Exact Periodic Wave, Bisoliton, and Various Breather Solutions for the Zakharov Equations

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The Zakharov equations, which involve the interactions between Langmuir and ion acoustic waves in plasma, are analytically studied. By using the method of Exp-function, the periodic wave, bisoliton, Akhmediev breather, Ma breather, and Peregrine breather of the Zakharov equations are obtained. These results presented in this paper enrich the diversity of solution structures of the Zakharov equations. Furthermore, based on the numerical simulations of these solutions, some physics analysis of bisolitons and various breathers are given.

1. Introduction

In this paper, we consider the following Zakharov equations:

\[ n_{tt} - c_s^2 n_{xx} = \beta (|E|^2)_{xx}, \]
\[ iE_t + \alpha E_{xx} = \delta nE, \]  

(1)

where \( n \) is the perturbed number density of the ion (in the low-frequency response), \( E \) is the slow variation amplitude of the electric field intensity, \( c_s \) is the thermal transportation velocity of the electron-ion, \( \alpha \neq 0, \beta \neq 0, \) and \( \delta \neq 0 \) [1]. The Zakharov equation is one of the classical models describing the interactions between high- and low-frequency waves and governing nonlinear dynamics of travelling waves. In the interaction of laser-plasma, (1) plays crucial roles. In recent years, many exact solutions of (1) have been successfully obtained by different methods in [2–7].

Recently, some important mathematical physics equations have been widely studied [8–12]. In particular, the investigation of the rogue wave phenomenon of nonlinear partial differential equation achieved some significant advances. Many rogue waves have been obtained [13–18]. Here we continue to analyze the rogue wave phenomenon and study the existence of rogue waves of (1).

Firstly, we set \( \xi = k_1 x + k_2 t; k_1 \) and \( k_2 \) are arbitrary real constants. Equation (1) is transformed into the following equations:

\[ n = \frac{\beta k_1^2}{k_2^2 - c_s^2 k_1^2} |E|^2, \]
\[ iE_t + \alpha E_{xx} = \delta nE. \]  

(2)

Next (2) is reduced to the \((1 + 1)\)-dimensional nonlinear Schrödinger equation:

\[ iE_t + \alpha E_{xx} + \gamma |E|^2 E = 0, \]  

(3)

where \( \gamma = -\delta \beta k_1^2/(k_2^2 - c_s^2 k_1^2). \)
Then, to find the exact solutions of the Zakharov equations, we assume that

\[ E(x,t) = re^{i\sigma^2 t} \left( 1 + \frac{A(x,t) + iB(x,t)}{C(x,t)} \right), \tag{4} \]

where \( r \) is a real constant. Substituting (4) into (3), we have

2\( yr^2 A(x,t) + 2\alpha A(x,t) C_x(x,t)^2 \)
+ \( yr^2 A(x,t)^3 - 2\alpha A_x(x,t) C_x(x,t) C(x,t) \)
+ \( \alpha A_{xx}(x,t) C(x,t)^2 - \alpha A(x,t) C_{xx}(x,t) C(x,t) \)
+ \( B(x,t) C(x,t) C_x(x,t) + 3\alpha r^2 A(x,t)^3 C(x,t) \)
+ \( yr^2 B(x,t)^2 C(x,t) - B_x(x,t) C(x,t)^2 \)
+ \( yr^2 A(x,t) B(x,t)^2 + i(\alpha A(x,t)^3 B(x,t) \)
- \( A(x,t) C(x,t) C_x(x,t) \)
- \( \alpha B(x,t) C_{xx}(x,t) C(x,t) + yr^2 B(x,t)^3 \)
+ \( \alpha B_x(x,t) C(x,t)^2 + A_x(x,t) C(x,t)^2 \)
+ \( 2\alpha B(x,t) C_x(x,t)^2 + 2yr^2 A(x,t) B(x,t) C(x,t) \)
- \( 2\alpha B_x(x,t) C_x(x,t) C(x,t) = 0, \tag{5} \)

where \( A(x,t), B(x,t), \) and \( C(x,t) \) are real functions. We separate the imaginary and real parts of (5) and have

2\( yr^2 A(x,t) C(x,t)^2 + 2\alpha A(x,t) C_x(x,t)^2 \)
+ \( yr^2 A(x,t)^3 - 2\alpha A_x(x,t) C_x(x,t) C(x,t) \)
+ \( \alpha A_{xx}(x,t) C(x,t)^2 - \alpha A(x,t) C_{xx}(x,t) C(x,t) \)
+ \( B(x,t) C(x,t) C_x(x,t) + 3\alpha r^2 A(x,t)^3 C(x,t) \)
+ \( yr^2 B(x,t)^2 C(x,t) - B_x(x,t) C(x,t)^2 \)
+ \( yr^2 A(x,t) B(x,t)^2 = 0, \tag{6} \)

