Research Article

A Conforming Triangular Plane Element with Rotational Degrees of Freedom

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This paper presents a novel way to formulate the triangular plane element with rotational degrees of freedom (RDOF). The linear distribution of rotational displacement is assumed. The conforming displacement along the sides based on the rotational displacement assumption is derived, and the triangular plane element TR3 for isotropic material is formulated. By using the explicit integral formulae of the triangular element, the matrices used in the proposed plane element TR3 are calculated efficiently. The benchmark examples showed their high accuracy and high efficiency.

1. Introduction

The triangular plane element is widely used [1–3]. In order to improve the performance of the triangular plane element, rotational degrees of freedom (RDOF) has been introduced. The other advantage of plane elements with RDOF is that the singularity of global stiffness can be avoided in the analysis of shell structures. Mervyn and Terrence introduced the RDOF to describe the displacement of the flat shell element [4]. Mohr constructed the hybrid membrane element with RDOF [5]. Allman et al. presented some rational displacement-type plane elements with RDOF [6–10]. Many researchers improve accuracy of the membrane element by this way [11–15], and the plane elements with RDOF have been widely used in analysis of shell structure.

Long et al. studied the generalized conforming plane elements [16–20] with RDOF. Taking advantage of the conforming boundary displacement, the nonconforming displacement model can be involved freely by the generalized conforming equations [17–20]. Chen et al. proved that the hybrid element using the balanced stress fields can play the same role in the element formulation as the generalized conforming element introducing the associated generalized conforming equations [20]. On the other hand, the hybrid

element employed analytical trial functions of stress needs conforming boundary displacement [21–26].

This paper studies the triangular plane element based on the rational assumption of the rotation displacement field. There are mainly four steps as follows in Section 2.

- (a) The first step assumes the rational distribution of the rotational displacement (in Section 2.1).
- (b) The second step gives the displacement field based on the assumed rotational displacement (in Section 2.2).
- (c) The third step formulates the stiffness matrix of the element TR3 based on the associated strain fields (in Section 2.3).
- (d) The fourth step calculates the stiffness matrix using the explicit integral formulae of the triangular element (in Section 2.4).

In Section 3, some numerical examples are shown as benchmark to study the accuracy and the efficiency of the proposed element model TR3.

In Section 4, some conclusions are given.

2. The Formulation of the Triangular Element TR3 with RDOF

2.1. The Rational Distribution of the Rotational Displacement. The rotational displacement field in the triangular element with three nodes can be assumed as follows:

$$\theta_z = L_1\theta_{z1} + L_2\theta_{z2} + L_3\theta_{z3}, \quad (1)$$

where

$$\begin{aligned} L_1 &= \frac{(a_1 + b_1x + c_1y)}{2A}, \\ L_2 &= \frac{(a_2 + b_2x + c_2y)}{2A}, \\ L_3 &= \frac{(a_3 + b_3x + c_3y)}{2A}, \end{aligned} \quad (2)$$

where

$$\begin{aligned} a_1 &= x_2y_3 - x_3y_2, & a_2 &= x_3y_1 - x_1y_3, \\ a_3 &= x_1y_2 - x_2y_1, \\ b_1 &= y_2 - y_3, & b_2 &= y_3 - y_1, \\ b_3 &= y_1 - y_2, \\ c_1 &= x_3 - x_2, & c_2 &= x_1 - x_3, \\ c_3 &= x_2 - x_1, \end{aligned} \quad (3)$$

$$2A = x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2,$$

in which x_1, x_2 , and x_3 and y_1, y_2 , and y_3 are coordinates of three nodes in the triangular element. θ_{z1}, θ_{z2} , and θ_{z3} are rotational displacement of three nodes.

2.2. The Distribution of the Displacement. According to the definition of the rotation displacement in the elasticity, the relationship between displacement symbols u, v and the rotational displacement θ_z can be denoted as

$$\theta_z = \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right). \quad (4)$$

From (1),

$$\begin{aligned} \theta_z &= (a_1\theta_{z1} + a_2\theta_{z2} + a_3\theta_{z3}) + (b_1\theta_{z1} + b_2\theta_{z2} + b_3\theta_{z3})x \\ &\quad + (c_1\theta_{z1} + c_2\theta_{z2} + c_3\theta_{z3})y \\ &= A_z + B_zx + C_zy. \end{aligned} \quad (5)$$

