Research Article

Remarks on the Regularity Criterion of the Navier-Stokes Equations with Nonlinear Damping

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This paper is concerned with the regularity criterion of weak solutions to the three-dimensional Navier-Stokes equations with nonlinear damping in critical weak L^q spaces. It is proved that if the weak solution satisfies $\int_0^T ((\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)})/(1 + \ln(e + \|\nabla u\|_{L^2}^2)) ds < \infty$, $q > 3/2$, then the weak solution of Navier-Stokes equations with nonlinear damping is regular on $(0, T]$.

1. Introduction

In this study we consider the Cauchy problem of the three-dimensional Navier-Stokes equations with the nonlinear damping

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla \pi + |u|^{r-2} u &= \Delta u, \\ \nabla \cdot u &= 0, \end{aligned} \quad (1)$$

together with the initial data

$$u(x, 0) = u_0, \quad (2)$$

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ and $\pi(x, t)$ denote the unknown velocity fields and the unknown pressure of the fluid. $|u|^{r-2}u$, $r > 2$ is the nonlinear damping. Moreover,

$$\nabla = (\partial_1, \partial_2, \partial_3) \quad (3)$$

represents the gradient operator,

$$\Delta = \nabla \cdot \nabla = \sum_{i=1}^3 \frac{\partial^2 u(x, t)}{\partial x_i^2} \quad (4)$$

denotes the Laplacian operator, and

$$(u \cdot \nabla) u = \sum_{i=1}^3 u_i \frac{\partial u}{\partial x_i}, \quad (5)$$

$$\nabla \cdot u = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}.$$

The mathematical model (1) is from the resistance to the motion of the flows. It describes various physical situations such as drag or friction effects, porous media flow, and some dissipative mechanisms [1, 2]. When the nonlinear damping term $|u|^{r-2}u$ in (1) disappears, the system reduces the classic Navier-Stokes equations [3, 4]

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla \pi &= \Delta u, \\ \nabla \cdot u &= 0. \end{aligned} \quad (6)$$

In the mathematical viewpoint, therefore, Navier-Stokes equations with the nonlinear damping are a modification of the classic Navier-Stokes equations. There is a large literature on the well-posedness and large time behavior for solutions of Navier-Stokes equations with the nonlinear damping (see [1, 5, 6]).

However it is not known whether the weak solution of Navier-Stokes equations with the nonlinear damping (1) is

regular or smooth for a given smooth and compactly supported initial velocity u_0 . Fortunately, the regularity of weak solutions for Navier-Stokes equations with the nonlinear damping (1) can be derived when certain growth conditions are satisfied. This is known as a regularity criterion problem. Recently, Zhou [7] studied the regularity criterion for weak solutions for Navier-Stokes equations with the nonlinear damping (1) in critical Lebesgue spaces. That is, if a weak solution u of Navier-Stokes equations with the nonlinear damping (1) satisfies

$$\nabla u \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{for } \frac{2}{p} + \frac{3}{q} = 2, \quad 3 < p \leq \infty, \quad (7)$$

then the weak solution is smooth on $(0, T]$.

Since Navier-Stokes equations with the nonlinear damping (1) are a modification of the classic Navier-Stokes equations, it is necessary to mention some regularity criteria of weak solutions for Navier-Stokes equations and related fluid models [8, 9]. As for this direction, the first result of Navier-Stokes equations is studied by He [10] and improved by Dong and Zhang [11], Pokorý [12, 13], and Zhou [14]. One may also refer to some interesting regularity criteria on related fluid models (see [15] and the references therein).

The main purpose of this paper is to investigate the regularity criteria of weak solutions with the aid of two components of velocity fields in critical weak L^q space. To do so, we recall the definition of the weak solution of Navier-Stokes equations with the nonlinear damping (1).

