Research Article

Some Properties of Complex Fuzzy Integral

Cuilian You and Weiqing Wang

College of Mathematics and Information Science, Hebei University, Baoding 071002, China

Correspondence should be addressed to Cuilian You; yycclian@163.com

Received 28 December 2014; Revised 16 April 2015; Accepted 17 April 2015

Academic Editor: Mohamed Abd El Aziz

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In order to simplify the calculations of complex fuzzy integral (complex Liu integral), this paper aims to discuss the properties of this kind of complex fuzzy integral. Firstly, some properties of one-dimensional complex fuzzy integral are given. Then the properties above are extended to the case of multidimensional complex fuzzy integral.

1. Introduction

Many problems in the world are hard to be described statically. To deal with dynamic fuzzy phenomena, Liu [1] proposed the concept of fuzzy process with credibility measure. A detailed survey on credibility theory may be found in Liu [2]. As for more concepts and results, interested reader may consult Liu [3]. Subsequently, Liu process, Liu integral, and Liu formula were presented by Liu [1], which are the bases of fuzzy calculus. Later some researches surrounding Liu process have been done. You et al. [4] studied multidimensional Liu process, Liu formula, and Liu integral. Some properties of Liu integral were discussed by You and Wang [5], then these properties were extended to multidimensional Liu integral by You and Huo [6]. Lipschitz continuity of Liu process was proved by Dai [7]. Besides, You et al. [8] derived existence and uniqueness theorems for two special fuzzy differential equations. Chen and Qin [9] extended the existence and uniqueness theorem to the general fuzzy differential equations.

As for applications in finance, to describe the variance of the price of a stock in fuzzy market, fuzzy differential equation is an important tool. Based on Liu process, Liu [1] initiated a stock model named Liu's stock model. Considering that the price of a stock is a solution of a fuzzy differential equation, Qin and Li [10] deduced option pricing formula for European option. Other kinds of stock model were presented by Gao [11] and Peng [12]. Applying the knowledge of fuzzy differential equations, Qin and Li [13] studied the problems of trading strategies. While in the solving process of fuzzy

differential equation, fuzzy integral (Liu integral) is needed. As for applications in control, it is known that classical control system is described by a differential equation, if there exist some fuzzy factors, fuzzy differential equations must be taken into consideration. Based on Liu process, Zhu [14] gave the optimal equation in the field of fuzzy control. Qin et al. [15] applied the fuzzy control to production planning problem.

As we know, there exist complex stochastic processes in many fields, such as signal analysis, thermodynamics, and fluid dynamics. Complex stochastic process is different from the two-dimensional stochastic process since it can reveal the relation of its real and imaginary parts. However, if the uncertainty factor is dominated by fuzzy factors in the above systems, how can we deal with them? In order to answer this question, Yang [16] defined a complex fuzzy variable whose real and imaginary parts are both fuzzy variables. Similarly the complex fuzzy process is introduced. In particular, the properties of complex fuzzy process were given by Qin and Wen [17].

Despite the importance of complex fuzzy process, there are so few literatures on this topic, considering these reasons, in this paper, based on credibility theory, the properties of complex fuzzy integral will be presented. For this purpose, the paper is organized as follows: Some basic definitions and properties of Liu integral and the definitions of complex fuzzy process are recalled in Section 2. Section 3 demonstrates the properties of one-dimensional complex fuzzy integral. The definition of m -dimensional complex fuzzy integral is given in Section 4, then some properties are obtained. Finally a brief conclusion is given in Section 5.

2. Preliminaries

In this section, some basic definitions and properties of credibility theory are recalled.

Let Θ be a nonempty set and $\mathcal{P}$ the power set of Θ . For any $A \in \mathcal{P}$, a credibility measure $\text{Cr}\{A\}$ describes the chance that fuzzy event occurs. In 2006, Liu [3] proved that a set function Cr is a credibility measure if and only if

- (1) $\text{Cr}\{\Theta\} = 1$;
- (2) $\text{Cr}\{A\} \leq \text{Cr}\{B\}$, whenever $A \subset B$;
- (3) $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}$;
- (4) $\text{Cr}\{\cup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any collection A_i in $\mathcal{P}$ with $\sup_i \text{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}, \text{Cr})$ is called a credibility space in Liu [3].

