Research Article

Sampled-Data Control of Nonlinear Systems with Quantization

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This paper is concerned with sampled-data control problem for a class of nonlinear systems with input quantization. The nonlinear system is converted into a linear-like system with modeling error by using partition of unity method. A time-dependent Lyapunov functional is introduced to capture the characteristic of nonlinear sampled-data systems and the exponential stability conditions are derived by the use of inequality techniques. The desired sampled-data controller is then synthesized. An example is provided to illustrate the effectiveness and benefits of the proposed scheme.

1. Introduction

In modern control systems, such as computer-based control systems, the continuous-time plant is usually controlled by a discrete-time controller with sample and hold devices. Such control systems are referred to as sampled-data systems. The analysis and synthesis of sampled-data systems have been a research focus for nearly three decades and many results have been reported in the literature; see, for example, [1–5] and the references therein. Among these references, lifting technique method and impulse model method are two main approaches. In the past years, some papers have considered the modeling of continuous-time systems with sampled-control in the form of continuous-time systems with delayed control input, which was introduced in [6]. Compared with the other two approaches, this input delay approach can be applied to systems with nonuniform uncertain sampling and can cope with the system with parameter uncertainties. Many results related to this approach have been reported in the literature. For example, stabilization and stability analysis were investigated in [7–13], and filtering problems were addressed in [14, 15]. Improvements were provided in [16–20], where new time-dependent Lyapunov-Krasovskii functionals (LKFs) were defined to capture the characteristic of sampled-data systems.

The aforementioned results about sampled-data systems are mainly on linear systems or linear systems with nonlinear terms because the nonlinear dynamics are extremely difficult

to deal with. Most of the existing results for nonlinear sampled-data systems are based on certain T-S fuzzy models. The T-S fuzzy model-based technique is an efficient approach for taking advantage of modern linear sampled-data control theory to nonlinear control. In [21–26], based on input delay approach, the considered systems were simply treated as ordinary continuous-time systems with a bounded fast-varying delay. The work in [27, 28] has improved the results by applying an improved Lyapunov functional, which captures the characteristic of fuzzy sampled-data systems. It should be pointed out that the fuzzy controller design is based on the assumption that the fuzzy model exactly matches the plant and the modeling error may be neglected despite that the existence of modeling error may cause the instability of the system [29].

On the other hand, partition of unity is an important concept in differential geometry and is close to a group of open covering sets. The partition of unity method has proved that certain finite linear combinations of partition of unity have the ability to approximate continuous functions at any arbitrary precision on a compact region of Euclid space, which was first introduced in [30]. When a nonlinear system is well approximated by using partition of unity, the remaining control problems become easier [31]. Compared with T-S fuzzy approach, the main difference lies in that the former is more flexible in the modeling of nonlinear systems. In addition, the fuzzy method ignores the modeling error, while partition of unity method yields an equivalent

model with modeling error, which may better represent the characteristics of the origin nonlinear system. Recently, the control problem for nonlinear systems approximated by using partition of unity has gained some scattered research attention [31–33]. However, there rarely exists published work addressing sampled-control problem for nonlinear systems approximated by using partition of unity method despite its potential in practical applications. It is, therefore, an important purpose of this paper to close such a gap by making the first challenge to deal with the sampled-data control problem for a class of nonlinear systems approximated by using partition of unity method.

On the other hand, the aforementioned publications do not take into account the quantizer effects. In real computer-based control systems, quantization is an indispensable step because data transmissions cannot be performed with infinite precision due to the limited communication capacity [34–36]. On this account, some efforts have been made on sampled-data control with quantization. In [37–39], the periodic sampling technique was used to obtain a discrete-time system for modeling the real plant. However, such a discrete-time model might not capture the intersample behavior of the real system, especially for the case when the sampling period is not periodic. To the best of our knowledge, the sampled-control problem with quantization has not been fully investigated, especially for nonlinear systems with variable sampling period.

Motivated by the above discussions, in this paper, we aim to investigate the sampled-data control problem for a class of affine nonlinear systems with quantization. Following the idea of [31–33], partition of unity method is adopted to transform the nonlinear system into an equivalent linear-like system with modeling error. Based on the proposed time-dependent Lyapunov functional, sufficient conditions have been presented for the existence of desired sampled-data controller. The main contribution of this paper is mainly threefold: (1) the problem addressed is new in the sense that this paper represents the first of few attempts to deal with sampled-data control for nonlinear systems by using partition of unity method; (2) quantization and variable sampling are considered simultaneously; and (3) an LMI framework based quantized sampled-data controller design scheme is proposed for nonlinear systems. An example is given to show the effectiveness of the proposed method.

2. Problem Formulation

Consider the following nonlinear system:

$$\dot{x}(t) = F(x(t))x(t) + G(x(t))u(t), \quad (1)$$

where $x(t) \in \bar{U} \subset R^n$ is the state vector and $\bar{U}$ is a compact set in R^n ; input $u(t) \in R^m$; $F(x) = (f_{ij}(x))_{n \times n}$ and $G(x) = (g_{il}(x))_{n \times m}$ are function matrices defined on $\bar{U}$.

Assumption 1. $F(x) = (f_{ij}(x))_{n \times n}$ and $G(x) = (g_{il}(x))_{n \times m}$ are continuous on $\bigcup_{j=1}^N U_j$, where $\{U_i\}$, $i = 1, \dots, N$, is an open covering of $\bar{U}$.

Lemma 2 (see [30]). *For m given real continuous functions $\xi_i = \xi_i(x)$ defined in a compact domain $\bar{U} \subset R^n$ and m arbitrary scalars $\varepsilon_i > 0$ ($1 \leq i \leq m$), there exists a partition of unity $\{\alpha_j\}$, $j = 1, 2, \dots, N$, subordinated to the open covering $\{U_i\}$ of the compact region $\bar{U} \subset R^n$ and $m \times N$ real numbers γ_{ij} , such that*

$$\sup_{x \in \bar{U}} \left| \xi_i(x) - \sum_{j=1}^N \gamma_{ij} \alpha_j(x) \right| < \varepsilon_i. \quad (2)$$

Now, we rewrite system (1) into a mathematically equivalent linear-like system with modeling error using Lemma 2.

