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Research Article

Measuring Loss-Based Process Capability Index L_e and Its Generation L_e'' with Fuzzy Numbers

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Loss-based process capability indices are appropriate and realistic tools in order to measure the process capability. Among them, index L_e and its generation L_e'' are well-known loss-based process capability indices, whose concepts are based on the worth (the opposite concept of loss). Sometimes, in order to calculate L_e and L_e'' there are some uncertainties in observations, so fuzzy logic can be employed to manage the uncertainties. This paper investigates fuzzification of process capability index $\widetilde{L}_e$ and its generation $\widetilde{L}_e''$. In order to find the membership function of process capability indices $\widetilde{L}_e$ and $\widetilde{L}_e''$, the α-cuts of fuzzy observation were employed. Then with an example of fuzzy process capability index, $\widetilde{L}_e$ and $\widetilde{L}_e''$ were calculated and compared. Results showed that fuzzy $\widetilde{L}_e''$ was more sensitive compared with $\widetilde{L}_e$ and was increased while the target departs (asymmetric tolerance). This example also showed that, with departure from the target, variation of fuzzy $\widetilde{L}_e''$ and consequently its fuzziness were increased.

1. Introduction

Nowadays manufacturers are so interested in understanding the capability of their processes in order to improve them [1]. Process capability can be defined as a measurement of inherent variability in a process compared to the specification requirements of the product [2–4]. The process capability indices (PCIs) C_p, C_{pk}, $k(C_a)$, C_{pu}, and C_{pl} have been described as five original PCIs by Sullivan [5] who had observed the usage of these indices at Japanese manufacturing facilities. Since PCIs present the capability of process, loss-based PCIs are more appropriate indices in order to measure the real capability of a process. The most well-known loss-based PCIs are C_{pm}, C_{pmk} [6], L_e, and L_e'' [7].

Fuzzy set theory which was introduced by Zadeh [8] has had a wide application in many fields such as industrial and production management [9, 10]. Fuzzy set theory and fuzzy logic also have been applied in measuring of the fuzzy process capability indices and till now there are some papers about some fuzzy PCIs such as C_p, C_{pk}, and C_{pm}. This paper is an extension of Tsai and Chen [11] and Chen et al. [12] for the process capability index $\widetilde{L}_e$ and its generation $\widetilde{L}_e''$ of fuzzy numbers and their differences.

The organization of this paper is as follows. Section 2 reviews the literature of loss-based process capability indices and fuzzy PCIs. Section 3 presents the α-cuts of fuzzy estimation for index L_e with fuzzy data. Similarly Section 4 describes fuzzy $\widetilde{L}_e''$ and Section 5 illustrates a numerical example. Finally Sections 6 and 7 deal with conclusions and suggestions for further research.

2. Literature Review

Kane [3] developed C_p and C_{pk} indices, which are commonly used in industries to evaluate single quality characteristics in mass production. Since C_p and C_{pk} indices do not take into account the difference between the processes mean and its target value and the loss of process (rejects and reworks), Chan et al. [13] considered this difference to develop C_{pm} and later Pearn et al. [14] developed C_{pmk}. The indices C_{pm} and C_{pmk} are initial loss-based PCIs which were based on Taguchi loss function [7]. Another well-known loss-based PCI is L_e which was proposed by Johnson [15]. This index was inspired from the process losses similar to C_{pm} and concept of relative expected squared error. This process capability index actually

uses the concept of worth (the opposite concept of loss). It was assumed that a quality characteristic (X) can achieve its maximum worth (W_T), when $X = T$. So when X moves away from the target, the worth becomes zero and then negative. The worth function can be defined by $W_T - k(X-T)^2$, for $W_T \geq k(X-T)^2$, and it will be zero when $|X - T| = (W_T/k)^{1/2}$. So in order to define the specification limits of C_{pm}, Johson [15] used this concept and defined $(T - (W_T/k)^{1/2}, T + (W_T/k)^{1/2})$ for specification limits of C_{pm}. Johson [15] defined ratio of worth (W_T) to maximum worth as follows:

$$W(X) = 1 - \frac{k(X-T)^2}{W_\tau} = 1 - \frac{(X-T)^2}{\Delta^2}, \quad (1)$$

where $\Delta = (W_T/k)^{1/2}$ and $(X-T)^2/\Delta^2$ are the relative loss. Then L_e is defined as follows [15]:

$$L_e = \frac{(d/3\Delta)^2}{C_{pm}^2}, \quad (2)$$

where $d = (\text{USL} - \text{LSL})/2$. The function Δ is related to ratio of worth. For example, supposing that $\Delta = (\text{USL} - \text{LSL})/2$, then L_e is the index which has the negative relationship with C_{pm} ($L_e = (3C_{pm})^{-2}$). Because the C_{pm} is a PCI between $[0, +\infty]$, then L_e will be a loss function between $[0, +\infty]$. The index L_e is defined as the ratio of the expected quadratic loss and the square of the half specification width as follows:

$$L_e = \int_{-\infty}^{\infty} \left[\frac{(X-T)^2}{d^2} \right] dF(x) = \frac{\sigma^2 + (\mu - T)^2}{d^2}, \quad (3)$$

where d^2 is a constant parameter derived from d^2 standard table. Then L_e can be defined as follows:

$$L_e = \frac{\sigma^2 + (\mu - T)^2}{d^2} = \left(\frac{\sigma}{d}\right)^2 + \left[\frac{(\mu - T)}{d}\right]^2. \quad (4)$$

Tsui [16] expressed the above equation as $L_e = L_{pe} + L_{ot}$. So Tsui [16] interpreted L_e as a summation of potential relative expected loss (L_{pe}) and the relative off-target squared deviation (L_{ot}).