From (4) and (5), we can assume

$$\begin{aligned} u(x, y) &= \frac{A_z}{2} + \frac{B_zxy}{2} + \frac{C_z}{4}y^2 + f_u(x, y), \\ v(x, y) &= -\frac{A_z}{2} - \frac{B_z}{4}x^2 - \frac{C_zxy}{2} + f_v(x, y), \end{aligned} \quad (6)$$

where we can give two assumptions,

$$\begin{aligned} f_u(x, y) &= u_0 + u_x x + u_y y, \\ f_v(x, y) &= v_0 + v_x x + v_y y. \end{aligned} \quad (7)$$

On the other hand, we can define the displacement fields in the triangular element as

$$\begin{aligned} u &= g_u(u_1, u_2, u_3, \theta_{z1}, \theta_{z2}, \theta_{z3}, x, y), \\ v &= g_v(v_1, v_2, v_3, \theta_{z1}, \theta_{z2}, \theta_{z3}, x, y). \end{aligned} \quad (8)$$

According to (6)–(8), the nodal conforming requirement can be given as

$$\begin{aligned} u_1 &= u(x_1, y_1), & u_2 &= u(x_2, y_2), & u_3 &= u(x_3, y_3), \\ v_1 &= v(x_1, y_1), & v_2 &= v(x_2, y_2), & v_3 &= v(x_3, y_3). \end{aligned} \quad (9)$$

Then, we can find the relationship between nodal displacement symbols u_1, u_2, u_3, v_1, v_2 , and v_3 and inner parameters u_0, u_x, u_y, v_0, v_x , and v_y ; it is a linear translation matrix. By solving the associated equations (9), we can get the distribution of the displacement as follows:

$$\begin{aligned} u &= u_1L_1 + u_2L_2 + u_3L_3 + \theta_{z1} \left(\frac{a_1}{2}y + \frac{c_1}{4}y^2 + \frac{b_1}{2}xy \right) \\ &\quad + \theta_{z2} \left(\frac{a_2}{2}y + \frac{c_2}{4}y^2 + \frac{b_2}{2}xy \right) \\ &\quad + \theta_{z3} \left(\frac{a_3}{2}y + \frac{c_3}{4}y^2 + \frac{b_3}{2}xy \right) \\ &\quad + \frac{\theta_{z1}}{4(2A)} \left\{ x(y_2 - y_3) [2x_1y_1(y_3 - y_2) \right. \\ &\quad \left. + x_2(y_1 + y_2)(y_1 - y_3) \right. \\ &\quad \left. + x_3(y_1 + y_3)(y_2 - y_1)] \right. \\ &\quad \left. + y(x_2 - x_3) \right. \\ &\quad \left. \cdot [(x_1 - x_2)(y_1 - y_3)^2 + (x_3 - x_1)(y_1 - y_2)^2] \right. \\ &\quad \left. - [x_1x_2y_3(2y_1 - y_2)(y_2 - y_3) \right. \\ &\quad \left. + x_1x_3y_2(y_3 - 2y_1)(y_2 - y_3) \right. \\ &\quad \left. + x_2^2y_1y_3(y_3 - y_1) + x_3^2y_1y_2(y_2 - y_1) \right. \\ &\quad \left. + x_2x_3y_1[(y_1 + y_2 + y_3)(y_2 + y_3) \right. \\ &\quad \left. - 6y_2y_3] \right] \left. \right\} \\ &\quad + \frac{\theta_{z2}}{4(2A)} \left\{ x(y_3 - y_1) [2x_2y_2(y_1 - y_3) \right. \\ &\quad \left. + x_3(y_2 + y_3)(y_2 - y_1) \right. \\ &\quad \left. + x_1(y_2 + y_1)(y_3 - y_2)] \right. \end{aligned}$$