From Lemma 2, for every continuous function $f_{ij}(x)$ or $g_{il}(x)$, $i, j = 1, \dots, n$, $l = 1, \dots, m$, there exists an open covering $\{U_i\}$ of $\bar{U} \subset R^n$ and a partition of unity subordinated to it, such that

$$f_{ij}(x) = \sum_{k=1}^N \lambda_{ij}^k \alpha_k(x) + e_{ij}(x), \quad (3)$$

$$g_{il}(x) = \sum_{k=1}^N \bar{\lambda}_{il}^k \alpha_k(x) + \bar{e}_{il}(x),$$

where $e_{ij}(x)$, $\bar{e}_{il}(x)$ are approximation errors, satisfying

$$\begin{aligned} \sup_{x \in \bar{U}} |e_{ij}(x)| &< \varepsilon_{ij}, \\ \sup_{x \in \bar{U}} |\bar{e}_{il}(x)| &< \bar{\varepsilon}_{il}, \end{aligned} \quad (4)$$

where $\varepsilon_{ij} > 0$, $\bar{\varepsilon}_{il} > 0$.

Choose $\lambda_{ij}^k = f_{ij}(x^k)$, $\bar{\lambda}_{il}^k = g_{il}(x^k)$, where x^k is a group of sample values subordinated to open covering $\{U_k\}$, respectively, $x^k = (x^{k1}, \dots, x^{kn})^T$, and $k \in \mathfrak{R} = \{1, \dots, N\}$, $i = 1, \dots, n$, $l = 1, \dots, m$.

From (3), we obtain

$$\begin{aligned} F(x) &= (f_{ij}(x))_{n \times n} = \left(\sum_{k=1}^N \lambda_{ij}^k \alpha_k(x) + e_{ij}(x) \right)_{n \times n} \\ &= \sum_{k=1}^N \alpha_k(x) A_k + (e_{ij}(x))_{n \times n}, \end{aligned} \quad (5)$$

$$G(x) = (g_{il}(x))_{n \times m} = \left(\sum_{k=1}^N \bar{\lambda}_{il}^k \alpha_k(x) + \bar{e}_{il}(x) \right)_{n \times m}$$

$$= \sum_{k=1}^N \alpha_k(x) B_k + (\bar{e}_{il}(x))_{n \times m},$$

where $A_k = (\lambda_{ij}^k)_{n \times n}$, $B_k = (\bar{\lambda}_{il}^k)_{n \times m}$. Then, system (1) can be converted into the following equivalent linear model with error:

$$\dot{x}(t) = \sum_{k=1}^N \alpha_k(x) [A_k x(t) + B_k u(t)] + e, \quad (6)$$

where the error vector $e = (e_{ij}(x))_{n \times n} x(t) + (\bar{e}_{il}(x))_{n \times m} u(t)$.

Remark 3. If the open covering $\{U_k\}$ of a compact $\bar{U}$ in (3) is composed of some rectangular domains $\{U_k\} = \prod_{i=1}^n (a_i^k, b_i^k)$, $k = 1, \dots, N$, the partition of unity $\{\alpha_k\}$ subordinated to $\{U_k\}$ can be chosen as the following collection of C^∞ functions $\alpha_k(x) = g^k(x) / \sum_{k=1}^N g^k(x)$, where the function $g^k(x) = \prod_{i=1}^n g_i^k(x_i)$,

$$g_i(x_i) = \begin{cases} e^{-(1/(x_i-a_i)+1/(b_i-x_i))^2}, & x_i \in (a_i, b_i) \\ 0, & x_i \notin (a_i, b_i). \end{cases} \quad (7)$$

For more details, say, for example, the open covering $\{U_k\}$ is composed of some open sphere domains or some open rectangular domains and open sphere domains together; one may refer to [22].

Assumption 4. $(e_{ij}(x))_{n \times n}, (\bar{e}_{il}(x))_{n \times m}$ are norm-bounded and satisfy

$$\|(e_{ij}(x))_{n \times n}\| \leq a, \quad \|(\bar{e}_{il}(x))_{n \times m}\| \leq b, \quad (8)$$

where a and b are positive real constants.

For sampled-data control, only discrete measurements of $x(t)$ are available for control purposes, and the control signal is assumed to be generated by using a zero-order-hold (ZOH) function with a sequence of hold times

$$0 = t_0 < t_1 < \dots < t_k < \dots < \lim_{k \rightarrow +\infty} t_k = +\infty. \quad (9)$$

Moreover, the sampling is not required to be periodic, and the only assumption is that the distance between any two consecutive sampling instants belongs to an interval. Specifically, it is assumed that

$$t_{k+1} - t_k = h_k \in [h_1, h_2]. \quad (10)$$

For all $k \geq 0$, where $h_2 \geq h_1 > 0$ represents the upper and lower bounds of sampling interval, respectively.

In this paper, we propose the following state feedback controller:

$$u(t) = \sum_{j=1}^N \alpha_j(x) K_j x(t_k) \quad t_k \leq t < t_{k+1}. \quad (11)$$

It is assumed that the output signals of controller (11) are passed via a quantizer and the quantizer is denoted as $f(\cdot) = [f_1(\cdot), f_2(\cdot), \dots, f_m(\cdot)]^T$, which is assumed to be symmetric; that is, $f(-v) = -f(v)$. The set of quantized levels is described by

$$\mathfrak{R} = \{\pm u_i, i = 0, \pm 1, \pm 2, \dots\} \cup \{0\}. \quad (12)$$

Theoretically, many different forms are possible for a quantizer. There is always a need to find a compromise between the performance and simplicity for real-world sampled-data control applications. In this paper, we are interested in logarithmic static and time-invariant quantizer,

which is relatively simple and suitable for sampled-data systems. According to [40, 41], a quantizer is called logarithmic if the set of quantized levels is characterized by

$$\mathfrak{R} = \{\pm u_i, u_i = \rho_i u_0, i = 0, \pm 1, \pm 2, \dots\} \cup \{\pm u_0\} \cup \{0\} \quad (13)$$

$$0 < \rho_i < 1 \quad u_0 > 0.$$

For the logarithmic quantizer, the associated quantizer $f(\cdot)$ is defined as follows:

$$f(v) = \begin{cases} u_i, & \text{if } \frac{1}{1+\sigma} u_i < v \leq \frac{1}{1-\sigma} u_i, \quad v > 0, \\ 0, & \text{if } v = 0, \\ -f(v), & \text{if } v < 0, \end{cases} \quad (14)$$

where

$$\sigma = \frac{1-\rho}{1+\rho}. \quad (15)$$

Then, considering the above quantization behavior, we obtain the following quantized sampled-data controller:

$$\bar{u}(t) = (I + \Lambda) \sum_{j=1}^N \alpha_j(x) K_j x(t_k), \quad (16)$$

where $\Lambda = \text{diag}\{\Lambda_1, \Lambda_2, \dots, \Lambda_m\}$ and $\Lambda_s \in [-\sigma, \sigma]$, $s = 1, \dots, m$.