Definition 1 (Liu [1]). A fuzzy process C_t is said to be a Liu process if

- (1) $C_0 = 0$,
- (2) C_t has stationary and independent increments,
- (3) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2 \left(1 + \exp \left(\frac{\pi |x - et|}{\sqrt{6}t} \right) \right)^{-1}, \quad -\infty < x < \infty. \quad (1)$$

The parameters e and σ are called drift and diffusion coefficients, respectively. Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Definition 2 (Liu [1]). Let X_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1 \leq i \leq k} |t_{p+1} - t_p|$. Then the Liu integral of X_t with respect to C_t is

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{p=1}^k X_{t_p} (C_{t_{p+1}} - C_{t_p}) \quad (2)$$

provided that the limit exists almost surely and is a fuzzy variable.

Theorem 3 (You and Wang [5]). Let fuzzy process X_t be monotonous and bounded with respect to t . Then Liu integral $\int_a^b X_t dC_t$ exists.

Theorem 4 (You and Wang [5]). Let X_t be a continuous fuzzy process. Then integral $\int_a^b X_t dC_t$ exists.

Theorem 5 (You and Wang [5]). Let $a < k < b$. If fuzzy process X_t is Liu integrable on any closed interval $[a, k]$ and $[k, b]$, then X_t is integrable on closed interval $[a, b]$, and

$$\int_a^b X_t dC_t = \int_a^k X_t dC_t + \int_k^b X_t dC_t. \quad (3)$$

Theorem 6 (You and Huo [6]). If X_t, Y_t are monotonous and bounded fuzzy processes, then $\int_a^b X_t Y_t dC_t$ exists.

Definition 7 (Qin and Wen [17]). Let T be an index set and let $(\Theta, \mathcal{P}, \text{Cr})$ be a credibility space. A complex fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \text{Cr})$ to the set of complex numbers.

Definition 8 (Qin and Wen [17]). If C_{1t} and C_{2t} are independent Liu processes, then $C_t = C_{1t} + iC_{2t}$ is called a complex Liu process. Here i is an imaginary number.

Theorem 9 (Qin and Wen [17]). A process X_t is a complex fuzzy process if and only if there exist two fuzzy processes X_{1t} and X_{2t} such that $X_t = X_{1t} + iX_{2t}$.

Definition 10 (Qin and Wen [17]). Let $X_t = X_{1t} + iX_{2t}$ be a complex fuzzy process and let $C_t = C_{1t} + iC_{2t}$ be a standard complex Liu process. Then the complex fuzzy integral of X_t with respect to C_t is defined by

$$\begin{aligned} \int_a^b X_t dC_t &= \int_a^b (X_{1t} dC_{1t} - X_{2t} dC_{2t}) \\ &\quad + i \int_a^b (X_{1t} dC_{2t} + X_{2t} dC_{1t}). \end{aligned} \quad (4)$$

We will discuss some properties of complex fuzzy integral in next two sections, which will be useful for the calculations of fuzzy integral.

3. One-Dimensional Complex Fuzzy Integral

Theorem 11. If $X_t = X_{1t} + iX_{2t}$ is a complex fuzzy process, X_{1t} and X_{2t} are two continuous fuzzy processes, and $C_t = C_{1t} + iC_{2t}$ is a complex Liu process, then Liu integral $\int_a^b X_t dC_t$ exists.