$$\begin{aligned}
& + y(x_3 - x_1) \\
& \cdot [(x_2 - x_3)(y_2 - y_1)^2 + (x_1 - x_2)(y_2 - y_3)^2] \\
& - [x_2 x_3 y_1 (2y_2 - y_3)(y_3 - y_1) \\
& + x_1 x_2 y_3 (y_1 - 2y_2)(y_3 - y_1) \\
& + x_3^2 y_2 y_1 (y_1 - y_2) + x_1^2 y_2 y_3 (y_3 - y_2) \\
& + x_3 x_1 y_2 [(y_1 + y_2 + y_3)(y_1 + y_3) \\
& - 6y_1 y_3]] \} \\
& + \frac{\theta_{z3}}{4(2A)} \{ x(y_1 - y_2) [2x_3 y_3 (y_2 - y_1) \\
& + x_1 (y_3 + y_1)(y_3 - y_2) \\
& + x_2 (y_3 + y_2)(y_1 - y_3)] \\
& + y(x_1 - x_2) \\
& \cdot [(x_3 - x_1)(y_3 - y_2)^2 + (x_2 - x_3)(y_3 - y_1)^2] \\
& - [x_3 x_1 y_2 (2y_3 - y_1)(y_1 - y_2) \\
& + x_3 x_2 y_1 (y_2 - 2y_3)(y_1 - y_2) \\
& + x_1^2 y_3 y_2 (y_2 - y_3) + x_2^2 y_3 y_1 (y_1 - y_3) \\
& + x_2 x_1 y_3 [(y_1 + y_2 + y_3)(y_1 + y_2) \\
& - 6y_1 y_2]] \} , \\
v = & v_1 L_1 + v_2 L_2 + v_3 L_3 + \theta_{z1} \left(\frac{a_1}{2} x + \frac{b_1}{4} x^2 + \frac{c_1}{2} xy \right) \\
& - \theta_{z2} \left(\frac{a_2}{2} x + \frac{b_2}{4} x^2 + \frac{c_2}{2} xy \right) \\
& - \theta_{z3} \left(\frac{a_3}{2} x + \frac{b_3}{4} x^2 + \frac{c_3}{2} xy \right) \\
& + \frac{\theta_{z1}}{4(2A)} \{ -y(x_2 - x_3) [2x_1 y_1 (x_3 - x_2) \\
& + y_2 (x_1 + x_2)(x_1 - x_3) \\
& + y_3 (x_1 + x_3)(x_2 - x_1)] \\
& - x(y_2 - y_3) \\
& \cdot [(y_1 - y_2)(x_1 - x_3)^2 + (y_3 - y_1)(x_1 - x_2)^2] \\
& + [y_1 y_2 x_3 (2x_1 - x_2)(x_2 - x_3) \\
& + y_1 y_3 x_2 (x_3 - 2x_1)(x_2 - x_3) \\
& + y_2^2 x_1 x_3 (x_3 - x_1) + y_3^2 x_1 x_2 (x_2 - x_1) \\
& + y_2 y_3 x_1 [(x_1 + x_2 + x_3)(x_2 + x_3) - 6x_2 x_3]] \}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\theta_{z2}}{4(2A)} \{ -y(x_3 - x_1) [2y_2 x_2 (x_1 - x_3) \\
& + y_3 (x_2 + x_3)(x_2 - x_1) \\
& + y_1 (x_2 + x_1)(x_3 - x_2)] \\
& - x(y_3 - y_1) \\
& \cdot [(y_2 - y_3)(x_2 - x_1)^2 + (y_1 - y_2)(x_2 - x_3)^2] \\
& + [y_2 y_3 x_1 (2x_2 - x_3)(x_3 - x_1) \\
& + y_1 y_2 x_3 (x_1 - 2x_2)(x_3 - x_1) \\
& + y_3^2 x_2 x_1 (x_1 - x_2) + y_1^2 x_2 x_3 (x_3 - x_2) \\
& + y_3 y_1 x_2 [(x_1 + x_2 + x_3)(x_1 + x_3) - 6x_1 x_3]] \} \\
& + \frac{\theta_{z3}}{4(2A)} \{ -y(x_1 - x_2) [2y_3 x_3 (x_2 - x_1) \\
& + y_1 (x_3 + x_1)(x_3 - x_2) \\
& + y_2 (x_3 + x_2)(x_1 - x_3)] \\
& - x(y_1 - y_2) \\
& \cdot [(y_3 - y_1)(x_3 - x_2)^2 + (y_2 - y_3)(x_3 - x_1)^2] \\
& + [y_3 y_1 x_2 (2x_3 - x_1)(x_1 - x_2) \\
& + y_3 y_2 x_1 (x_2 - 2x_3)(x_1 - x_2) \\
& + y_1^2 x_3 x_2 (x_2 - x_3) + y_2^2 x_3 x_1 (x_1 - x_3) \\
& + y_2 y_1 x_3 [(x_1 + x_2 + x_3)(x_1 + x_2) - 6x_1 x_2]] \} .
\end{aligned} \tag{10}$$

It can be proved that the assumed displacement is conforming.

2.3. The Stiffness Matrix of the Plane Element TR3 with RDOF. From (10), taking into account the definition of the strain field, we can get the strain matrix about the nodal displacement. It can be denoted as

$$[B] = [[B_1] \quad [B_2] \quad [B_3]] ,$$

$$[B_1] = \begin{bmatrix} \frac{b_1}{2A} & 0 & \frac{b_1(2y - y_1 - b_x)}{4} \\ 0 & \frac{c_1}{2A} & \frac{-c_1(2x - x_1 - b_y)}{4} \\ \frac{c_1}{2A} & \frac{b_1}{2A} & \frac{(x_1 b_1 + x_2 b_3 + x_3 b_2)}{4} \end{bmatrix} ,$$