\( yrA(x,t)^2 B(x,t) - A(x,t) C(x,t) C_x(x,t) \)
- \( \alpha B(x,t) C_{xx}(x,t) C(x,t) + yr^2 B(x,t)^3 \)
+ \( \alpha B_x(x,t) C(x,t)^2 + A_x(x,t) C(x,t)^2 \)
+ \( 2\alpha B(x,t) C_x(x,t)^2 + 2yr^2 A(x,t) B(x,t) C(x,t) \)
- \( 2\alpha B_x(x,t) C_x(x,t) C(x,t) = 0. \)

We suppose that \( A(x,t), B(x,t), \) and \( C(x,t) \) are the following functions:

\[
A(x,t) = a_1 e^{p(Vx + Kt)} + a_2 e^{-p(Vx + Kt)} + a_3 e^{q(Vx + Lz)} + a_4 e^{-q(Vx + Lz)},
\]

\[
B(x,t) = b_1 e^{p(Vx + Kt)} + b_2 e^{-p(Vx + Kt)} + b_3 e^{q(Vx + Lz)} + b_4 e^{-q(Vx + Lz)},
\]

\[
C(x,t) = c_1 e^{p(Vx + Kt)} + c_2 e^{-p(Vx + Kt)} + c_3 e^{q(Vx + Lz)} + c_4 e^{-q(Vx + Lz)},
\]

where \( a_i, b_i, c_i (i = 1, \ldots, 4), p, q, V, K, W, \) and \( L \) are the constants to be determined later. We substitute (7) into (6) and equate all coefficients of \( e^{mp(Vx + Kt)} e^{mq(Vx + Lz)} (m, n = -3, \ldots, 3) \) to 0, and then we obtain a set of algebraic equations for \( a_i, b_i, c_i (i = 1, \ldots, 4), p, q, V, K, W, \) and \( L \). With the aid of Maple, we have

\[
a_1 = 0,
\]
\[
a_2 = 0,
\]
\[
a_3 = \frac{b_3^2}{\sqrt{4c_1^2 - b_4^2}},
\]
\[
a_4 = \frac{b_4^2}{\sqrt{4c_1^2 - b_4^2}},
\]
\[
b_1 = 0,
\]
\[
b_2 = 0,
\]
\[
b_3 = -b_4,
\]
\[
c_1 = c_2,
\]
\[
c_3 = -\frac{2c_2^2}{\sqrt{4c_1^2 - b_4^2}},
\]
\[
c_4 = -\frac{2c_2^2}{\sqrt{4c_1^2 - b_4^2}},
\]
\[
V = \frac{rb_4 \sqrt{2(y/\alpha)}}{2pc_2},
\]
\[
K = 0,
\]
\[
W = 0,
\]
\[
L = \frac{yr^2 b_4 \sqrt{4c_1^2 - b_4^2}}{2c_2^2 q}.
\]

Substituting (8) and (7) into (4), we can obtain the solution of (3) as follows:

\[
E(x,t) = re^{i\sigma^2 t} \left( 1 + \frac{b_4^2 \cosh \left( \frac{yr^2 b_4 \sqrt{4c_1^2 - b_4^2}}{2c_2^2} t \right) + ib_4 \sqrt{4c_1^2 - b_4^2} \sinh \left( \frac{yr^2 b_4 \sqrt{4c_1^2 - b_4^2}}{2c_2^2} t \right)}{c_2 \sqrt{4c_1^2 - b_4^2} \cosh \left( \frac{r \sqrt{2(y/\alpha)b_4/2c_2} x \right) - 2c_2^2 \cosh \left( \frac{yr^2 b_4 \sqrt{4c_1^2 - b_4^2}}{2c_2^2} t \right)} \right). \tag{9} \]
Finally, according to \( n = \left( \frac{\beta k_1^2}{k_2^2 - c_1^2 k_2^2} \right) |E|^2 \), we obtain the exact solution of (1) as follows:

\[
E(\chi, t) = \exp^{ir\gamma t} \left( 1 + \frac{b_4^2 \cosh \left( \sqrt{r^2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} \right)}{c_2 \sqrt{4c_2^2 - b_4^2}} \cosh \left( \frac{r \sqrt{-2 (\gamma / \alpha) b_4 / 2c_2}}{x} \right) - 2c_2^2 \cosh \left( \frac{r \sqrt{2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} }{x} \right) \right),
\]

\[
n(\chi, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_2^2} |E(\chi, t)|^2,
\]

where \( y = -\delta k_1^2 / (k_2^2 - c_1^2 k_2^2) \). Here (10) is a unified solution formula of (1) which can produce periodic wave solution, bisoliton solution, and a series of breather solutions.