Substituting the quantized controller (16) into (6), we obtain the following closed-loop system:

$$\dot{x}(t) = \sum_{i=1}^N \sum_{j=1}^N \alpha_i(x) \alpha_j(x) [A_i x(t) + B_i (I + \Lambda) K_j x(t_k)] + e \quad (17)$$

$$t_k \leq t < t_{k+1},$$

where $e = (e_{ij}(x))_{n \times n} x(t) + (\bar{e}_{ij}(x))_{n \times m} (I + \Lambda) K_j x(t_k)$.

The goal of this paper is to design quantized sampled-data controller (16) to make the closed-loop system (17) be exponentially stable; that is, there exist two constants $\alpha > 0$ and $\beta > 0$ such that

$$\|x(t)\| \leq \beta e^{-\alpha t} \|x(0)\|, \quad \forall t \geq 0. \quad (18)$$

Throughout this paper, the following lemma will be used.

Lemma 5. *Considering system (17), the following inequality holds:*

$$\|x(t)\|^2 \leq \vartheta \|x(t_k)\|^2, \quad t_k \leq t < t_{k+1}, \quad (19)$$

where $\vartheta_1 = \max_{i \in \mathfrak{R}} \{\|B_i\|\}$, $\vartheta_2 = \max_{i \in \mathfrak{R}} \{\|K_i\|\}$, $\vartheta_3 = \max_{i \in \mathfrak{R}} \{\|A_i\|\}$, and $\vartheta = 5(1 + (1 + \sigma)h_2^2 \vartheta_1 \vartheta_2 + (1 + \sigma)b^2 h_2^2 \vartheta_2) e^{5(a^2 + \vartheta_3)h_2^2}$.

Proof. For any $t \in [t_k, t_{k+1})$, we obtain from (17) that

$$\begin{aligned} \|x(t)\| \leq & \left\| \int_{t_k}^t \sum_{i=1}^N \sum_{j=1}^N \alpha_i(x) \alpha_j(x) B_i (I + \Lambda) K_j x(t_k) d\theta \right\| \\ & + \|x(t_k)\| + \left\| \int_{t_k}^t \sum_{i=1}^N \alpha_i(x) A_i x(\theta) d\theta \right\| \\ & + \left\| \int_{t_k}^t (e_{ij}(x))_{n \times n} x(\theta) d\theta \right\| \\ & + \left\| \int_{t_k}^t (\bar{e}_{ij}(x))_{n \times m} (I + \Lambda) K_j x(t_k) d\theta \right\|. \end{aligned} \quad (20)$$

Applying the Cauchy-Schwarz inequality, we find from (20) that

$$\begin{aligned} \|x(t)\|^2 \leq & 5 \left\| \int_{t_k}^t \sum_{i=1}^N \sum_{j=1}^N \alpha_i(x) \alpha_j(x) B_i (I + \Lambda) K_j x(t_k) d\theta \right\|^2 \\ & + 5 \|x(t_k)\|^2 + 5 \left\| \int_{t_k}^t \sum_{i=1}^N \alpha_i(x) A_i x(\theta) d\theta \right\|^2 \\ & + 5 \left\| \int_{t_k}^t (e_{ij}(x))_{n \times n} x(\theta) d\theta \right\|^2 \\ & + 5 \left\| \int_{t_k}^t (\bar{e}_{ij}(x))_{n \times m} (I + \Lambda) K_j x(t_k) d\theta \right\|^2. \end{aligned} \quad (21)$$

Using the Cauchy-Schwarz inequality again, we have

$$\begin{aligned} \|x(t)\|^2 \leq & 5h_2 \int_{t_k}^t \left\| \sum_{i=1}^N \sum_{j=1}^N \alpha_i(x) \alpha_j(x) B_i (I + \Lambda) K_j x(t_k) \right\|^2 d\theta \\ & + 5 \|x(t_k)\|^2 + 5h_2 \int_{t_k}^t \left\| \sum_{i=1}^N \alpha_i(x) A_i x(\theta) \right\|^2 d\theta \\ & + 5h_2 \int_{t_k}^t \left\| (e_{ij}(x))_{n \times n} x(\theta) \right\|^2 d\theta \\ & + 5h_2 \int_{t_k}^t \left\| (\bar{e}_{ij}(x))_{n \times m} (I + \Lambda) K_j x(t_k) \right\|^2 d\theta. \end{aligned} \quad (22)$$

Thus,

$$\begin{aligned} \|x(t)\|^2 \leq & 5(1 + \sigma) h_2 \vartheta_1 \vartheta_2 \int_{t_k}^t \|x(t_k)\|^2 d\theta \\ & + 5 \|x(t_k)\|^2 + 5h_2 \vartheta_3 \int_{t_k}^t \|x(\theta)\|^2 d\theta \\ & + 5a^2 h_2 \int_{t_k}^t \|x(\theta)\|^2 d\theta \\ & + 5(1 + \sigma) b^2 h_2 \vartheta_2 \int_{t_k}^t \|x(t_k)\|^2 d\theta \\ \leq & 5 \left(1 + (1 + \sigma) h_2^2 \vartheta_1 \vartheta_2 + (1 + \sigma) b^2 h_2^2 \vartheta_2 \right) \|x(t_k)\|^2 \\ & + 5(a^2 + \vartheta_3) h_2 \int_{t_k}^t \|x(\theta)\|^2 d\theta. \end{aligned} \quad (23)$$

Applying the Gronwall-Bellman lemma, we can conclude that (19) holds. This completes the proof. $\square$

3. Main Results

In this section, the problem of quantized sampled-data control will be studied for nonlinear systems by using partition of unity. Some results are provided as follows.