Proof. It follows from the definition of complex Liu integral that

$$\begin{aligned} \int_a^b X_t dC_t &= \int_a^b (X_{1t} + iX_{2t}) d(C_{1t} + iC_{2t}) \\ &= \int_a^b X_{1t} (dC_{1t} + idC_{2t}) \\ &\quad + \int_a^b iX_{2t} (dC_{1t} + idC_{2t}) \\ &= \int_a^b X_{1t} dC_{1t} + i \int_a^b X_{1t} dC_{2t} + i \int_a^b X_{2t} dC_{1t} \\ &\quad - \int_a^b X_{2t} dC_{2t}. \end{aligned} \quad (5)$$

According to Theorem 4, $\int_a^b X_{1t} dC_{1t}$, $\int_a^b X_{1t} dC_{2t}$, $\int_a^b iX_{2t} dC_{1t}$, and $\int_a^b X_{2t} dC_{2t}$ exist and are finite, then Liu integral $\int_a^b X_t dC_t$ exists. $\square$

Theorem 12. Let X_t be a complex fuzzy process and $X_t = X_{1t} + iX_{2t}$, where X_{1t} and X_{2t} are monotonous and bounded with

respect to t . Assume that $C_t = C_{1t} + iC_{2t}$ and C_{1t}, C_{2t} are both standard Liu processes. Then integral $\int_a^b X_t dC_t$ exists.

Proof. It follows from the definition of complex Liu integral that

$$\begin{aligned} \int_a^b X_t dC_t &= \int_a^b X_{1t} dC_{1t} - \int_a^b X_{2t} dC_{2t} + i \int_a^b X_{2t} dC_{1t} \\ &\quad + i \int_a^b X_{1t} dC_{2t}, \end{aligned} \quad (6)$$

and by using Theorem 3, we know $\int_a^b X_{1t} dC_{1t}$, $\int_a^b X_{1t} dC_{2t}$, $\int_a^b iX_{2t} dC_{1t}$, and $\int_a^b X_{2t} dC_{2t}$ exist and are finite. Thus Liu integral $\int_a^b X_t dC_t$ exists. $\square$

Theorem 13. Suppose that $X_t = X_{1t} + iX_{2t}$ is Liu integrable complex fuzzy process and $C_t = C_{1t} + iC_{2t}$ is a complex Liu process. Then

$$\int_a^b kX_t dC_t = k \int_a^b X_t dC_t \quad (7)$$

for any complex number k .

Proof. It follows from Definition 8 that

$$\begin{aligned} \int_a^b kX_t dC_t &= \int_a^b (kX_{1t} + ikX_{2t}) d(C_{1t} + iC_{2t}) \\ &= \int_a^b kX_{1t} dC_{1t} + \int_a^b ikX_{2t} dC_{1t} + \int_a^b ikX_{1t} dC_{2t} \\ &\quad + \int_a^b ikX_{2t} dC_{2t} = k \int_a^b X_{1t} dC_{1t} + ik \int_a^b X_{2t} dC_{1t} \\ &\quad + ik \int_a^b X_{1t} dC_{2t} - k \int_a^b X_{2t} dC_{2t} = k \left(\int_a^b X_{1t} dC_{1t} \right. \\ &\quad \left. - \int_a^b X_{2t} dC_{2t} + i \int_a^b iX_{2t} dC_{1t} + i \int_a^b X_{1t} dC_{2t} \right) \\ &= k \int_a^b X_t dC_t. \end{aligned} \quad (8)$$

The theorem is proved. $\square$

Theorem 14. Suppose that $X_t = X_{1t} + iX_{2t}$, $Y_t = Y_{1t} + iY_{2t}$ are Liu integrable complex fuzzy processes and $C_t = C_{1t} + iC_{2t}$ is a complex Liu process. Then

$$\int_a^b (X_t + Y_t) dC_t = \int_a^b X_t dC_t + \int_a^b Y_t dC_t. \quad (9)$$

Proof. It follows from the definition of complex Liu integral that

$$\begin{aligned} \int_a^b (X_t + Y_t) dC_t &= \int_a^b ((X_{1t} + Y_{1t}) + i(X_{2t} + Y_{2t})) dC_t \end{aligned}$$

$$\begin{aligned} &= \int_a^b (X_{1t} + Y_{1t}) dC_t + i \int_a^b (X_{2t} + Y_{2t}) dC_t \\ &= \int_a^b X_{1t} dC_t + \int_a^b Y_{1t} dC_t + i \int_a^b X_{2t} dC_t \\ &\quad + i \int_a^b Y_{2t} dC_t \\ &= \int_a^b X_{1t} dC_t + i \int_a^b X_{2t} dC_t + \int_a^b Y_{1t} dC_t \\ &\quad + i \int_a^b Y_{2t} dC_t = \int_a^b X_t dC_t + \int_a^b Y_t dC_t. \end{aligned} \quad (10)$$