The index L_e has some weak points. This index uses expected relative loss and considers the proximity of the target value, while users need to specify the target and the distance from the target at which the product would have zero worth to quantify the process loss. Another weak point of L_e is that it does not take into account the asymmetric tolerances. Pearn et al. [17] with an example showed this weak point and proposed L_e'' which is derived from L_e. The index L_e'' is defined as follows:

$$L_e'' = \left(\frac{A}{d^*}\right)^2 + \left(\frac{\sigma}{d^*}\right)^2, \quad (5)$$

where $A = \max\{(\mu - T)\cdot d/D_u, (T - \mu)\cdot d/D_l\}$, $D_u = \text{USL} - T$, $D_l = T - \text{LSL}$, and $d^* = \min\{D_u, D_l\}$. If the tolerances are symmetric ($T = m$), then $A = |\mu - T|$, $D_l = D_u = d$, and $d^* = d = (\text{USL} - \text{LSL})/2$, where m is the midpoint of tolerance. Pearn et al. [17] also proposed the following estimator for L_e'':

$$\widehat{L}_e'' = \left(\frac{\widehat{A}}{d^*}\right)^2 + \left(\frac{S}{d^*}\right)^2, \quad (6)$$

where $\widehat{A} = \max\{(\overline{X} - T)\cdot(d/D_u), (T - \overline{X})\cdot(d/D_l)\}$ and the mean μ is estimated by the sample mean and σ by sample standard deviation. There are some other PCIs such as C_{pk}'' [18], but since L_e'' is a loss-based PCI, it can take into account the target and have more reliable and reasonable outputs.

Some research on fuzzy process capability indices has been conducted. An initial study by Yongting [19] defined and explained fuzzy quality and analysis on fuzzy probability. Fuzzy α-cuts method is another methodology that has been used vastly in fuzzy PCIs studies. This methodology was introduced for the first time by Lee [20]. Chen et al. [21] introduced a fuzzy evaluation of C_{pm} for selecting a supplier. They employed Mamdani inference method [22] to interpret the different amount of C_{pm} for fuzzy data. Their methodology was based on the usage of confidence intervals. Tsai and Chen [11] had a survey about making a decision to evaluate a process capability index C_p with fuzzy numbers. Their methodology was using α-cuts method based on Kao and Liu [23] in order to find the membership function of the fuzzy $\widetilde{C}_p$. In another study Chen et al. [12] used this methodology in order to calculate the fuzzy process capability $\widetilde{C}_{pm}$. Buckley's estimation approach [24–26] is another methodology which was employed by Parchami and Mashinchi [27] to estimate process capability indices C_p, C_{pk}, and C_{pm}.

Recently these fuzzy indices have been used in real projects; for example, Kaya and Kahraman [28] developed fuzzy robust process capability indices for risk assessment of air pollution.

Most of the studies are based on fuzzy data and in this study we also assumed fuzzy observations with the methodology of Tsai and Chen [11] and Chen et al. [12] for fuzzy indices $\widetilde{L}_e$ and $\widetilde{L}_e''$.

3. Measuring Process Capability Index L_e with Fuzzy Data

Fuzzy logic can be employed to manage the uncertainties. These uncertainties could exist in the concepts of data which are being used to measure the capability of processes. Different relevant uncertainties are as follows:

(1) specification limits cannot always be represented by crisp numbers and are defined by fuzzy numbers [28];

(2) interpretation of a capable process is sometimes fuzzy [21];

(3) the specification data and observations can be fuzzy [11, 12, 27, 29].

The first uncertainty is not common, because specification limits are usually defined exactly by customers. The second uncertainty also is a kind of estimation and has error, so, in this paper, we considered the last kind of uncertainty. If

specification data is fuzzy, the process capability indices will be fuzzy. For simplicity, we supposed that our observations were triangular fuzzy numbers defined as:

$$\widetilde{X}_i = [X_{il}, X_{im}, X_{iu}], \quad \text{for } i = 1, 2, \ldots, n, \tag{7}$$

where n is the number of fuzzy parameters. We can assume fuzzy observations as a fuzzy set. Each observation or number in a fuzzy set has a membership function. So a fuzzy set consists of data and their memberships. For example, set A and fuzzy variable X are ordered in pairs as follows [30]:

$$Z = \{(x, \mu_Z(x)) \mid x \in X\}, \tag{8}$$

where $\mu_Z(x)$ represents the membership function of variable X in Z. The membership function of a fuzzy variable is the most important concept of a fuzzy variable. So in order to measure $\widetilde{L}_e$, its membership function must be achieved:

$$\widetilde{L}_e = \widetilde{L}_{pe} + \widetilde{L}_{ot}, \tag{9}$$

$$\widetilde{L}_e = \left(\frac{\widetilde{S}}{d}\right) + \left[\frac{\left(\widetilde{\overline{X}} - T\right)}{d}\right]^2, \tag{10}$$

where $\widetilde{\overline{X}}$ and $\widetilde{S}^2$ are defined as

$$\begin{aligned} \widetilde{S}^2 &\approx \frac{1}{n-1}\sum_{i=1}^{n}\left(\widetilde{x}_i - \widetilde{\overline{x}}\right)^2 \quad \text{for } i = 1, 2, \ldots, n \\ \widetilde{\overline{X}} &\approx \sum_{i=1}^{n}\frac{\widetilde{x}_i}{n} \quad \text{for } i = 1, 2, \ldots, n, \end{aligned} \tag{11}$$

where n is the number of fuzzy parameters. According to (10), calculation of $\widetilde{L}_e$ and its membership function is so complicated. This complexity has two main reasons:

(1) there are two fuzzy functions ($\widetilde{\overline{X}}$ and $\widetilde{S}^2$) in (10) which makes this equation more complicated;

(2) the relationship between the index $\widetilde{L}_e$ and the fuzzy observations (x) is nonlinear.