Definition 1. A measurable function $u(x, t)$ is called a weak solution of Navier-Stokes equations with the nonlinear damping (1) on $(0, T)$ if u satisfies the following properties:

- (i) $u \in L^\infty(0, T; L^2(\mathbb{R}^3))$ and $\nabla u \in L^2(0, T; L^2(\mathbb{R}^3))$;
- (ii) for any $\phi \in C_0^\infty(\mathbb{R}^3 \times [0, T))$ with $\nabla \cdot \phi = 0$

$$\int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \phi - \nabla u \cdot \nabla \phi + \nabla \phi : u \otimes u + |u|^{r-2} u \phi) dx dt \quad (8)$$

$$= - \int_{\mathbb{R}^3} u_0 \phi(0) dx;$$

- (iii) $\nabla \cdot u = 0$ in the distribution space $\mathcal{D}'((0, T) \times \mathbb{R}^3)$;
- (iv) u satisfies the energy inequality for $0 \leq t \leq T$

$$\begin{aligned} \|u(t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, s)|^2 dx ds \\ + 2 \int_0^t \int_{\mathbb{R}^3} |u(x, s)|^p dx ds \leq \|u_0\|_{L^2}^2. \end{aligned} \quad (9)$$

The main result on the regularity criteria of the weak solutions of Navier-Stokes equations with the nonlinear damping (1) is now read.

Theorem 2. Suppose $u_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ and u is a weak solution of Navier-Stokes equations with the nonlinear

damping (1) in $(0, T)$. If any two components of velocity fields satisfy

$$\int_0^T \frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} ds < \infty, \quad q > \frac{3}{2}, \quad (10)$$

then u is smooth on $(0, T]$.

This result improves the earlier regularity criterion involving (7). Furthermore, Theorem 2 also implies the following regularity criterion for weak solutions of Navier-Stokes equations with the nonlinear damping (1).

Theorem 3. Suppose $u_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ and u is a weak solution of Navier-Stokes equations with the nonlinear damping (1) in $(0, T)$. If any two components of velocity fields satisfy

$$\int_0^T \|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)} ds < \infty, \quad q > \frac{3}{2}, \quad (11)$$

then u is smooth on $(0, T]$.

Remark 4. The main idea in the proof of Theorem 2 is borrowing from the argument of previous results on classic Navier-Stokes equations [16] and together with energy methods.

2. Preliminaries

To start with, let us recall the definitions of some functional spaces. $L^q(\mathbb{R}^3)$ with $1 \leq q \leq \infty$ is a Lebesgue space under the norm

$$\|g\|_{L^q} = \begin{cases} \left(\int_{\mathbb{R}^3} |g(x)|^q dx \right)^{1/q}, & 1 \leq q < \infty, \\ \text{ess sup}_{x \in \mathbb{R}^3} |g(x)|, & q = \infty, \end{cases} \quad (12)$$

and $\dot{H}^m(\mathbb{R}^3)$ the Hilbert space

$$\{g \in L^2(\mathbb{R}^3); \|\nabla^m g\|_{L^2} < \infty\}. \quad (13)$$

To define the Lorenz space $L^{p,q}(\mathbb{R}^3)$ with $1 \leq p, q \leq \infty$, $g \in L^{p,q}(\mathbb{R}^3)$ if and only if

$$\|g\|_{L^{p,q}} = \left(\int_0^\infty t^q (m(g, t))^{q/p} \frac{dt}{t} \right)^{1/q} < \infty \quad \text{for } 1 \leq q < \infty, \quad (14)$$

$$\|g\|_{L^{p,\infty}} = \sup_{t \geq 0} \left(t (m(g, t))^{1/p} \right) < \infty \quad \text{for } q = \infty,$$

where

$$m(f, t) := m\{x \in \mathbb{R}^3 : |g(x)| > t\} \quad (15)$$

is Lebesgue measure of the set $\{x \in \mathbb{R}^3 : |g(x)| > t\}$.

Actually Lorentz space $L^{p,q}(\mathbb{R}^3)$ may be alternatively defined by real interpolation (see Triebel [17])

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3), L^{p_2}(\mathbb{R}^3))_{\theta,q} \quad (16)$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad 1 \leq p_1 < p < p_2 \leq \infty, \quad (17)$$

$$0 < \theta < 1.$$

In particular, $\|g\|_{L^{q,\infty}}$ is equivalent to the norm

$$\sup_{0 < |S| < \infty} |S|^{1/q-1} \int_S |g(x)| dx. \quad (18)$$

Furthermore, the definition implies the continuous relationship

$$L^q(\mathbb{R}^3) \hookrightarrow L^{q,\infty}(\mathbb{R}^3), \quad 1 < q < \infty. \quad (19)$$

In fact it is easy to check and thus it is readily seen that

$$|x|^{-3/q} \in L^q(\mathbb{R}^3), \quad (20)$$

$$\text{but } |x|^{-3/q} \in L^{q,\infty}(\mathbb{R}^3).$$

We also recall the Hölder inequality in Lorentz space which plays an important role in the next section.