2. Exact Periodic Wave, Bisoliton, and Various Breather Solutions for (1)

In this section, according to (10), when we select different parameters in (10), we can obtain different structure exact solutions of (1) as follows:

\[
E(\chi, t) = \exp^{i r \gamma t} \left( 1 + \frac{b_4^2 \cosh \left( \sqrt{r^2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} \right)}{c_2 \sqrt{4c_2^2 - b_4^2}} \cosh \left( \frac{r \sqrt{-2 (\gamma / \alpha) b_4 / 2c_2}}{x} \right) - 2c_2^2 \cosh \left( \frac{r \sqrt{2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}}}{x} \right) \right),
\]

\[
n(\chi, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_2^2} |E(\chi, t)|^2.
\]

Case 1 (the periodic wave solution). When \( \alpha \gamma < 0 \) and \( 4c_2^2 - b_4^2 > 0 \), we obtain the following periodic solution:

\[
E(\chi, t) = \exp^{i r \gamma t} \left( 1 + \frac{b_4^2 \cosh \left( \sqrt{r^2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} \right)}{c_2 \sqrt{4c_2^2 - b_4^2}} \cosh \left( \frac{r \sqrt{-2 (\gamma / \alpha) b_4 / 2c_2}}{x} \right) - 2c_2^2 \cosh \left( \frac{r \sqrt{2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} }{x} \right) \right),
\]

\[
n(\chi, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_2^2} |E(\chi, t)|^2.
\]

Case 2 (the bisoliton solution). When \( \alpha \gamma < 0 \) and \( 4c_2^2 - b_4^2 > 0 \), we obtain the following bisoliton solution:

\[
E(\chi, t) = \exp^{i r \gamma t} \left( 1 + \frac{b_4^2 \cosh \left( \sqrt{r^2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} \right)}{c_2 \sqrt{4c_2^2 - b_4^2}} \cosh \left( \frac{r \sqrt{-2 (\gamma / \alpha) b_4 / 2c_2}}{x} \right) - 2c_2^2 \cosh \left( \frac{r \sqrt{2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} }{x} \right) \right),
\]

\[
n(\chi, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_2^2} |E(\chi, t)|^2.
\]

Case 3 (various breather solutions). (1) When \( \alpha \gamma > 0 \) and \( 4c_2^2 - b_4^2 > 0 \), we obtain the following Akhmediev breather solution:

\[
E(\chi, t) = \exp^{i r \gamma t} \left( 1 + \frac{b_4^2 \cosh \left( \sqrt{r^2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} \right)}{c_2 \sqrt{4c_2^2 - b_4^2}} \cosh \left( \frac{r \sqrt{-2 (\gamma / \alpha) b_4 / 2c_2}}{x} \right) - 2c_2^2 \cosh \left( \frac{r \sqrt{2 b_4 \sqrt{4c_2^2 - b_4^2 / 2c_2^2}} }{x} \right) \right),
\]

\[
n(\chi, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_2^2} |E(\chi, t)|^2.
\]
(2) When $\alpha \gamma > 0$, we set $b_4 = ib$, and we obtain the following Ma breather solution:

$$E(x, t) = re^{i\gamma t} \left( 1 - \frac{b^2 \cos \left( \frac{(\gamma r^2 b \sqrt{4c_1^2 + b^2}/2c_1^2} t \right) + ib \sqrt{4c_1^2 + b^2} \sin \left( \left(\frac{\gamma r^2 b \sqrt{4c_1^2 + b^2}/2c_1^2} t \right) \right)}{c_2 \sqrt{4c_1^2 + b^2} \cosh \left( \left(\frac{r\sqrt{-2 (\gamma/\alpha)b/2c_1^2}}{\gamma/\alpha} \right)x \right) - 2c_1^2 \cos \left( \left(\frac{\gamma r^2 b \sqrt{4c_1^2 + b^2}/2c_1^2} t \right) \right)} \right),$$

$$n(x, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_1^2} |E(x, t)|^2.$$ (14)

(3) When $\alpha \gamma > 0$, we set $c_2 > 0$ and $b_4 \rightarrow 0$, and we obtain the following Peregrine breather solution:

$$E(x, t) = re^{i\gamma t} \left( 1 - \frac{\alpha + i2\alpha \gamma r^2 t}{r^2 \gamma x^2 + \alpha \gamma r^2 t^2} \right),$$

$$n(x, t) = \frac{\beta k_1^2}{k_2^2 - c_1^2 k_1^2} |E(x, t)|^2.$$ (15)

To verify the correctness of our results, by using the numerical simulation method, the 3D graphics of the periodic waves, bisolitons, and various breathers of (1) are given in Figures 1–10.