In order to overcome this problem, according to the methodology of Tsai and Chen [11] and Chen et al. [12], the α-cuts of the fuzzy observation based on Kao and Liu [23] can be used. Consequently, the exact form of the membership function can be derived by taking the membership functions of the α-cut. The α-cuts of the fuzzy observation $\widetilde{X}_i$ were presented as follows:

$$\begin{aligned} (X_i)_\alpha &= \left[(X_i)_\alpha^L, (X_i)_\alpha^U\right] \\ &= \left\lfloor \min_{x_i}\left\{x_i \in X \mid \mu_{\widetilde{X}_i}(x_i) \geq \alpha\right\}, \right. \\ &\qquad \left. \max_{x_i}\left\{x_i \in X \mid \mu_{\widetilde{X}_i}(x_i) \geq \alpha\right\}\right\rfloor, \end{aligned} \tag{12}$$

where X is the crisp universal set o, $(X_i)_\alpha^L$ is alpha cut of lower part of parameter, and $(X_i)_\alpha^U$ is alpha cut of upper part of parameter, in which $\widetilde{X}_i$ is defined. For a typical α, the yield of X_i will be $(X_i)_\alpha^L \leq X_i \leq (X_i)_{\alpha\,i}^U$. On the other hand, stated observations are as triangular fuzzy numbers denoted by the triplet $A = (a_l, a, a_u)$ and these observations have the shape of a triangle as shown in the Figure 1.

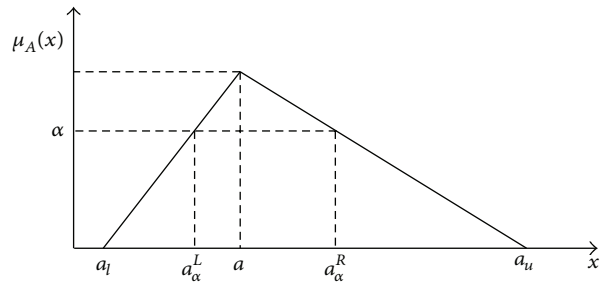

Figure 1: A triangular fuzzy number $A = (a_l, a, a_u)$.

Each α-cut of the triangular fuzzy number is $A_\alpha = [a_l, a, a_u]$ which is a closed interval as follows:

$$\begin{aligned} A_\alpha = \left[a_\alpha^L, a_\alpha^R\right] = \left[(a - a_l)\alpha + a_l, -(a_u - a)\alpha + a_u\right], \\ \alpha \in (0, 1]. \end{aligned} \tag{13}$$

Similarly $(\widetilde{S})_\alpha = \lfloor(\widetilde{S})_\alpha^L, (\widetilde{S})_\alpha^U\rfloor$ are the α-cuts of $\widetilde{S}$, $(\widetilde{\overline{X}})_\alpha = [(\widetilde{\overline{X}})_\alpha^L, (\widetilde{\overline{X}})_\alpha^U]$ are the α-cuts of $\widetilde{\overline{X}}$, and $(\widetilde{L}_e)_\alpha = \lfloor(\widetilde{L}_e)_\alpha^L, (\widetilde{L}_e)_\alpha^U\rfloor$ are the α-cuts of $\widetilde{L}_e$. According to (10) and (12) and Kao and Liu [23], lower and upper bounds of the α-cuts of $\widetilde{L}_e$ can be obtained as follows:

$$\left(\widetilde{L}_e\right)_\alpha^L = \min_{(X_i)_\alpha^L \leq X_i \leq (X_i)_{\alpha\,i}^U}\left[\left(\frac{\widetilde{S}}{d}\right)^2 + \left[\frac{(\widetilde{\overline{X}} - T)}{d}\right]^2\right], \tag{14a}$$

$$\left(\widetilde{L}_e\right)_\alpha^U = \max_{(X_i)_\alpha^L \leq X_i \leq (X_i)_{\alpha\,i}^U}\left[\left(\frac{\widetilde{S}}{d}\right)^2 + \left[\frac{(\widetilde{\overline{X}} - T)}{d}\right]^2\right]. \tag{14b}$$

Since (14a) and (14b) denote nonlinear relationship between $\widetilde{X}_i$ and α-cuts of $\widetilde{L}_e$, nonlinear programming package, LINGO [31], can be employed. It was shown in part 5 that each α-cut of $\widetilde{L}_e$ for different α from 0 to 1 must be calculated. Finally, the membership function $\mu_{C_{pm}}$ is constructed as

$$\mu_{\widetilde{L}_e}\left(\widetilde{L}_e\right) = \begin{cases} L\left(\widetilde{L}_e\right), & \left(\widetilde{L}_e\right)_0^L \leq \widetilde{L}_e \leq \left(\widetilde{L}_e\right)_1^L, \\ 1, & \left(\widetilde{L}_e\right)_1^L \leq \widetilde{L}_e \leq \left(\widehat{C}_{pmk}\right)_1^U, \\ R\left(\widetilde{L}_e\right), & \left(\widetilde{L}_e\right)_1^U \leq \widetilde{L}_e \leq \left(\widetilde{L}_e\right)_0^U. \end{cases} \tag{15}$$