$$[B_2] = \begin{bmatrix} \frac{b_2}{2A} & 0 & \frac{b_2(2y - y_2 - b_x)}{4} \\ 0 & \frac{c_2}{2A} & \frac{-c_2(2x - x_2 - b_y)}{4} \\ \frac{c_2}{2A} & \frac{b_2}{2A} & \frac{(x_1b_3 + x_2b_2 + x_3b_2)}{4} \end{bmatrix},$$

$$[B_3] = \begin{bmatrix} \frac{b_3}{2A} & 0 & \frac{b_3(2y - y_3 - b_x)}{4} \\ 0 & \frac{c_3}{2A} & \frac{-c_3(2x - x_3 - b_y)}{4} \\ \frac{c_3}{2A} & \frac{b_3}{2A} & \frac{(x_1b_2 + x_2b_1 + x_3b_3)}{4} \end{bmatrix},$$
(11)

in which

$$b_x = \frac{(x_1y_1b_1 + x_2y_2b_2 + x_3y_3b_3)}{2A},$$

$$b_y = \frac{(x_1y_1c_1 + x_2y_2c_2 + x_3y_3c_3)}{2A}.$$
(12)

Then, the stiffness matrix can be obtained:

$$[K] = \iint_A [B]^T [D] [B] dA, \quad (13)$$

where

$$[D] = \frac{E}{1 - \mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 - \mu}{2} \end{bmatrix}. \quad (14)$$

2.4. The Stiffness Matrix Using the Explicit Integral Formulae. The strain matrix of the triangular plane element TR3 with RDOF can be presented in the natural coordinates of the triangular element. The stress trial functions contain the variables x and y , which can be written as

$$x = x_1L_1 + x_2L_2 + x_3L_3,$$

$$y = y_1L_1 + y_2L_2 + y_3L_3. \quad (15)$$

Then, the matrix $[K]$ can be expressed in the natural coordinates L_1 , L_2 , and L_3 . To the integral formula of the natural coordinates in the triangular element, we have

$$\iint_A L_1^a L_2^b L_3^c dx dy = \frac{a!b!c!}{(a+b+c+2)!} 2A,$$

$$\int_l L_1^a L_2^b ds = \frac{a!b!}{(a+b+1)!} l_{12}, \quad (16)$$

where A is the area of the triangular element. Utilizing the explicit integral formulae of (16) in the triangular element, the explicit formula of the proposed elements in this paper can be obtained.

FIGURE 1: Cantilever beam subjected to force/moment.

FIGURE 2: Schemes of the mesh.

FIGURE 3: Cook's skew shell.

TABLE 1: Deflection V_{Cz} /m of the cantilever beam subjected to M_x .

Element	Mesh			
	2 × 2	2 × 4	2 × 8	2 × 16
CST	0.0128	0.0489	0.1654	0.4058
GT9	0.1045	0.3156	0.6371	0.8510
GT9M	0.1049	0.3191	0.6504	0.8692
TR3	0.9906	0.9903	0.9911	0.9961

3. Benchmark of the Proposed Element TR3

3.1. Cantilever Beam. As shown in Figure 1, a slender cantilever beam has the length b of 32 m, the height h of 2 m, and the thickness t of 1 m. It is made of the isotropy material, which has the material parameters $E = 768$ Pa, $\mu = 0.25$. Figure 2 shows four schemes of the mesh to the proposed cantilever beam.

The load is an in-plane moment $M_x = 1$ N·m on the end side C. Table 1 gives the results of the displacement V_{Cz} obtained by the proposed element TR3. Comparing with other elements, TR3 can give higher precise results.

3.2. Cook's Skew Beam. As showed in Figure 3, the Cook's skew beam [10] is studied. The thickness t of the shell is 1 m.

TABLE 2: Deflections (m) of the central point C in different mesh scheme.

Element	Mesh					
	2×2	4×4	8×8	16×16	32×32	64×64
CST	11.99	18.28	22.02	23.41	/	/
GT9	20.08	22.71	23.61	23.87	23.94	23.96
GT9M	20.36	22.80	23.63	23.88	23.94	23.96
TR3	19.66	22.46	23.47	23.81	23.92	23.95

It is also made of the isotropy, whose material parameters $E = 1$ Pa, $\mu = 0.3333$, and is subjected to the uniformly distributed load which has the sum of $P = 1$ N. Table 2 gives the deflection of the central point C in different mesh scheme.

4. Conclusion

This paper present a new type of plane element with RDOF: the assumptions of the rotational displacement and the displacement are proved to be useful. The accuracy and the efficiency of the proposed element model TR3 are shown in the benchmark problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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