Theorem 6. Consider system (1) satisfying Assumptions 1 and 4, for given scalar $\lambda > 0$, if there exist matrices $P > 0$, $U > 0$, $\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0$, $X_1, X_2, X_3, X_4, X_5, G_1, G_2$, and $H^{ij} = [H_1^{ij} \ H_2^{ij} \ H_3^{ij} \ H_4^{ij}]$ and constants $\gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0$, and $\gamma_4 > 0$ such that, for any $i, j \in \mathfrak{R}$, the following two equations hold:

$$\begin{aligned} & \Phi_1^{ij}(\bar{h}) \\ & = \begin{bmatrix} \Xi_{11}^{ij} + \Theta_{11}(\bar{h}) & \Xi_{12}^{ij} + \Theta_{12}(\bar{h}) & \Xi_{13}^{ij} + \Theta_{13}(\bar{h}) & \Xi_{14}^{ij} + \Theta_{14}(\bar{h}) \\ * & \Xi_{22}^{ij} + \bar{h}U & \Xi_{23}^{ij} + \Theta_{23}(\bar{h}) & \bar{h}X_3 \\ * & * & \Xi_{33}^{ij} + \Theta_{33}(\bar{h}) & \Xi_{34}^{ij} + 2\lambda\bar{h}X_4 \\ * & * & * & \Xi_{44}^{ij} + \Theta_{44}(\bar{h}) \end{bmatrix} \\ & < 0, \end{aligned} \quad (24)$$

$$\begin{aligned} & \Phi_2^{ij}(\bar{h}) \\ & = \begin{bmatrix} \Xi_{11}^{ij} & \Xi_{12}^{ij} & \Xi_{13}^{ij} & \Xi_{14}^{ij} \\ * & \Xi_{22}^{ij} & \Xi_{23}^{ij} & 0 \\ * & * & \Xi_{33}^{ij} & \Xi_{34}^{ij} \\ * & * & * & \Xi_{44}^{ij} \end{bmatrix} + \bar{h} \begin{bmatrix} H_1^{ijT} \\ H_2^{ijT} \\ H_3^{ijT} \\ H_4^{ijT} \end{bmatrix} e^{2\lambda h_2 U^{-1}} \begin{bmatrix} H_1^{ijT} \\ H_2^{ijT} \\ H_3^{ijT} \\ H_4^{ijT} \end{bmatrix}^T < 0, \\ & \bar{h} = h_1, h_2, \end{aligned} \quad (25)$$

where

$$\begin{aligned}
\Xi_{11}^{ij} &= 2\lambda P + H_1^{ij} + H_1^{ijT} - \frac{X_1 + X_1^T}{2} + G_1 A_i \\
&\quad + A_i^T G_1^T + (\gamma_1 + \gamma_2) G_1 G_1^T + a^2 \gamma_1^{-1} I + a^2 \gamma_3^{-1} I, \\
\Xi_{12}^{ij} &= P + H_2^{ij} + H_1^{ijT} - G_1 + A_i^T G_2^T, \\
\Xi_{13}^{ij} &= X_1 - X_2 + H_3^{ij} - H_1^{ijT} + G_1 B_i (I + \Lambda) K_j, \\
\Xi_{14}^{ij} &= -X_3 + H_4^{ij}, \\
\Xi_{22}^{ij} &= -G_2 - G_2^T + (\gamma_3 + \gamma_4) G_2 G_2^T, \\
\Xi_{23}^{ij} &= -H_2^{ijT} + G_2 B_i (I + \Lambda) K_j, \\
\Xi_{33}^{ij} &= X_2 + X_2^T - \frac{X_1 + X_1^T}{2} - H_3^{ij} - H_3^{ijT} \\
&\quad + b^2 \gamma_2^{-1} K_j^T (I + \Lambda)^T (I + \Lambda) K_j \\
&\quad + b^2 \gamma_4^{-1} K_j^T (I + \Lambda)^T (I + \Lambda) K_j, \\
\Xi_{34}^{ij} &= -X_4 - e^{-2\lambda h_2} R_2^T - H_4^{ij}, \\
\Xi_{44}^{ij} &= -\frac{X_5 + X_5^T}{2} - \frac{e^{-2\lambda h_2}}{h_2} R_1, \\
\Theta_{11}^{ij}(\bar{h}) &= \lambda \bar{h} (X_1 + X_1^T) + \bar{h} R_1 + \bar{h} X_3 + \bar{h} X_3^T, \\
\Theta_{12}^{ij}(\bar{h}) &= \bar{h} \frac{X_1 + X_1^T}{2}, \\
\Theta_{13}^{ij}(\bar{h}) &= 2\lambda \bar{h} (-X_1 + X_2) + \bar{h} R_2 + \bar{h} X_4^T, \\
\Theta_{14}^{ij}(\bar{h}) &= 2\lambda \bar{h} X_3 + \bar{h} \frac{X_5 + X_5^T}{2}, \\
\Theta_{23}^{ij}(\bar{h}) &= \bar{h} (-X_1 + X_2), \\
\Theta_{33}^{ij}(\bar{h}) &= 2\lambda \bar{h} \left(-X_2 + X_2^T + \frac{X_1 + X_1^T}{2} \right) + \bar{h} R_3 - \bar{h} e^{-2\lambda h_2} R_3, \\
\Theta_{44}^{ij}(\bar{h}) &= \lambda \bar{h} (X_5 + X_5^T).
\end{aligned} \tag{26}$$

Then, system (1) is exponentially stable.

Proof. Consider the following Lyapunov functional for system (17):

$$V(t) = \sum_{l=1}^4 V_l(t), \quad t_k \leq t < t_{k+1}, \tag{27}$$

where

$$\begin{aligned}
V_1(t) &= e^{2\lambda t} x^T(t) P x(t), \\
V_2(t) &= (t_{k+1} - t) \int_{t_k}^t e^{2\lambda s} \dot{x}^T(s) U \dot{x}(s) ds, \\
V_3(t) &= (t_{k+1} - t) \int_{t_k}^t e^{2\lambda s} \begin{bmatrix} x(s) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(s) \\ x(t_k) \end{bmatrix} ds, \\
V_4(t) &= (t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix}^T H \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix} \tag{28}
\end{aligned}$$

with

$$H = \begin{bmatrix} \frac{X_1 + X_1^T}{2} & -X_1 + X_2 & X_3 \\ * & -X_2 + X_2^T + \frac{X_1 + X_1^T}{2} & X_4 \\ * & * & \frac{X_5 + X_5^T}{2} \end{bmatrix}. \tag{29}$$

Note that $V_2(t)$, $V_3(t)$, and $V_4(t)$ vanish before t_k and after t_k . So, $V(t)$ is continuous in time since $\lim_{t \rightarrow t_k} V(t) = V(t_k)$.