4. Measuring the Process Capability Index L_e'' with Fuzzy Data

As stated in (5) Pearn et al. [17] proposed L_e'' which is derived from L_e. The index L_e'', which is defined as follows, is a combination of two separate indices L_{pe}'' and L_{ot}'':

$$L_e'' = \left(\frac{\widehat{A}}{d^*}\right)^2 + \left(\frac{\sigma}{d^*}\right)^2 = L_{pe}'' + L_{ot}''. \tag{16}$$

In order to estimate the fuzzy membership of L_e'', the fuzzy membership of $\widetilde{L}_{pe}''$ and $\widetilde{L}_{ot}''$ was calculated and then was combined to each other. Using (16), the fuzzy $\widetilde{L}_e''$ can be calculated as follows:

$$\widetilde{L}_e'' = \widetilde{L}_{pe}'' + \widetilde{L}_{ot}'' = \left(\frac{\widetilde{S}}{d^*}\right)^2 + \left(\frac{\widetilde{A}}{d^*}\right)^2, \tag{17}$$

where the statistics $\widetilde{\widetilde{X}}$ and $\widetilde{S}^2$ are defined as stated in (11). Since, in the above equation, there are two fuzzy functions ($\widetilde{\widetilde{X}}$ and $\widetilde{S}^2$) and also nonlinear relationship between the index $\widetilde{L}_e''$ and the fuzzy observations (x), the calculation of $\widetilde{L}_e''$ is so complicated, and so, according to the methodology of Tsai and Chen [11] and Chen et al. [12], the α-cuts of the fuzzy observation can be used. Consequently, the exact form of the membership function can be derived by taking the membership functions of the α-cut. Similarly $(\widetilde{L}_e'')_\alpha = \lfloor (\widetilde{L}_e'')_\alpha^L, (\widetilde{L}_e'')_\alpha^U \rfloor$ are the α-cuts of $\widetilde{L}_e''$ and according to Kao and Liu [23] lower and upper bounds of the α-cuts of $\widetilde{L}_e''$ can be obtained as follows:

$$\left(\widetilde{L}_e''\right)_\alpha^L = \min_{(X_i)_\alpha^L \le X_i \le (X_i)_{\alpha i}^U} \left[\left(\frac{\widetilde{S}}{d^*}\right)^2 + \left[\frac{\widetilde{A}}{d^*}\right]^2\right], \tag{18a}$$

$$\left(\widetilde{L}_e''\right)_\alpha^U = \max_{(X_i)_\alpha^L \le X_i \le (X_i)_{\alpha i}^U} \left[\left(\frac{\widetilde{S}}{d^*}\right)^2 + \left[\frac{\widetilde{A}}{d^*}\right]^2\right]. \tag{18b}$$

Similarly LINGO [31] can be employed in order to calculate $(\widetilde{L}_e'')_\alpha^L$ and $(\widetilde{L}_e'')_\alpha^U$ for various α and then the membership function, $\mu_{\widetilde{L}_e''}$, is constructed as

$$\mu_{\widetilde{L}_e''}\left(\widetilde{L}_e\right) = \begin{cases} L\left(\widetilde{L}_e''\right), & \left(\widetilde{L}_e''\right)_0^L \le \widetilde{L}_e'' \le \left(\widetilde{L}_e''\right)_1^L, \\ 1, & \left(\widetilde{L}_e''\right)_1^L \le \widetilde{L}_e'' \le \left(\widetilde{L}_e''\right)_1^U, \\ R\left(\widetilde{L}_e''\right), & \left(\widetilde{L}_e''\right)_1^U \le \widetilde{L}_e'' \le \left(\widetilde{L}_e''\right)_0^U. \end{cases} \tag{19}$$

5. A Numerical Example-Comparison with Other Fuzzy PCIs

In order to clarify the fuzzy process capability indices of $\widetilde{L}_e$ and $\widetilde{L}_e''$, we used the numerical example of Chen et al. [12] which was for the fuzzy PCI $\widetilde{C}_{pm}$. In this example, there are 30 fuzzy observations which are shown in Table 1.

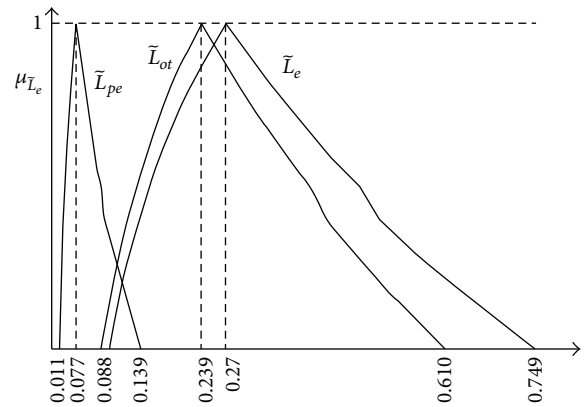

Figure 2: Membership function of $\widetilde{L}_{pe}$, $\widetilde{L}_{ot}$, and $\widetilde{L}_e$.