3. Conclusion

To summarize, by using the Exp-function method, we have obtained the periodic wave solution, bisoliton solution, and various breather solutions of the Zakharov equations. Firstly, by using the appropriate transformation, the Zakharov equations are transformed into the $(1+1)$-dimensional nonlinear Schrödinger equation. Then, based on the Exp-function method, a unified solution formula of the $(1+1)$-dimensional nonlinear Schrödinger equation is given. Under different parameter conditions, the exact solutions of the Zakharov equations which contain the periodic wave solution, bisoliton solution, Akhmediev breather solution, Ma breather solution, and Peregrine breather solution are obtained. The results greatly enrich the diversity of exact solutions of the Zakharov equations.

Secondly, based on the numerical simulations of these solutions, we obtain some physics behind our results. Figures 3 and 4 show the bisolitons which move to each other; they meet and have an elastic collision. When the elastic collision occurs, the amplitude is obviously increased. After separation, they do the reverse motion. Figures 5 and 6 show Akhmediev breather; it yields the breather which affects spatial direction. Figures 7 and 8 show Ma breather; it yields the breather which affects time direction. Figures 9 and 10 show Peregrine breather which is a high amplitude wave. With the increase of time, the wavelength becomes larger, and the amplitude decreases gradually. Finally, it becomes a plane wave.
Figure 3: The 3D graphics of $|E(x,t)|$ in (12) ($\alpha = -2, \beta = 1, \delta = -1, c_4 = 1, \gamma = 4, k_1 = 2, k_2 = \sqrt{5}, r = 2, b_4 = 2$, and $c_2 = 5$).

Figure 4: The 3D graphics of $n(x,t)$ in (12) ($\alpha = -2, \beta = 1, \delta = -1, c_4 = 1, \gamma = -4, k_1 = 2, k_2 = \sqrt{5}, r = 2, b_4 = 2$, and $c_2 = 5$).

Figure 7: The 3D graphics of $|E(x,t)|$ in (13) ($\alpha = -2, \beta = 1, \delta = 1, c_4 = 1, \gamma = -4, k_1 = 2, k_2 = \sqrt{5}, r = 2, b_4 = 2$, and $c_2 = 5$).

Figure 8: The 3D graphics of $n(x,t)$ in (14) ($\alpha = -2, \beta = 1, \delta = 1, c_4 = 1, \gamma = -4, k_1 = 2, k_2 = \sqrt{5}, r = 2, b_4 = 2$, and $c_2 = 5$).
Finally, when $\alpha = 1/2$ and $\gamma = 1$, based on the work of Akhmediev et al. [19, 20], we can obtain the following higher-order rational rogue wave solutions of (3):

$$E(x, t) = \sqrt{\frac{1}{\gamma}} \left( 1 - \frac{G + iH}{D} \right) e^{\psi}, \quad \gamma > 0,$$

(16)

where $G = (4x^2 + 12\alpha x^2 + 72\alpha^2 t^2 + 48\alpha^2 t^4 + 80\alpha^2 t^4 - 3\alpha^2)/16\alpha^2$, $H = t(4x^2 + 8\alpha t^2 + 16\alpha^2 t^4 + 16\alpha^2 t^4 - 15\alpha^2 - 12\alpha x^2)/8\alpha^2$, and $D = (8x^6 + \alpha(12 + 48t^2)x^4 + 6\alpha^2(4t^2 - 3)x^2 + \alpha^3(9 + 64t^6 + 432t^8 + 396t^{12}))/192\alpha^2$. Then we have the following higher-order rational rogue wave solutions of (1):

$$E(x, t) = \sqrt{\frac{1}{\gamma}} \left( 1 - \frac{G + iH}{D} \right) e^{\psi},$$

$$n(x, t) = \frac{\beta k_2^2}{k_2^2 - c_2^2 k_1^2} |E(x, t)|^2.$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References