Calculating the derivative of $V(t)$ along the trajectories of (17), we obtain

$$\begin{aligned}
\dot{V}_1(t) &= 2\lambda e^{2\lambda t} x^T(t) P x(t) + 2e^{2\lambda t} x^T(t) P \dot{x}(t), \\
\dot{V}_2(t) &= (t_{k+1} - t) e^{2\lambda t} \dot{x}^T(t) U \dot{x}(t) - \int_{t_k}^t e^{2\lambda s} \dot{x}^T(s) U \dot{x}(s) ds \\
&\leq (t_{k+1} - t) e^{2\lambda t} \dot{x}^T(t) U \dot{x}(t) - \int_{t_k}^t e^{2\lambda s} \dot{x}^T(s) U \dot{x}(s) ds \\
&\leq (t_{k+1} - t) e^{2\lambda t} \dot{x}^T(t) U \dot{x}(t) \\
&\quad - e^{2\lambda t} \int_{t_k}^t e^{-2\lambda h} \dot{x}^T(s) U \dot{x}(s) ds, \\
\dot{V}_3(t) &= - \int_{t_k}^t e^{2\lambda s} \begin{bmatrix} x(s) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(s) \\ x(t_k) \end{bmatrix} ds \\
&\quad + (t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix} \\
&\leq -e^{2\lambda t} \int_{t_k}^t e^{-2\lambda h} \begin{bmatrix} x(s) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(s) \\ x(t_k) \end{bmatrix} ds \\
&\quad + (t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= -e^{2\lambda t} \int_{t_k}^t e^{-2\lambda h} x^T(s) R_1 x(s) ds \\
&\quad - 2e^{2\lambda t} x^T(t_k) e^{-2\lambda h} R_2^T \int_{t_k}^t x(s) ds \\
&\quad - (t_{k+1} - t) e^{2\lambda t} x^T(t_k) e^{-2\lambda h} R_3 x(t_k) \\
&\quad + (t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix} \\
&\leq -e^{2\lambda t} \int_{t_k}^t x^T(s) ds \frac{e^{-2\lambda h}}{h} R_1 \int_{t_k}^t x(s) ds \\
&\quad - 2e^{2\lambda t} x^T(t_k) e^{-2\lambda h} R_2^T \int_{t_k}^t x(s) ds \\
&\quad - (t_{k+1} - t) e^{2\lambda t} x^T(t_k) e^{-2\lambda h} R_3 x(t_k) \\
&\quad + (t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}, \\
\dot{V}_4(t) &= -e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix}^T \mathbf{H} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix} \\
&\quad + 2\lambda(t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix}^T \mathbf{H} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix} \\
&\quad + 2(t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix}^T \mathbf{H} \begin{bmatrix} \dot{x}(t) \\ 0 \\ x(t) \end{bmatrix} \\
&\leq -e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix}^T \mathbf{H} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix} \\
&\quad + 2\lambda(t_{k+1} - t) e^{2\lambda t} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix}^T \mathbf{H} \begin{bmatrix} x(t) \\ x(t_k) \\ \int_{t_k}^t x(s) ds \end{bmatrix} \\
&\quad + 2(t_{k+1} - t) e^{2\lambda t} x^T(t) \frac{X_1 + X_1^T}{2} \dot{x}(t) \\
&\quad + 2(t_{k+1} - t) e^{2\lambda t} x^T(t_k) (-X_1 + X_2) \dot{x}(t) \\
&\quad + 2(t_{k+1} - t) e^{2\lambda t} \int_{t_k}^t x^T(s) ds X_3^T \dot{x}(t)
\end{aligned}$$

$$\begin{aligned}
&+ 2(t_{k+1} - t) e^{2\lambda t} x^T(t) X_3 x(t) \\
&+ 2(t_{k+1} - t) e^{2\lambda t} x^T(t_k) X_4 x(t) \\
&+ 2(t_{k+1} - t) e^{2\lambda t} \int_{t_k}^t x^T(s) ds \frac{X_5 + X_5^T}{2} x(t).
\end{aligned} \tag{30}$$

Based on Schur complement, it can be found that for any appropriately dimensioned matrix H^{ij} , $i, j \in \mathfrak{R}$,

$$\int_{t_k}^t \begin{bmatrix} \xi(t) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} H^{ijT} e^{2\lambda h} U^{-1} H^{ij} & H^{ijT} \\ * & e^{-2\lambda h} U \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{x}(s) \end{bmatrix} ds \geq 0, \tag{31}$$

where

$$\xi(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) & \dot{x}^T(t_k) & \int_{t_k}^t x^T(s) ds \end{bmatrix}^T. \tag{32}$$

From (31), we have

$$\begin{aligned}
&- \int_{t_k}^t e^{-2\lambda h} \dot{x}^T(s) U \dot{x}(s) ds \\
&\leq (t - t_k) \xi^T(t) H^{ijT} e^{2\lambda h} U^{-1} H^{ij} \xi(t) \\
&\quad + 2\xi^T(t) H^{ijT} x(t) - 2\xi^T(t) H^{ijT} x(t_k).
\end{aligned} \tag{33}$$

It is noted that, for any appropriately dimensioned matrices G_1 and G_2 , the following equation holds:

$$\begin{aligned}
0 &= 2e^{2\lambda t} \left[x^T(t) G_1 + \dot{x}^T(t) G_2 \right] \\
&\quad \cdot \left[-\dot{x}(t) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \left[A_i x(t) + B_i (I + \Lambda) K_j x(t_k) \right] + e \right].
\end{aligned} \tag{34}$$

