varepsilon(\beta, \gamma) d\beta d\gamma = 1, \quad (50)$$

we obtain

$$\begin{aligned}
J_{12} &= 2^2 \iint_{\omega_T \times \omega_T} F(\alpha, \zeta, \alpha, \zeta, \phi(\alpha, \zeta), \psi(\alpha, \zeta)) \\
& \cdot \left(\int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{+\infty} \lambda_\varepsilon(\beta, \gamma) d\beta d\gamma \right) d\alpha d\zeta = 4 \\
& \cdot \iint_{\omega_T} F(t, x, t, x, v_1(t, x), v_2(t, x)) dt dx.
\end{aligned} \quad (51)$$

Thus, we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \iiint_{\omega_T \times \omega_T} F_\varepsilon \\
& = 4 \iint_{\omega_T} F(t, x, t, x, v_1(t, x), v_2(t, x)) dt dx.
\end{aligned} \quad (52)$$

Similarly, the integrand of the third term in (46) can be represented as

$$\begin{aligned}
& \bar{F}_\varepsilon = \text{sign}(v_1(t, x) - v_2(\tau, y)) \\
& \cdot [\Psi(t, x, v_1(t, x)) - \Psi(\tau, y, v_2(\tau, y))] g \lambda_\varepsilon \\
& = \bar{F}(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \lambda_\varepsilon \left(\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right).
\end{aligned} \quad (53)$$

Then

$$\begin{aligned}
& \iiint_{\omega_T \times \omega_T} \bar{F}_\varepsilon dt dx d\tau dy \\
& = \iiint_{\omega_T \times \omega_T} \{ \bar{F}(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \\
& - \bar{F}(t, x, t, x, v_1(t, x), v_2(t, x)) \} \lambda_\varepsilon dt dx d\tau dy \\
& + \iiint_{\omega_T \times \omega_T} \bar{F}(t, x, t, x, v_1(t, x), v_2(t, x)) \\
& \cdot \lambda_\varepsilon dt dx d\tau dy = J_{21}(\varepsilon) + J_{22}.
\end{aligned} \quad (54)$$

It holds that

$$\begin{aligned}
& |J_{21}(\varepsilon)| \\
& \leq c \left[\varepsilon + \frac{1}{\varepsilon^2} \iiint_{\Pi} |v_1(t, x) - v_2(\tau, y)| dt dx d\tau dy \right].
\end{aligned} \quad (55)$$

Using Lemma 3, it yields $J_{21}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Repeating the steps as before, we have

$$J_{22} = 4 \iint_{\omega_T} \bar{F}(t, x, t, x, v_1(t, x), v_2(t, x)) dt dx. \quad (56)$$

From (46) to (56), we get

$$\begin{aligned} & \iint_{\omega_T} \{ |v_1(t, x) - v_2(t, x)| \eta_t \\ & + \text{sign}(v_1(t, x) - v_2(t, x)) \\ & \cdot [P(t, x, v_1(t, x)) - P(\tau, y, v_2(\tau, y))] \eta_x \} dt dx \\ & + \left| \iint_{\omega_T} \text{sign}(v_1(t, x) - v_2(t, x)) \right. \\ & \cdot [\Psi(t, x, v_1) - \Psi(t, x, v_2)] \eta dt dx \Big| \geq 0. \end{aligned} \quad (57)$$

We set

$$\begin{aligned} g(t) &= \int_{-\infty}^{+\infty} |v_1(t, x) - v_2(t, x)| dx, \\ \mu_\varepsilon(\sigma) &= \int_{-\infty}^{\sigma} \delta_\varepsilon(\sigma) d\sigma. \end{aligned} \quad (58)$$

Take two numbers $\rho, \tau \in (0, T_1)$ and $\rho < \tau$. In (57), we set

$$\begin{aligned} \eta(t, x) &= [\mu_\varepsilon(t - \rho) - \mu_\varepsilon(t - \tau)] \chi(t, x), \\ \varepsilon &< \min(\rho, T_1 - \tau), \end{aligned} \quad (59)$$

in which

$$\chi(t, x) = \chi_\theta(t, x) = 1 - \mu_\theta(|x| + Nt - R_1 + \theta), \quad (60)$$

where θ is a small positive constant and $\chi(t, x) = 0$ outside the cone Ω . When $\theta \rightarrow 0, R_1 \rightarrow +\infty$, we observe that $\chi_\theta \rightarrow 1$. By the definition of the number N , we have

$$0 = \chi_t + N|\chi_x| \geq \chi_t + N\chi_x, \quad (t, x) \in \Omega. \quad (61)$$