The proof is completed. $\square$

Remark 15. It follows from Theorems 13 and 14 that

$$\int_a^b (k_1 X_t + k_2 Y_t) dC_t = k_1 \int_a^b X_t dC_t + k_2 \int_a^b Y_t dC_t, \quad (11)$$

for any complex numbers k_1 and k_2 .

Theorem 16. If $X_t = X_{1t} + iX_{2t}$, $Y_t = Y_{1t} + iY_{2t}$, where X_{1t} , X_{2t} , Y_{1t} , and Y_{2t} are monotonous and bounded fuzzy processes, and $C_t = C_{1t} + iC_{2t}$ is a complex Liu process, then $\int_a^b X_t Y_t dC_t$ exists.

Proof. It follows from the definition of complex Liu integral that

$$\begin{aligned} \int_a^b (X_t Y_t) dC_t &= \int_a^b ((X_{1t} Y_{1t} - X_{2t} Y_{2t}) + i(X_{1t} Y_{2t} + X_{2t} Y_{1t})) dC_t \\ &= \int_a^b (X_{1t} Y_{1t} - X_{2t} Y_{2t}) dC_{1t} \\ &\quad - \int_a^b (X_{1t} Y_{2t} + X_{2t} Y_{1t}) dC_{2t} \\ &\quad + i \int_a^b (X_{1t} Y_{1t} - X_{2t} Y_{2t}) dC_{2t} \\ &\quad + i \int_a^b (X_{1t} Y_{2t} + X_{2t} Y_{1t}) dC_{1t} \\ &= \int_a^b X_{1t} Y_{1t} dC_{1t} - \int_a^b X_{2t} Y_{2t} dC_{1t} - \int_a^b X_{1t} Y_{2t} dC_{2t} \\ &\quad - \int_a^b X_{2t} Y_{1t} dC_{2t} + i \int_a^b X_{1t} Y_{1t} dC_{2t} \\ &\quad - i \int_a^b X_{2t} Y_{2t} dC_{2t} + i \int_a^b X_{1t} Y_{2t} dC_{1t} \\ &\quad + i \int_a^b X_{2t} Y_{1t} dC_{1t}, \end{aligned} \quad (12)$$

and by using Theorem 6, we have $\int_a^b X_{1t}Y_{1t}dC_{1t}$, $\int_a^b X_{2t}Y_{2t}dC_{1t}$, $\int_a^b X_{1t}Y_{2t}dC_{2t}$, $\int_a^b X_{2t}Y_{1t}dC_{2t}$, $\int_a^b X_{1t}Y_{1t}dC_{2t}$, $\int_a^b X_{2t}Y_{2t}dC_{2t}$, $\int_a^b X_{1t}Y_{2t}dC_{1t}$, and $\int_a^b X_{2t}Y_{1t}dC_{1t}$ exist. Hence $\int_a^b X_tY_t dC_t$ exists. $\square$

Theorem 17. Suppose that $a < k < b$. Let $X_t = X_{1t} + iX_{2t}$ be a Liu integrable complex fuzzy process and $C_t = C_{1t} + iC_{2t}$ a complex Liu process. Then