On the other hand, for any arbitrary $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, and $\gamma_4 > 0$, we can get from (8) that

$$\begin{aligned}
&x^T(t) G_1 e + e^T G_1^T x(t) \\
&= 2x^T(t) G_1 (e_{ij}(x))_{n \times n} x(t) \\
&\quad + 2 \sum_{j=1}^N \alpha_j x^T(t) G_1 (\bar{e}_{ij}(x))_{n \times m} (I + \Lambda) K_j x(t_k) \\
&\leq \gamma_1 x^T(t) G_1 G_1^T x(t) \\
&\quad + \gamma_1^{-1} \left\| (e_{ij}(x))_{n \times n} \right\|^2 x^T(t) x(t) \\
&\quad + \gamma_2 x^T(t) G_1 G_1^T x(t)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^N \alpha_j \gamma_2^{-1} \left\| \left(\bar{e}_{ij}(x) \right)_{n \times n} \right\|^2 x^T(t_k) K_j^T (I + \Lambda)^T \\
& \quad \cdot (I + \Lambda) K_j x(t_k) \\
& \leq \gamma_1 x^T(t) G_1 G_1^T x(t) + \gamma_2 x^T(t) G_1 G_1^T x(t) \\
& \quad + a^2 \gamma_1^{-1} x^T(t) x(t) \\
& \quad + b^2 \gamma_2^{-1} \sum_{j=1}^N \alpha_j x^T(t_k) K_j^T (I + \Lambda)^T (I + \Lambda) K_j x(t_k), \\
& \dot{x}^T(t) G_2 e + e^T G_2^T \dot{x}(t) \\
& = 2 \dot{x}^T(t) G_2 \left(e_{ij}(x) \right)_{n \times n} x(t) \\
& \quad + 2 \sum_{j=1}^N \alpha_j \dot{x}^T(t) G_2 \left(\bar{e}_{ij}(x) \right)_{n \times m} (I + \Lambda) K_j x(t_k) \\
& \leq \gamma_3 \dot{x}^T(t) G_2 G_2^T \dot{x}(t) + \gamma_4 \dot{x}^T(t) G_2 G_2^T \dot{x}(t) \\
& \quad + a^2 \gamma_3^{-1} x^T(t) x(t) \\
& \quad + b^2 \gamma_4^{-1} \sum_{j=1}^N \alpha_j x^T(t_k) K_j^T (I + \Lambda)^T (I + \Lambda) K_j x(t_k).
\end{aligned} \tag{35}$$

Adding the right side of (9) into $\dot{V}(t)$, we obtain from (30), (33), and (35) that for $t \in [t_k, t_{k+1})$

$$\begin{aligned}
\dot{V}(t) & \leq e^{2\lambda t} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \xi^T(t) \\
& \quad \cdot \left[\frac{t_{k+1} - t}{h_k} \Phi_1^{ij}(h_k) + \frac{t_{k+1} - t}{h_k} \Phi_2^{ij}(h_k) \right] \xi(t).
\end{aligned} \tag{36}$$

From (24) and (25), it is easy to get

$$\begin{aligned}
\Phi_1^{ij}(h_k) & = \frac{h_k - h_1}{h_2 - h_1} \Phi_1^{ij}(h_2) + \frac{h_2 - h_k}{h_2 - h_1} \Phi_1^{ij}(h_1) < 0, \\
\Phi_2^{ij}(h_k) & = \frac{h_k - h_1}{h_2 - h_1} \Phi_2^{ij}(h_2) + \frac{h_2 - h_k}{h_2 - h_1} \Phi_2^{ij}(h_1) < 0.
\end{aligned} \tag{37}$$

Then, we obtain from (36) and (37) that

$$\dot{V}(t) < 0, \quad t \in [t_k, t_{k+1}). \tag{38}$$

Thus, it follows that for $t \in [t_k, t_{k+1})$

$$V(t) \leq V(t_k) \leq V(t_{k-1}) \leq \dots \leq V(0). \tag{39}$$

Applying Lemma 5 and (39), we have for $t \in [t_k, t_{k+1})$

$$\begin{aligned}
\|x(t)\|^2 & \leq \vartheta \|x(t_k)\|^2 \\
& = \frac{\vartheta}{\lambda_{\min}(P)} e^{2\lambda t_k} \lambda_{\min}(P) \|x(t_k)\|^2 \\
& \leq \frac{\vartheta}{\lambda_{\min}(P)} e^{2\lambda t_k} x^T(t_k) P x(t_k) \\
& = \frac{\vartheta}{\lambda_{\min}(P)} e^{-2\lambda t_k} V(t_k) \\
& \leq \frac{\vartheta}{\lambda_{\min}(P)} e^{-2\lambda t} e^{2\lambda(t-t_k)} V(0) \\
& \leq e^{2\lambda h_2} \frac{\vartheta}{\lambda_{\min}(P)} e^{-2\lambda t} \lambda_{\max}(P) \|x(0)\|^2.
\end{aligned} \tag{40}$$

From (40), we get

$$\|x(t)\|^2 \leq \vartheta \|x(t_k)\|^2 \leq e^{\lambda h_2} \sqrt{\frac{\vartheta \lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\lambda t} \|x(0)\|. \tag{41}$$

Then, system (17) is exponentially stable. This completes the proof. $\square$

Remark 7. Theorem 6 provides a stability criterion for system (1). Inspired by [19, 20], the time-dependent Lyapunov functional is adopted in the proof of Theorem 6, which can capture the characteristic of sampled-data systems and reduce the conservatism. In addition, unlike most of the existing works, the obtained results depend not only on the upper, but also on the lower bound of the variable sampling pattern, which can fully adopt the actual sampling pattern and reduce conservatism.

Now, based on Theorem 6, we focus on the quantized sampled-data controller design for system (17). The following theorem presents a sufficient condition of the existence of the desired controller.

Theorem 8. Consider system (1) satisfying Assumptions 1 and 4, for given scalar $\lambda > 0$, if there exist matrices $\hat{P} > 0$, $\hat{U} > 0$, $\begin{bmatrix} \hat{R}_1 & \hat{R}_2 \\ * & \hat{R}_3 \end{bmatrix} > 0$, $\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5, \hat{G}$, and $\hat{H}^{ij} = [\hat{H}_1^{ij} \ \hat{H}_2^{ij} \ \hat{H}_3^{ij} \ \hat{H}_4^{ij}]$ and constants $\gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0, \gamma_4 > 0$, and $\gamma_5 > 0$ such that, for any $i, j \in \mathfrak{R}$, (42) and (43) hold:

$$\widehat{\Phi}_1^{ij}(\bar{h}) = \begin{bmatrix} \Psi_1^{ij}(\bar{h}) & \Gamma_1^{ij} & \Gamma_1^{ij} & \Gamma_2^{ij} & \Gamma_2^{ij} & \Gamma_2^{ij} \\ * & -\gamma_1 I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_3 I & 0 & 0 & 0 \\ * & * & * & -\gamma_2 I + \sigma^2 \gamma_5 I & \sigma^2 \gamma_5 I & 0 \\ * & * & * & * & -\gamma_4 I + \sigma^2 \gamma_5 I & 0 \\ * & * & * & * & * & -\gamma_5 I \end{bmatrix} < 0, \quad (42)$$