Lemma 5 (O'Neil [18]). *Let $f \in L^{p_1,q_1}(\mathbb{R}^3)$ and $g \in L^{p_2,q_2}(\mathbb{R}^3)$ with*

$$1 \leq p_1, \quad p_2 \leq \infty, \quad (21)$$

$$1 \leq q_1, \quad q_2 \leq \infty.$$

Then $fg \in L^{p,q}(\mathbb{R}^3)$ satisfies the Hölder inequality of Lorentz spaces

$$\|fg\|_{L^{p,q}} \leq C \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \quad (22)$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad (23)$$

$$\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}.$$

3. A Priori Estimates

In this section we will prove a priori estimates for smooth solutions of (1) described in the following.

Theorem 6. *Let $T > 0$, let $u_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$, and let u be a local smooth solution of the Navier-Stokes equations with the nonlinear damping (1). If u also satisfies (11), namely,*

$$\int_0^T \frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} ds < \infty, \quad q > \frac{3}{2}, \quad (24)$$

then the a priori estimate

$$\sup_{0 < t < T} \|u(t)\|_{L^4} \leq C \quad (25)$$

holds true.

Proof of Theorem 6. Multiplying both sides of the Navier-Stokes equations with the nonlinear damping (1) with Δu and integrating in $\mathbb{R}^3$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\ & + (r-1) \int_{\mathbb{R}^3} |u|^{r-2} |\nabla u|^2 dx \\ & = - \int_{\mathbb{R}^3} u \cdot \nabla u \Delta u dx, \end{aligned} \quad (26)$$

where we have used

$$\int_{\mathbb{R}^3} \nabla p \Delta u dx = - \int_{\mathbb{R}^3} p \Delta (\operatorname{div} u) dx = 0. \quad (27)$$

For the right hand side of (26) we have

$$\begin{aligned} - \int_{\mathbb{R}^3} u \cdot \nabla u \Delta u dx &= - \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx \\ &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k (u_i \partial_i u_j) \partial_k u_j dx \\ &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx \\ &\quad + \frac{1}{2} \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i (\partial_k u_j \partial_k u_j) dx \\ &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx \\ &= \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx \\ &\quad + \sum_{j=1}^2 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx \\ &\quad + \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 dx \\ &= \sum_{m=1}^3 I_m, \end{aligned} \quad (28)$$

where we have used the fact that the divergence-free condition

$$\sum_{k=1}^3 \partial_k u_k = 0. \quad (29)$$

For I_1 , we have

$$\begin{aligned} |I_1| &\leq \left| \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx \right| \\ &\leq \int_{\mathbb{R}^3} (|\nabla u_1| + |\nabla u_2|) |\nabla u|^2 dx. \end{aligned} \quad (30)$$

For I_2 , similarly we obtain

$$\begin{aligned} |I_2| &\leq \left| \sum_{j=1}^2 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx \right| \\ &\leq \int_{\mathbb{R}^3} (|\nabla u_1| + |\nabla u_2|) |\nabla u|^2 dx. \end{aligned} \quad (31)$$

Finally for I_3 , applying the fact

$$\begin{aligned} \partial_3 u_3 &= -\partial_2 u_2 - \partial_1 u_1, \\ |I_3| &\leq \left| \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 dx \right| \\ &\leq \left| \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 (-\partial_2 u_2 - \partial_1 u_1) \partial_k u_3 dx \right| \\ &\leq \int_{\mathbb{R}^3} (|\nabla u_1| + |\nabla u_2|) |\nabla u|^2 dx. \end{aligned} \quad (32)$$

Plugging the estimates I_i into the right hand side of (26), it follows that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\ &\quad + (r-1) \int_{\mathbb{R}^3} |u|^{r-2} |\nabla u|^2 dx \\ &\leq C \int_{\mathbb{R}^3} (|\nabla u_1| + |\nabla u_2|) |\nabla u| |\nabla u| dx. \end{aligned} \quad (33)$$