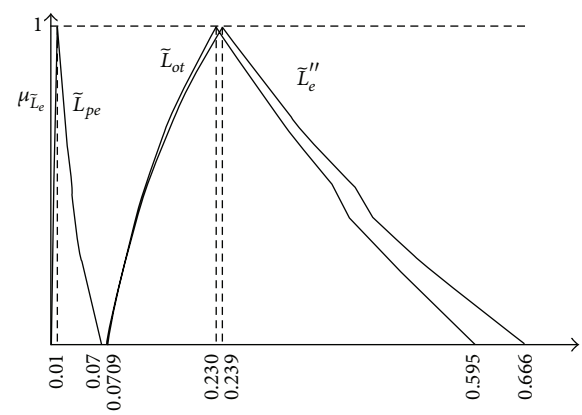

Figure 3: Membership function of $\widetilde{L}_{pe}$, $\widetilde{L}_{ot}$, and $\widetilde{L}_e''$.

The production specifications for a particular product are as follows: USL = 6.4, LSL = 5.5, $M = 5.95$, and $T = 6$. In order to measure the fuzzy process capability indices $\widetilde{L}_e$ and $\widetilde{L}_e''$, α-cuts of the fuzzy observations must be calculated. So, based on (9), α-cuts of these fuzzy observations for 11 various α were calculated and shown in Table 2. Note that in Table 2 just α-cuts of first 5 observations were mentioned and because of their similarity other α-cuts were omitted.

According to (11) the functions $\widetilde{\widetilde{X}}$ and $\widetilde{S}^2$ can be calculated. Equation (7) stated the α-cut of fuzzy number or variable. So some functions such as $\widetilde{\widetilde{X}}$, $\widetilde{S}^2$, and A also can have fuzzy α-cuts. Calculation of these functions is not useful directly, but it can help in order to define the appropriate operations research model in LINGO [31]. The results of α-cuts of $\widetilde{\widetilde{X}}$, $\widetilde{S}^2$, and $\widetilde{A}$ are shown in Table 3.

Using (11), α-cuts of $\widetilde{L}_e$ could be calculated. The results were shown in Table 4. In order to calculate α-cuts of $\widetilde{L}_e$, at first, α-cuts of $\widetilde{L}_{ot}$ and $\widetilde{L}_{pe}$ were calculated and then they were added together. Figure 2 shows the relation of $\widetilde{L}_{ot}$ and $\widetilde{L}_{pe}$ more clearly.

Now in order to calculate $\widetilde{L}_e''$, we calculate α-cuts 9 of $(\widetilde{L}_{ot}'')_\alpha^L$, $(\widetilde{L}_{ot}'')_\alpha^U$, $(\widetilde{L}_{pe}'')_\alpha^L$, and $(\widetilde{L}_{pe}'')_\alpha^U$. The results are shown in Table 5 and Figure 3 for various α.

Table 1: An example consisted of 30 fuzzy triangular observations.

$\overline{X}_1 = [5.85 \quad 6.15 \quad 6.35]$	$\overline{X}_{11} = [5.86 \quad 6.04 \quad 6.25]$	$\overline{X}_{21} = [5.50 \quad 5.81 \quad 5.99]$
$\overline{X}_2 = [5.79 \quad 5.9 \quad 5.98]$	$\overline{X}_{12} = [6.13 \quad 6.23 \quad 6.33]$	$\overline{X}_{22} = [5.60 \quad 5.92 \quad 6.05]$
$\overline{X}_3 = [5.71 \quad 5.83 \quad 5.99]$	$\overline{X}_{13} = [5.95 \quad 6.05 \quad 6.19]$	$\overline{X}_{23} = [5.50 \quad 5.75 \quad 5.95]$
$\overline{X}_4 = [6.05 \quad 6.18 \quad 6.32]$	$\overline{X}_{14} = [5.6 \quad 5.65 \quad 5.70]$	$\overline{X}_{24} = [5.84 \quad 6.03 \quad 6.15]$
$\overline{X}_5 = [5.89 \quad 6.06 \quad 6.23]$	$\overline{X}_{15} = [5.65 \quad 5.74 \quad 5.84]$	$\overline{X}_{25} = [6.05 \quad 6.30 \quad 6.50]$
$\overline{X}_6 = [6.01 \quad 6.10 \quad 6.25]$	$\overline{X}_{16} = [5.70 \quad 5.77 \quad 5.83]$	$\overline{X}_{26} = [6.25 \quad 6.35 \quad 6.45]$
$\overline{X}_7 = [6.15 \quad 6.20 \quad 6.30]$	$\overline{X}_{17} = [6.23 \quad 6.32 \quad 6.40]$	$\overline{X}_{27} = [5.65 \quad 5.86 \quad 6.05]$
$\overline{X}_8 = [5.64 \quad 5.81 \quad 6.05]$	$\overline{X}_{18} = [5.60 \quad 5.70 \quad 5.80]$	$\overline{X}_{28} = [5.70 \quad 5.87 \quad 5.95]$
$\overline{X}_9 = [5.8 \quad 5.90 \quad 5.98]$	$\overline{X}_{19} = [5.85 \quad 5.95 \quad 6.05]$	$\overline{X}_{29} = [5.75 \quad 5.95 \quad 6.15]$
$\overline{X}_{10} = [6.01 \quad 6.12 \quad 6.24]$	$\overline{X}_{20} = [5.90 \quad 6.00 \quad 6.10]$	$\overline{X}_{30} = [6.10 \quad 6.23 \quad 6.46]$

Table 2: Different α-cuts of the first five observations.