$$\widehat{\Phi}_2^{ij}(\bar{h}) = \begin{bmatrix} \Psi_1^{ij}(\bar{h}) & \Gamma_1^{ij} & \Gamma_1^{ij} & \Gamma_2^{ij} & \Gamma_2^{ij} & \Gamma_2^{ij} & \sqrt{\bar{h}} \widehat{H}^{ijT} \\ * & -\gamma_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & -\gamma_3 I & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_2 I + \sigma^2 \gamma_5 I & \sigma^2 \gamma_5 I & 0 & 0 \\ * & * & * & * & -\gamma_4 I + \sigma^2 \gamma_5 I & 0 & 0 \\ * & * & * & * & * & -\gamma_5 I & 0 \\ * & * & * & * & * & * & -e^{2\lambda h_2} \widehat{U} \end{bmatrix} < 0, \quad (43)$$

where

$$\Psi_1^{ij}(\bar{h})$$

$$= \begin{bmatrix} \widehat{\Theta}_{11}^{ij} + \widehat{\Theta}_{11}(\bar{h}) & \widehat{\Theta}_{12}^{ij} + \widehat{\Theta}_{12}(\bar{h}) & \widehat{\Theta}_{13}^{ij} + \widehat{\Theta}_{13}(\bar{h}) & \widehat{\Theta}_{14}^{ij} + \widehat{\Theta}_{14}(\bar{h}) \\ * & \widehat{\Theta}_{22}^{ij} + \bar{h} \widehat{U} & \widehat{\Theta}_{23}^{ij} + \widehat{\Theta}_{23}(\bar{h}) & \bar{h} \widehat{X}_3 \\ * & * & \widehat{\Theta}_{33}^{ij} + \widehat{\Theta}_{33}(\bar{h}) & \widehat{\Theta}_{34}^{ij} + 2\lambda \bar{h} \widehat{X}_4 \\ * & * & * & \widehat{\Theta}_{44}^{ij} + \widehat{\Theta}_{44}(\bar{h}) \end{bmatrix},$$

$$\Psi_2^{ij}(\bar{h}) = \begin{bmatrix} \widehat{\Theta}_{11}^{ij} & \widehat{\Theta}_{12}^{ij} & \widehat{\Theta}_{13}^{ij} & \widehat{\Theta}_{14}^{ij} \\ * & \widehat{\Theta}_{22}^{ij} & \widehat{\Theta}_{23}^{ij} & 0 \\ * & * & \widehat{\Theta}_{33}^{ij} & \widehat{\Theta}_{34}^{ij} \\ * & * & * & \widehat{\Theta}_{44}^{ij} \end{bmatrix},$$

$$\Gamma_1^{ij} = [a \widehat{G}^T \ 0 \ 0 \ 0]^T,$$

$$\Gamma_2^{ij} = [0 \ 0 \ b \widehat{K}_j \ 0]^T,$$

$$\begin{aligned} \widehat{\Theta}_{11}^{ij} &= 2\lambda \widehat{P} + \widehat{H}_1^{ij} + \widehat{H}_1^{ijT} - \frac{\widehat{X}_1 + \widehat{X}_1^T}{2} + \tau_1 \widehat{A}_i \widehat{G}^T \\ &\quad + \tau_1 \widehat{G} \widehat{A}_i^T + \tau_1^2 (\gamma_1 + \gamma_2) I + \gamma_5 \sigma^2 \tau_1^2 B_i B_i^T, \end{aligned}$$

$$\widehat{\Theta}_{12}^{ij} = \widehat{P} + \widehat{H}_2^{ij} + \widehat{H}_1^{ijT} - \tau_1 \widehat{G}^T + \tau_2 \widehat{G} \widehat{A}_i^T + \gamma_5 \sigma^2 \tau_1 \tau_2 B_i B_i^T,$$

$$\widehat{\Theta}_{13}^{ij} = \widehat{X}_1 - \widehat{X}_2 + \widehat{H}_3^{ij} - \widehat{H}_1^{ijT} + \tau_1 B_i \widehat{K}_j,$$

$$\widehat{\Theta}_{14}^{ij} = -\widehat{X}_3 + \widehat{H}_4^{ij},$$

$$\widehat{\Theta}_{22}^{ij} = -\tau_2 \widehat{G}^T - \tau_2 \widehat{G} + \tau_2^2 (\gamma_3 + \gamma_4) I + \gamma_5 \sigma^2 \tau_2^2 B_i B_i^T,$$

$$\widehat{\Theta}_{23}^{ij} = -\widehat{H}_2^{ijT} + \tau_2 B_i \widehat{K}_j,$$

$$\widehat{\Theta}_{33}^{ij} = \widehat{X}_2 + \widehat{X}_2^T - \frac{\widehat{X}_1 + \widehat{X}_1^T}{2} - \widehat{H}_3^{ij} - \widehat{H}_3^{ijT},$$

$$\widehat{\Theta}_{34}^{ij} = -\widehat{X}_4 - e^{-2\lambda h} \widehat{R}_2^T - \widehat{H}_4^{ij},$$

$$\widehat{\Theta}_{44}^{ij} = -\frac{\widehat{X}_5 + \widehat{X}_5^T}{2} - \frac{e^{-2\lambda h_2}}{h_2} \widehat{R}_1,$$

$$\widehat{\Theta}_{11}^{ij}(\bar{h}) = \lambda \bar{h} (\widehat{X}_1 + \widehat{X}_1^T) + \bar{h} \widehat{R}_1 + \bar{h} \widehat{X}_3 + \bar{h} \widehat{X}_3^T,$$

$$\widehat{\Theta}_{12}^{ij}(\bar{h}) = \bar{h} \frac{\widehat{X}_1 + \widehat{X}_1^T}{2},$$

$$\widehat{\Theta}_{13}^{ij}(\bar{h}) = 2\lambda \bar{h} (-\widehat{X}_1 + \widehat{X}_2) + \bar{h} \widehat{R}_2 + \bar{h} \widehat{X}_4^T,$$

$$\widehat{\Theta}_{14}^{ij}(\bar{h}) = 2\lambda \bar{h} \widehat{X}_3 + \bar{h} \frac{\widehat{X}_5 + \widehat{X}_5^T}{2},$$

$$\widehat{\Theta}_{23}^{ij}(\bar{h}) = \bar{h} (-\widehat{X}_1 + \widehat{X}_2),$$

$$\widehat{\Theta}_{33}^{ij}(\bar{h}) = 2\lambda \bar{h} \left(-\widehat{X}_2 + \widehat{X}_2^T + \frac{\widehat{X}_1 + \widehat{X}_1^T}{2} \right) + \bar{h} \widehat{R}_3 - \bar{h} e^{-2\lambda h_2} \widehat{R}_3,$$

$$\widehat{\Theta}_{44}^{ij}(\bar{h}) = \lambda \bar{h} (\widehat{X}_5 + \widehat{X}_5^T).$$