$$\int_a^b X_t dC_t = \int_a^k X_t dC_t + \int_k^b X_t dC_t. \quad (13)$$

Proof. It follows from the definition of complex Liu integral and Theorem 5 that

$$\begin{aligned} \int_a^b X_t dC_t &= \int_a^b X_{1t} dC_{1t} - \int_a^b X_{2t} dC_{2t} + i \int_a^b X_{2t} dC_{1t} \\ &\quad + i \int_a^b X_{1t} dC_{2t} \\ &= \int_a^k X_{1t} dC_{1t} + \int_k^b X_{1t} dC_{1t} \\ &\quad - \left(\int_a^k X_{2t} dC_{2t} + \int_k^b X_{2t} dC_{2t} \right) \\ &\quad + i \left(\int_a^k X_{2t} dC_{1t} + \int_k^b X_{2t} dC_{1t} \right) \\ &\quad + i \left(\int_a^k X_{1t} dC_{2t} + \int_k^b X_{1t} dC_{2t} \right) \\ &= \int_a^k X_t dC_t + \int_k^b X_t dC_t. \end{aligned} \quad (14)$$

The theorem holds. $\square$

4. Multidimensional Complex Fuzzy Integral

Definition 18. If C_{pt} , $p = 1, 2, \dots, m$, are complex Liu processes, then $C_t = (C_{1t}, C_{2t}, \dots, C_{mt})^T$ is called an m -dimensional complex Liu process.

In particular, if C_{pt} , $p = 1, 2, \dots, m$, are standard complex Liu processes, $C_t = (C_{1t}, C_{2t}, \dots, C_{mt})^T$ is called an m -dimensional standard complex Liu process.

Definition 19. Let $C_t = (C_{1t}, C_{2t}, \dots, C_{mt})^T$ be an m -dimensional standard complex Liu process. Denote by $\mathcal{V}^{n \times m}$, the set of $n \times m$ matrices $v_t = [v_{pqt}]$, where each entry v_{pqt} is a Liu integrable complex fuzzy process, $v_{pqt} = v_{pq1t} + i v_{pq2t}$, $C_{pt} = C_{p1t} + i C_{p2t}$. Suppose $a < b$. If $v_t \in \mathcal{V}^{m \times n}$, then the

m -dimensional complex Liu integral is defined, using matrix notation

$$\int_a^b v_t dC_t = \int_a^b \begin{pmatrix} v_{11t} & v_{12t} & \cdots & v_{1mt} \\ v_{21t} & v_{22t} & \cdots & v_{2mt} \\ \vdots & \vdots & & \vdots \\ v_{n1t} & v_{n2t} & \cdots & v_{nmt} \end{pmatrix} \begin{pmatrix} dC_{1t} \\ dC_{2t} \\ \vdots \\ dC_{mt} \end{pmatrix} \quad (15)$$

as the $n \times 1$ matrix whose p th component is the following sum of complex Liu integral

$$\begin{aligned} \sum_{q=1}^m \int_a^b v_{pqt} dC_{qt} &= \sum_{q=1}^m \left(\int_a^b v_{pq1t} dC_{q1t} + i \int_a^b v_{pq1t} dC_{q2t} \right. \\ &\quad \left. + i \int_a^b v_{pq2t} dC_{q1t} - \int_a^b v_{pq2t} dC_{q2t} \right). \end{aligned} \quad (16)$$

In this case, v_t is called a complex Liu integrable matrix.

Example 20. Let $C_{11t}, C_{12t}, C_{21t}, C_{22t}, C_{23t}$ be Liu processes and $C_t = (C_{1t}, C_{2t})$, where $C_{it} = C_{i1t} + iC_{i2t}$, $i = 1, 2$. Then

$$v_t = \begin{pmatrix} t + iC_{11t} & C_{12t} \\ C_{21t} + it & C_{22t} + iC_{23t} \end{pmatrix} \quad (17)$$

is a complex Liu integrable matrix, and

$$\int_a^b v_t dC_t = \begin{pmatrix} m \\ n \end{pmatrix}, \quad (18)$$

where

$$\begin{aligned} m &= \int_a^b t dC_{11t} - \int_a^b C_{11t} dC_{12t} + \int_a^b C_{12t} dC_{11t} \\ &\quad + i \left(\int_a^b t dC_{12t} + \int_a^b C_{11t} dC_{11t} + \int_a^b C_{12t} dC_{12t} \right), \\ n &= \int_a^b C_{21t} dC_{11t} - \int_a^b t dC_{12t} + \int_a^b C_{22t} dC_{21t} \\ &\quad - \int_a^b C_{23t} dC_{22t} + i \left(\int_a^b C_{21t} dC_{12t} + \int_a^b t dC_{11t} \right. \\ &\quad \left. + \int_a^b C_{22t} dC_{22t} + \int_a^b C_{23t} dC_{21t} \right). \end{aligned} \quad (19)$$