L	Mid	U	α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
5.85	6.00	6.15	$(X_1)_\alpha^L$	5.85	5.88	5.91	5.94	5.97	6	6.03	6.06	6.09	6.12	6.15
			$(X_1)_\alpha^U$	6.35	6.33	6.31	6.29	6.27	6.25	6.23	6.21	6.19	6.17	6.15
5.79	5.845	5.9	$(X_2)_\alpha^L$	5.79	5.8	5.81	5.82	5.83	5.85	5.86	5.87	5.88	5.89	5.9
			$(X_2)_\alpha^U$	5.98	5.972	5.964	5.956	5.948	5.94	5.932	5.924	5.916	5.908	5.9
5.71	5.77	5.83	$(X_3)_\alpha^L$	5.71	5.72	5.73	5.75	5.76	5.77	5.78	5.79	5.81	5.82	5.83
			$(X_3)_\alpha^U$	5.99	5.974	5.958	5.942	5.926	5.91	5.894	5.878	5.862	5.846	5.83
6.05	6.115	6.18	$(X_4)_\alpha^L$	6.05	6.06	6.08	6.09	6.1	6.12	6.13	6.14	6.15	6.17	6.18
			$(X_4)_\alpha^U$	6.32	6.306	6.292	6.278	6.264	6.25	6.236	6.222	6.208	6.194	6.18
5.89	597.5	6.06	$(X_5)_\alpha^L$	5.89	5.91	5.92	5.94	5.96	5.98	5.99	6.01	6.03	6.04	6.06
			$(X_5)_\alpha^U$	6.23	6.213	6.196	6.179	6.162	6.145	6.128	6.111	6.094	6.077	6.06

Table 3: Results of α-cuts for 11 various α.

PCI	α										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$(\widetilde{S}^2)_\alpha^l$	0.0124	0.0138	0.0153	0.0171	0.0191	0.0214	0.024	0.0269	0.0303	0.0341	0.0383
$(\widetilde{S}^2)_\alpha^u$	0.1065	0.097	0.0881	0.0798	0.072	0.0678	0.0607	0.0547	0.0489	0.0434	0.0383
$\left(\widetilde{\overline{X}}^2\right)_\alpha^l$	5.8436	5.8585	5.8734	5.8882	5.9031	5.9180	5.9328	5.9477	5.9626	5.9774	5.9923
$\left(\widetilde{\overline{X}}^2\right)_\alpha^U$	6.1293	6.1156	6.1019	6.0882	6.0745	6.0608	6.0471	6.0334	6.0197	6.006	5.9923
$\left((\mu - T)\cdot \frac{d}{D_u}\right)_\alpha^l$	0.1455	0.1301	0.1146	0.0992	0.0838	0.0684	0.0530	0.0376	0.0222	0.0068	0.0087
$\left((\mu - T)\cdot \frac{d}{D_u}\right)_\alpha^u$	0.1760	0.1592	0.1424	0.1258	0.1090	0.0922	0.0756	0.0588	0.0421	0.0254	0.0087
$\left((T - \mu)\cdot \frac{d}{D_l}\right)_\alpha^l$	0.1164	0.1040	0.0917	0.0794	0.0670	0.0547	0.0424	0.0301	0.0177	0.0054	0.0069
$\left((T - \mu)\cdot \frac{d}{D_l}\right)_\alpha^u$	0.1408	0.1274	0.1139	0.1006	0.0872	0.0738	0.0605	0.0471	0.0337	0.0203	0.0069
$(\widetilde{A})_\alpha^l$	0.1455	0.1301	0.1146	0.0992	0.0838	0.0684	0.0530	0.0376	0.0222	0.0068	0.0087
$(\widetilde{A})_\alpha^u$	0.1760	0.1592	0.1424	0.1258	0.1090	0.0922	0.0756	0.0588	0.0421	0.0254	0.0087

Table 4: Results of α-cuts of $\widetilde{L}_e$ for 11 various α.

PCI	α										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$(\widetilde{L}_{ot})_\alpha^L$	0.0771	0.0854	0.0948	0.1056	0.1178	0.1315	0.1473	0.1651	0.1851	0.2074	0.2323
$(\widetilde{L}_{ot})_\alpha^U$	0.6100	0.5600	0.5124	0.4673	0.4249	0.4010	0.3615	0.3275	0.2944	0.2622	0.2323
$(\widetilde{L}_{pe})_\alpha^L$	0.0111	0.0122	0.0138	0.0154	0.0173	0.0197	0.0222	0.0249	0.0287	0.0330	0.0377
$(\widetilde{L}_{pe})_\alpha^U$	0.1398	0.1228	0.1078	0.0944	0.0824	0.0763	0.0663	0.0581	0.0506	0.0438	0.0377
$(\widetilde{L}_{e})_\alpha^L$	0.0882	0.0976	0.1086	0.1210	0.1351	0.1512	0.1695	0.190	0.2138	0.2404	0.27
$(\widetilde{L}_{e})_\alpha^U$	0.7498	0.6828	0.6202	0.5617	0.5073	0.4773	0.4278	0.3856	0.3450	0.306	0.27

Table 5: Results of α-cuts of $\widetilde{L}''_e$ for 11 various α.