Applying Hölder inequality and Young inequality, we have for the right hand side of (33)

$$\begin{aligned} &\int_{\mathbb{R}^3} (|\nabla u_1| + |\nabla u_2|) |\nabla u| |\nabla u| dx \\ &\leq C (\|\nabla u_1\|_{L^{q,\infty}} + \|\nabla u_2\|_{L^{q,\infty}}) \|\nabla u\|_{L^{2q/(q-1),2}}^2. \end{aligned} \quad (34)$$

Applying the Gagliardo-Nirenberg inequality in Lorentz spaces, that is,

$$\|\nabla u\|_{L^{2q/(q-1),2}}^{(2q-3)/2q} \leq C \|\nabla u\|_{L^2}^{(2q-3)/2q} \|\Delta u\|_{L^2}^{3/2q}, \quad (35)$$

thus we have from (33)

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\ &\quad + (r-1) \int_{\mathbb{R}^3} |u|^{r-2} |\nabla u|^2 dx \\ &\leq C (\|\nabla u_1\|_{L^{q,\infty}} + \|\nabla u_2\|_{L^{q,\infty}}) \|\nabla u\|_{L^2}^{(2q-3)/q} \|\Delta u\|_{L^2}^{3/q} \\ &\leq \frac{1}{2} \|\Delta u\|_{L^2}^2 \\ &\quad + C \left(\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)} \right) \|\nabla u\|_{L^2}^2 \end{aligned} \quad (36)$$

which implies

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\ &\leq C \left(\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)} \right) \|\nabla u\|_{L^2}^2. \end{aligned} \quad (37)$$

In particular,

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\ &\leq C \left(\frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} \right) \\ &\quad \cdot (1 + \ln(e + \|\nabla u\|_{L^2}^2)) \|\nabla u\|_{L^2}^2. \end{aligned} \quad (38)$$

Employing the Gronwall inequality, it follows that

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \exp \left\{ \int_0^T \left(\frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} \{1 + \ln(e + \|\nabla u\|_{L^2}^2)\} \right) dt \right\}. \quad (39)$$

Hence we have

$$\begin{aligned} \ln \left(e + \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) &\leq \ln \left(e + \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \right) \\ &\quad + \int_0^T \left(\frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} \{1 \right. \\ &\quad \left. + \ln(e + \|\nabla u\|_{L^2}^2)\} \right) dt. \end{aligned} \quad (40)$$

We take the Gronwall inequality into account again to get

$$\begin{aligned} \ln \left\{ e + \int_{\mathbb{R}^3} |\nabla u|^2 dx \right\} &\leq C(u_0) \\ &\quad \cdot \exp \left\{ \int_0^T \frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} ds \right\} \\ &< \infty. \end{aligned} \quad (41)$$

Hence we obtain a priori estimates of ∇u :

$$\operatorname{ess\,sup}_{0 < t < T} \int_{\mathbb{R}^3} |\nabla u|^2 dx < C(u_0). \quad (42)$$

□

4. Proof of Theorem 2

Under the a priori estimates in Theorem 6, we now are in a position to complete the proof of Theorem 2. Since $u_0 \in L^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, by the existence theorem of local strong solutions to the Navier-Stokes equations with nonlinear damping $r > 2$, there exist a constant $T^* > 0$ and a unique smooth solution $\tilde{u}$ of (1) satisfying (refer to [19])

$$\tilde{u} \in BC([0, T^*]; H^1), \quad \tilde{u}(x, 0) = u_0. \quad (43)$$

Note that the weak solution satisfies the energy inequality (9). It follows from the weak-strong uniqueness criterion that

$$\tilde{u} \equiv u \quad \text{on } [0, T^*]. \quad (44)$$

Thus it is sufficient to show that $T^* = T$. Suppose that $T^* < T$. Without loss of generality, we may assume that T^* is the maximal existence time for $\tilde{u}$. Since $\tilde{u} \equiv u$ on $[0, T^*)$, by the assumptions (11),

$$\int_0^T \frac{\|\nabla u_1\|_{L^{q,\infty}}^{2q/(2q-3)} + \|\nabla u_2\|_{L^{q,\infty}}^{2q/(2q-3)}}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} ds < \infty, \quad q > \frac{3}{2}. \quad (45)$$

Therefore it follows from (25) that the existence time of $\tilde{u}$ can be extended after $t = T^*$ which contradicts the maximality of $t = T^*$.

This completes the proof of Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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