Theorem 21. Let $v_t = [v_{pqt}]$ be $n \times m$ complex Liu integrable matrix. Then lv_t is complex integrable and

$$\int_a^b lv_t dC_t = l \int_a^b v_t dC_t, \quad (20)$$

for any complex number l .

Proof. Since v_{pqt} are complex Liu integrable fuzzy processes, for $p = 1, 2, 3, \dots$, $q = 1, 2, 3, \dots$, it follows from Theorem 13 that

$$\int_a^b lv_{pqt} dC_{qt} = l \int_a^b v_{pqt} dC_{qt}. \quad (21)$$

Hence lv_t is complex Liu integrable matrix, and

$$\begin{aligned} & \int_a^b lv_t dC_t \\ &= \int_a^b \begin{pmatrix} lv_{11t} & lv_{12t} & \cdots & lv_{1mt} \\ lv_{21t} & lv_{22t} & \cdots & lv_{2mt} \\ \vdots & \vdots & & \vdots \\ lv_{n1t} & lv_{n2t} & \cdots & lv_{nmt} \end{pmatrix} \begin{pmatrix} dC_{1t} \\ dC_{2t} \\ \vdots \\ dC_{mt} \end{pmatrix} \\ &= \begin{pmatrix} \int_a^b lv_{11t} dC_{1t} + \int_a^b lv_{12t} dC_{2t} + \cdots + \int_a^b lv_{1mt} dC_{mt} \\ \int_a^b lv_{21t} dC_{1t} + \int_a^b lv_{22t} dC_{2t} + \cdots + \int_a^b lv_{2mt} dC_{mt} \\ \vdots \\ \int_a^b lv_{n1t} dC_{1t} + \int_a^b lv_{n2t} dC_{2t} + \cdots + \int_a^b lv_{nmt} dC_{mt} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} l \left(\int_a^b v_{11t} dC_{1t} + \int_a^b v_{12t} dC_{2t} + \cdots + \int_a^b v_{1mt} dC_{mt} \right) \\ l \left(\int_a^b v_{21t} dC_{1t} + \int_a^b v_{22t} dC_{2t} + \cdots + \int_a^b v_{2mt} dC_{mt} \right) \\ \vdots \\ l \left(\int_a^b v_{n1t} dC_{1t} + \int_a^b v_{n2t} dC_{2t} + \cdots + \int_a^b v_{nmt} dC_{mt} \right) \end{pmatrix} \\ &= l \int_a^b v_t dC_t. \end{aligned} \quad (22)$$

The theorem holds. $\square$

Theorem 22. Let $u_t, v_t \in \mathcal{V}^{n \times m}$ be complex Liu integrable matrices. Then $u_t + v_t$ is complex Liu integrable matrix, and

$$\int_a^b (u_t + v_t) dC_t = \int_a^b v_t dC_t + \int_a^b u_t dC_t. \quad (23)$$

Proof. It follows from the definition of multidimensional complex Liu integral that