PCI	α										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$(\widetilde{L}''_{ot})_\alpha^L$	0.0769	0.0851	0.0945	0.1052	0.1173	0.1309	0.1465	0.1642	0.1839	0.2058	0.2303
$(\widetilde{L}''_{ot})_\alpha^U$	0.5951	0.5477	0.5022	0.4589	0.4181	0.3950	0.3567	0.3236	0.2913	0.2597	0.2303
$(\widetilde{L}''_{pe})_\alpha^L$	0.0010	0.0012	0.0015	0.0018	0.0023	0.0029	0.0036	0.0045	0.0057	0.0073	0.0092
$(\widetilde{L}''_{pe})_\alpha^U$	0.0709	0.0588	0.0485	0.0398	0.0324	0.0287	0.0230	0.0187	0.0149	0.0118	0.0092
$(\widetilde{L}''_{e})_\alpha^L$	0.0779	0.0863	0.0960	0.1070	0.1196	0.1338	0.1501	0.1687	0.1896	0.2131	0.2395
$(\widetilde{L}''_{e})_\alpha^U$	0.6660	0.6065	0.5507	0.4987	0.4505	0.4237	0.3797	0.3423	0.3062	0.2715	0.2395

As shown in Figure 2 the indices $\widetilde{L}_{ot}$, $\widetilde{L}_{pe}$, and $\widetilde{L}_e$ at $\alpha = 0$ have the intersection points with x-axis. For example, $\widetilde{L}_{pe}$ varies from 0.011 to 0.139 and its highest membership is in 0.077. So the membership functions of them are as follows:

$$\mu_{\widetilde{L}_{pe}}\left(\widehat{L}_{pe}\right) = \begin{cases} L\left(\widehat{L}_{pe}\right), & 0.011 \le \widehat{L}_{pe} \le 0.092, \\ 1, & \widehat{L}_{pe} = 0.092, \\ R\left(\widehat{L}_{pe}\right), & 0.092 \le \widehat{L}_{pe} \le 0.139, \end{cases}$$

$$\mu_{\widetilde{L}_{ot}}\left(\widehat{L}_{ot}\right) = \begin{cases} L\left(\widehat{L}_{ot}\right), & 0.077 \le \widehat{L}_{ot} \le 0.27, \\ 1, & \widehat{L}_{ot} = 0.27, \\ R\left(\widehat{L}_{ot}\right), & 0.27 \le \widehat{L}_{ot} \le 0.610, \end{cases} \tag{20}$$

$$\mu_{\widetilde{L}_{e}}\left(\widehat{L}_{e}\right) = \begin{cases} L\left(\widehat{L}_{e}\right), & 0.088 \le \widehat{L}_{e} \le 0.232, \\ 1, & \widehat{L}_{e} = 0.232, \\ R\left(\widehat{L}_{e}\right), & 0.232 \le \widehat{L}_{e} \le 0.749. \end{cases}$$

Similarly as shown in Figure 3 the indices $\widetilde{L}''_{pe}$, $\widetilde{L}''_{ot}$, and $\widetilde{L}''_e$ at $\alpha = 0$ have the intersection points with x-axis. So their membership functions are as follows:

$$\mu_{\widetilde{L}''_{pe}}\left(\widehat{L}''_{pe}\right) = \begin{cases} L\left(\widehat{L}''_{pe}\right), & 0.01 \le \widehat{L}''_{pe} \le 0.0377, \\ 1, & \widehat{L}''_{pe} = 0.0377, \\ R\left(\widehat{L}''_{pe}\right), & 0.0377 \le \widehat{L}''_{pe} \le 0.139, \end{cases}$$

$$\mu_{\widetilde{L}''_{ot}}\left(\widehat{L}''_{ot}\right) = \begin{cases} L\left(\widehat{L}''_{ot}\right), & 0.07 \le \widehat{L}''_{ot} \le 0.230, \\ 1, & \widehat{L}''_{ot} = 0.230, \\ R\left(\widehat{L}''_{ot}\right), & 0.230 \le \widehat{L}''_{ot} \le 0.595, \end{cases}$$

$$\mu_{\widetilde{L}''_{e}}\left(\widehat{L}''_{e}\right) = \begin{cases} L\left(\widehat{L}''_{e}\right), & 0.07 \le \widehat{L}''_{e} \le 0.239, \\ 1, & \widehat{L}''_{e} = 0.239, \\ R\left(\widehat{L}''_{e}\right), & 0.239 \le \widehat{L}''_{e} \le 0.666. \end{cases} \tag{21}$$

As shown in Figures 2 and 3, despite the triangular fuzzy observations, the membership functions of $\widetilde{L}_e$ and $\widetilde{L}''_e$ are not triangular functions, but as concluded from Figures 2 and 3 the indices $\widetilde{L}_e$ and $\widetilde{L}''_e$ such as other fuzzy PCIs have fuzzy behavior. In this example, fuzzy index $\widetilde{L}_e$ varies from 0.088 to 0.794. Similarly, fuzzy index $\widetilde{L}''_e$ varies from 0.07 to 0.666. The most probable value of $\widetilde{L}_e$ is 0.232, because this point has the maximum membership at 1 and the most probable quantity of $\widetilde{L}''_e$ is 0.239.

In order to compare the behavior of $\widetilde{L}_e$ and $\widetilde{L}''_e$, we shifted the target point (asymmetric tolerance). Initial target was 6 and in this example we supposed that target departs from 6 to 5.70. Then d^* will vary from 0.05 to 0.3. Similar methodology was employed in order to calculate fuzzy α-cuts of $\widetilde{L}_e$ and $\widetilde{L}''_e$. Results showed that $\widetilde{L}_e$ was constant and was not sensitive to

Table 6: Results of α-cuts of $\widetilde{L}_e''$ for various α for different $\overline{x}$.