Applying (57)–(60), we get

$$\begin{aligned} & \iint_{\omega_{T_1}} \{ |v_1(t, x) - v_2(t, x)| [\delta_\varepsilon(t - \rho) - \delta_\varepsilon(t - \tau)] \\ & \cdot \chi_\theta(t, x) \} dt dx + \int_0^{T_1} dt \\ & \cdot \int_{-\infty}^{+\infty} \{ |\Psi(t, x, v_1) - \Psi(t, x, v_2)| \\ & \cdot [\mu_\varepsilon(t - \rho) - \mu_\varepsilon(t - \tau)] \chi_\theta(t, x) \} dx \geq 0. \end{aligned} \quad (62)$$

In (62), sending $\theta \rightarrow 0, R_1 \rightarrow +\infty$ and using Lemma 2, we obtain

$$\begin{aligned} & \int_0^{T_1} [\delta_\varepsilon(t - \rho) - \delta_\varepsilon(t - \tau)] h(t) dt \\ & + c \int_0^{T_1} [\mu_\varepsilon(t - \rho) - \mu_\varepsilon(t - \tau)] g(t) dt \geq 0, \end{aligned} \quad (63)$$

where c is independent of ε .

Applying the properties of the function δ_ε for $\varepsilon \leq \min(\rho, T_1 - \rho)$, we get

$$\begin{aligned} & \left| \int_0^{T_1} \delta_\varepsilon(t - \rho) g(t) - g(\rho) dt \right| \\ & = \left| \int_0^{T_1} \delta_\varepsilon(t - \rho) [g(t) - g(\rho)] dt \right| \\ & \leq \frac{c}{\varepsilon} \int_{\rho-\varepsilon}^{\rho+\varepsilon} |g(t) - g(\rho)| dt. \end{aligned} \quad (64)$$

Then

$$\int_0^{T_1} \delta_\varepsilon(t - \rho) g(t) dt \rightarrow g(\rho) \quad \text{as } \varepsilon \rightarrow 0. \quad (65)$$

Let

$$\begin{aligned} W(\rho) &= \int_0^{T_1} \mu_\varepsilon(t - \rho) g(t) dt \\ &= \int_0^{T_1} dt \int_{-\infty}^{t-\rho} \delta_\varepsilon(\sigma) g(t) d\sigma. \end{aligned} \quad (66)$$

We observe that

$$W'(\rho) = - \int_0^{T_1} \delta_\varepsilon(t - \rho) g(t) dt. \quad (67)$$

Letting $\varepsilon \rightarrow 0$, it derives that

$$\begin{aligned} W'(\rho) &\rightarrow -g(\rho), \\ W(\rho) &\rightarrow W(0) - \int_0^\rho g(t) dt, \\ W(\tau) &\rightarrow W(0) - \int_0^\tau g(t) dt. \end{aligned} \quad (68)$$

Therefore, we have

$$W(\rho) - W(\tau) \rightarrow \int_\rho^\tau g(t) dt \quad \text{as } \varepsilon \rightarrow 0. \quad (69)$$

From (63)–(69), we obtain inequality

$$g(\rho) + c \int_\rho^\tau g(t) dt \geq g(\tau). \quad (70)$$

Choosing $\rho \rightarrow 0, \tau \rightarrow t$, we get

$$\begin{aligned} & \int_{-\infty}^{+\infty} |\phi(0, x) - \psi(0, x)| dx \\ & + c \int_0^t dt \int_{-\infty}^{+\infty} |\phi(t, x) - \psi(t, x)| dx \\ & \geq \int_{-\infty}^{+\infty} |\phi(t, x) - \psi(t, x)| dx. \end{aligned} \quad (71)$$

Applying the Gronwall inequality, we complete the proof of Theorem 6. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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