$$\begin{aligned} & \int_a^b (v_t + u_t) dC_t = \int_a^b \begin{pmatrix} v_{11t} + u_{11t} & v_{12t} + u_{12t} & \cdots & v_{1mt} + u_{1mt} \\ v_{21t} + u_{21t} & v_{22t} + u_{22t} & \cdots & v_{2mt} + u_{2mt} \\ \vdots & \vdots & & \vdots \\ v_{n1t} + u_{n1t} & v_{n2t} + u_{n2t} & \cdots & v_{nmt} + u_{nmt} \end{pmatrix} \begin{pmatrix} dC_{1t} \\ dC_{2t} \\ \vdots \\ dC_{mt} \end{pmatrix} \\ &= \begin{pmatrix} \int_a^b (v_{11t} + u_{11t}) dC_{1t} + \int_a^b (v_{12t} + u_{12t}) dC_{2t} + \cdots + \int_a^b (v_{1mt} + u_{1mt}) dC_{mt} \\ \int_a^b (v_{21t} + u_{21t}) dC_{1t} + \int_a^b (v_{22t} + u_{22t}) dC_{2t} + \cdots + \int_a^b (v_{2mt} + u_{2mt}) dC_{mt} \\ \vdots \\ \int_a^b (v_{n1t} + u_{n1t}) dC_{1t} + \int_a^b (v_{n2t} + u_{n2t}) dC_{2t} + \cdots + \int_a^b (v_{nmt} + u_{nmt}) dC_{mt} \end{pmatrix} \\ &= \begin{pmatrix} \left(\int_a^b v_{11t} dC_{1t} + \cdots + \int_a^b v_{1mt} dC_{mt} \right) + \left(\int_a^b u_{11t} dC_{1t} + \cdots + \int_a^b u_{1mt} dC_{mt} \right) \\ \left(\int_a^b v_{21t} dC_{1t} + \cdots + \int_a^b v_{2mt} dC_{mt} \right) + \left(\int_a^b u_{21t} dC_{1t} + \cdots + \int_a^b u_{2mt} dC_{mt} \right) \\ \vdots \\ \left(\int_a^b v_{n1t} dC_{1t} + \cdots + \int_a^b v_{nmt} dC_{mt} \right) + \left(\int_a^b u_{n1t} dC_{1t} + \cdots + \int_a^b u_{nmt} dC_{mt} \right) \end{pmatrix} \\ &= \begin{pmatrix} \int_a^b v_{11t} dC_{1t} + \int_a^b v_{12t} dC_{2t} + \cdots + \int_a^b v_{1mt} dC_{mt} \\ \int_a^b v_{21t} dC_{1t} + \int_a^b v_{22t} dC_{2t} + \cdots + \int_a^b v_{2mt} dC_{mt} \\ \vdots \\ \int_a^b v_{n1t} dC_{1t} + \int_a^b v_{n2t} dC_{2t} + \cdots + \int_a^b v_{nmt} dC_{mt} \end{pmatrix} \end{aligned}$$

$$+ \begin{pmatrix} \int_a^b u_{11t} dC_{1t} + \int_a^b u_{12t} dC_{2t} + \cdots + \int_a^b u_{1mt} dC_{mt} \\ \int_a^b u_{21t} dC_{1t} + \int_a^b u_{22t} dC_{2t} + \cdots + \int_a^b u_{2mt} dC_{mt} \\ \vdots \\ \int_a^b u_{n1t} dC_{1t} + \int_a^b u_{n2t} dC_{2t} + \cdots + \int_a^b u_{nmt} dC_{mt} \end{pmatrix} = \int_a^b v_t dC_t + \int_a^b u_t dC_t. \quad (24)$$

□

Remark 23. Let $u_t, v_t \in \mathcal{V}^{n \times m}$ be complex Liu integrable matrices. Then $u_t - v_t$ is complex Liu integrable matrix, and

$$\int_a^b (v_t - u_t) dC_t = \int_a^b v_t dC_t - \int_a^b u_t dC_t. \quad (25)$$

Remark 24. Let $u_t, v_t \in \mathcal{V}^{n \times m}$ be complex Liu integrable matrices. Then $k_1 u_t + k_2 v_t$ is complex Liu integrable matrix, and

$$\int_a^b (k_1 v_t + k_2 u_t) dC_t = k_1 \int_a^b v_t dC_t + k_2 \int_a^b u_t dC_t, \quad (26)$$

for any complex numbers k_1 and k_2 .