α	d^*	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.35	0.068	0.076	0.084	0.094	0.105	0.117	0.131	0.148	0.166	0.186	0.210
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.35	0.583	0.531	0.482	0.436	0.394	0.371	0.332	0.300	0.268	0.238	0.210
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.3	0.058	0.065	0.072	0.080	0.090	0.100	0.113	0.127	0.142	0.160	0.180
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.3	0.500	0.455	0.413	0.374	0.338	0.318	0.285	0.257	0.230	0.204	0.180
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.25	0.049	0.054	0.060	0.067	0.075	0.084	0.094	0.105	0.119	0.133	0.150
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.25	0.416	0.379	0.344	0.312	0.282	0.265	0.237	0.214	0.191	0.170	0.150
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.2	0.039	0.043	0.048	0.054	0.060	0.067	0.075	0.084	0.095	0.107	0.120
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.2	0.333	0.303	0.275	0.249	0.225	0.212	0.190	0.171	0.153	0.136	0.120
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.15	0.029	0.032	0.036	0.040	0.045	0.050	0.056	0.063	0.071	0.080	0.090
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.15	0.250	0.227	0.207	0.187	0.169	0.159	0.142	0.128	0.115	0.102	0.090
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.1	0.019	0.022	0.024	0.027	0.030	0.033	0.038	0.042	0.047	0.053	0.060
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.1	0.167	0.152	0.138	0.125	0.113	0.106	0.095	0.086	0.077	0.068	0.060
$\left(\widetilde{L}_e''\right)_\alpha^L$	0.05	0.010	0.011	0.012	0.013	0.015	0.017	0.019	0.021	0.024	0.027	0.030
$\left(\widetilde{L}_e''\right)_\alpha^U$	0.05	0.083	0.076	0.069	0.062	0.056	0.053	0.047	0.043	0.038	0.034	0.030

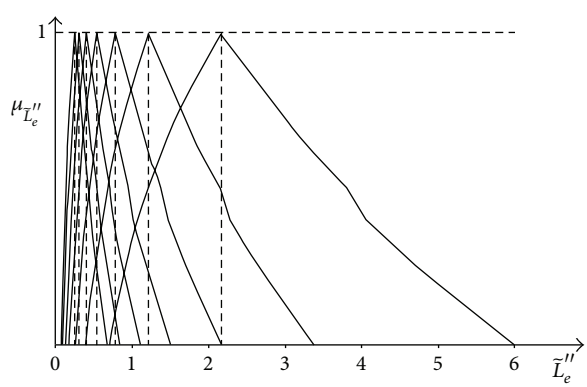

Figure 4: Results for fuzzy α-cuts of $\widetilde{L}_e''$, whereas the target departs.

the departure from target, while fuzzy index $\widetilde{L}_e''$ was sensitive to the changes of target. The results of fuzzy α-cuts of $\widetilde{L}_e''$ are shown in Table 6 and Figure 4.

As stated the index $\widetilde{L}_e''$ was more sensitive in process yield, so in this example, because of shift in process mean, $\widetilde{L}_e''$ is higher than $\widetilde{L}_e$. Figure 4 showed that with increasing the deviation of these fuzzy data the fuzzy $\widetilde{L}_e''$ was increased. Fuzzy $\widetilde{L}_e''$ was grown while the variation of this index also increased. For instance for $d^* = 0.05$ the fuzzy $\widetilde{L}_e''$ varies from 0.01 to 0.083 with yield of 0.073. When $d^* = 0.3$, then the fuzzy $\widetilde{L}_e''$ varies from 0.078 to 0.666 with yield of 0.588. This variation for a fuzzy index such as $\widetilde{L}_e''$ is so much and shows more fuzziness in results.

6. Conclusions

Loss-based process capability indices such as $\widetilde{L}_e$ and $\widetilde{L}_e''$ are more realistic and suitable tools to measure the capability of a process. This study investigated fuzzy process capability indices $\widetilde{L}_e$ and $\widetilde{L}_e''$. In order to find the membership functions of fuzzy process capability PCIs $\widetilde{L}_e$ and $\widetilde{L}_e''$, the methodology of Kao and Liu [23] for α-cuts of the fuzzy observations and converting them to a nonlinear model was employed. Moreover, with an example, the behaviors of fuzzy indices $\widetilde{L}_e$ and $\widetilde{L}_e''$ were investigated. In this example the target departed (asymmetric tolerance) gradually and results showed that fuzzy $\widetilde{L}_e''$ was increased and it was more sensitive compared with $\widetilde{L}_e$ (in case of asymmetric tolerance), but in this case when the target departed, $\widetilde{L}_e''$ was not a fully reliable fuzzy PCI because with departure from target, variation of fuzzy $\widetilde{L}_e''$ and consequently its fuzziness were increased.

7. Further Research

Fuzzy indices $\widetilde{L}_e$ and $\widetilde{L}_e''$ in the case of symmetric tolerance are the same. When tolerance is asymmetric, the fuzzy index $\widetilde{L}_e''$ is more sensitive because it takes into account the target, but with departure from the target (more asymmetric tolerance) variation of fuzzy $\widetilde{L}_e''$, and consequently its fuzziness was increased. So this fuzzy loss-based PCI also is not a reliable PCI and there is a necessity to develop a more accurate fuzzy loss-based PCI.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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