Theorem 25. Suppose that $a < k < b$. Let $v_t \in \mathcal{V}^{n \times m}$ denote the set of $n \times m$ complex matrices $v_t = [v_{pqt}]$, where each entry v_{pqt} is Liu integrable complex fuzzy process. If v_t is Liu integrable on $[a, k]$ and $[k, b]$, then v_t is Liu integrable on $[a, b]$, and

$$\int_a^b v_t dC_t = \int_a^k v_t dC_t + \int_k^b v_t dC_t. \quad (27)$$

Proof. Since v_t is Liu integrable on intervals $[a, k]$ and $[k, b]$, we have

$$\begin{aligned} \int_a^k v_t dC_t &= \int_a^k \begin{pmatrix} v_{11t} & v_{12t} & \cdots & v_{1mt} \\ v_{21t} & v_{22t} & \cdots & v_{2mt} \\ \vdots & \vdots & & \vdots \\ v_{n1t} & v_{n2t} & \cdots & v_{nmt} \end{pmatrix} \begin{pmatrix} dC_{1t} \\ dC_{2t} \\ \vdots \\ dC_{mt} \end{pmatrix}, \\ \int_k^b v_t dC_t &= \int_k^b \begin{pmatrix} v_{11t} & v_{12t} & \cdots & v_{1mt} \\ v_{21t} & v_{22t} & \cdots & v_{2mt} \\ \vdots & \vdots & & \vdots \\ v_{n1t} & v_{n2t} & \cdots & v_{nmt} \end{pmatrix} \begin{pmatrix} dC_{1t} \\ dC_{2t} \\ \vdots \\ dC_{mt} \end{pmatrix}, \\ \int_a^k v_t dC_t + \int_k^b v_t dC_t &= \int_a^k \begin{pmatrix} v_{11t} dC_{1t} + v_{12t} dC_{2t} + \cdots + v_{1mt} dC_{mt} \\ v_{21t} dC_{1t} + v_{22t} dC_{2t} + \cdots + v_{2mt} dC_{mt} \\ \vdots \\ v_{n1t} dC_{1t} + v_{n2t} dC_{2t} + \cdots + v_{nmt} dC_{mt} \end{pmatrix} \\ &\quad + \int_k^b \begin{pmatrix} v_{11t} dC_{1t} + v_{12t} dC_{2t} + \cdots + v_{1mt} dC_{mt} \\ v_{21t} dC_{1t} + v_{22t} dC_{2t} + \cdots + v_{2mt} dC_{mt} \\ \vdots \\ v_{n1t} dC_{1t} + v_{n2t} dC_{2t} + \cdots + v_{nmt} dC_{mt} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \left(\int_a^k v_{11t} dC_{1t} + \int_k^b v_{11t} dC_{1t} \right) + \cdots + \left(\int_a^k v_{1mt} dC_{mt} + \int_k^b v_{1mt} dC_{mt} \right) \\ \left(\int_a^k v_{21t} dC_{1t} + \int_k^b v_{21t} dC_{1t} \right) + \cdots + \left(\int_a^k v_{2mt} dC_{mt} + \int_k^b v_{2mt} dC_{mt} \right) \\ \vdots \\ \left(\int_a^k v_{n1t} dC_{1t} + \int_k^b v_{n1t} dC_{1t} \right) + \cdots + \left(\int_a^k v_{nmt} dC_{mt} + \int_k^b v_{nmt} dC_{mt} \right) \end{pmatrix}. \quad (28)$$

It follows from Theorem 17 that

$$\int_a^b v_t dC_t = \int_a^k v_t dC_t + \int_k^b v_t dC_t. \quad (29)$$

The proof is completed. $\square$

5. Conclusions

The main result in this paper can be summarized as follows: (a) the properties of *one*-dimensional complex fuzzy integral were deduced, containing the existence of complex fuzzy integral and linear properties of this integral; (b) the definition of *m*-dimensional complex integral was presented; (c) similar properties of multidimensional complex fuzzy integral were obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by Natural Science Foundation of China Grant no. 11201110, no. 61374184, and no. 11101115 and Outstanding Youth Science Fund of the Education Department of Hebei Province no. Y2012021.

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