(44)

Then, system (1) is exponentially stable, and the quantized sampled-data controller in (16) is given by

$$K_j = \widehat{K}_j \widehat{G}^{-T}. \quad (45)$$

Proof. Define

$$\begin{aligned} G_1 &= \tau_1 G, & G_2 &= \tau_2 G, & \widehat{G} &= G^{-1}, \\ \widehat{K}_j &= K_j \widehat{G}^T, & \widehat{P} &= \widehat{G} P \widehat{G}^T, & \widehat{U} &= \widehat{G} U \widehat{G}^T, \\ \widehat{R}_1 &= \widehat{G} R_1 \widehat{G}^T, & \widehat{R}_2 &= \widehat{G} R_2 \widehat{G}^T, & \widehat{R}_3 &= \widehat{G} R_3 \widehat{G}^T, \\ \widehat{X}_1 &= \widehat{G} X_1 \widehat{G}^T, & \widehat{X}_2 &= \widehat{G} X_2 \widehat{G}^T, & \widehat{X}_3 &= \widehat{G} X_3 \widehat{G}^T, \\ \widehat{X}_4 &= \widehat{G} X_4 \widehat{G}^T, & \widehat{H}_1^{ij} &= \widehat{G} H_1^{ij} \widehat{G}^T, & \widehat{H}_2^{ij} &= \widehat{G} H_2^{ij} \widehat{G}^T, \\ & & \widehat{H}_3^{ij} &= \widehat{G} H_3^{ij} \widehat{G}^T, & \widehat{H}_4^{ij} &= \widehat{G} H_4^{ij} \widehat{G}^T. \end{aligned} \quad (46)$$

Before and after multiplying (24) by $\text{diag}\{\widehat{G}, \widehat{G}, \widehat{G}, \widehat{G}\}$ and $\text{diag}\{\widehat{G}^T, \widehat{G}^T, \widehat{G}^T, \widehat{G}^T\}$, we get

$$\begin{aligned} & \overline{\Psi}_1^{ij}(\bar{h}) + a^2 \gamma_1^{-1} \Gamma_1^{ij} \Gamma_1^{ijT} + a^2 \gamma_3^{-1} \Gamma_1^{ij} \Gamma_1^{ijT} \\ & + b^2 \gamma_2^{-1} \Gamma_2^{ij} (I + \Lambda)^T (I + \Lambda) \Gamma_2^{ijT} \\ & + b^2 \gamma_4^{-1} \Gamma_2^{ij} (I + \Lambda)^T (I + \Lambda) \Gamma_2^{ijT} < 0, \end{aligned} \quad (47)$$

where

$$\overline{\Psi}_1^{ij}(\bar{h}) = \begin{bmatrix} \widehat{\Xi}_{11}^{ij} + \widehat{\Theta}_{11}(\bar{h}) - \gamma_5 \sigma^2 \tau_1^2 B_i B_i^T & \widehat{\Xi}_{12}^{ij} + \widehat{\Theta}_{12}(\bar{h}) - \gamma_5 \sigma^2 \tau_1 \tau_2 B_i B_i^T & \widehat{\Xi}_{13}^{ij} + \widehat{\Theta}_{13}(\bar{h}) + \tau_1 B_i \Lambda \widehat{K}_j & \widehat{\Xi}_{14}^{ij} + \widehat{\Theta}_{14}(\bar{h}) \\ * & \widehat{\Xi}_{22}^{ij} + \bar{h} \widehat{U} - \gamma_5 \sigma^2 \tau_1 \tau_2 B_i B_i^T & \widehat{\Xi}_{23}^{ij} + \widehat{\Theta}_{23}(\bar{h}) + \tau_2 B_i \Lambda \widehat{K}_j & \bar{h} \widehat{X}_3 \\ * & * & \widehat{\Xi}_{33}^{ij} + \widehat{\Theta}_{33}(\bar{h}) & \widehat{\Xi}_{34}^{ij} + 2\lambda \bar{h} \widehat{X}_4 \\ * & * & * & \widehat{\Xi}_{44}^{ij} + \Theta_{44}(\bar{h}) \end{bmatrix}. \quad (48)$$

Using Schur complement, we obtain

$$\overline{\Phi}_1^{ij}(\bar{h}) + \Gamma_3^{ij} \Lambda^T \Gamma_4^{ijT} + \Gamma_4^{ij} \Lambda \Gamma_3^{ijT} < 0, \quad (49)$$

where

$$\begin{aligned} \overline{\Phi}_1^{ij}(\bar{h}) &= \begin{bmatrix} \overline{\Psi}_1^{ij}(\bar{h}) & \Gamma_1^{ij} & \Gamma_1^{ij} & \Gamma_2^{ij} & \Gamma_2^{ij} \\ * & -\gamma_1 I & 0 & 0 & 0 \\ * & * & -\gamma_3 I & 0 & 0 \\ * & * & * & -\gamma_2 I & 0 \\ * & * & * & * & -\gamma_4 I \end{bmatrix}, \\ \overline{\Psi}_1^{ij}(\bar{h}) &= \begin{bmatrix} \widehat{\Xi}_{11}^{ij} + \widehat{\Theta}_{11}(\bar{h}) - \gamma_5 \sigma^2 \tau_1^2 B_i B_i^T & \widehat{\Xi}_{12}^{ij} + \widehat{\Theta}_{12}(\bar{h}) - \gamma_5 \sigma^2 \tau_1 \tau_2 B_i B_i^T & \widehat{\Xi}_{13}^{ij} + \widehat{\Theta}_{13}(\bar{h}) & \widehat{\Xi}_{14}^{ij} + \widehat{\Theta}_{14}(\bar{h}) \\ * & \widehat{\Xi}_{22}^{ij} + \bar{h} \widehat{U} - \gamma_5 \sigma^2 \tau_1 \tau_2 B_i B_i^T & \widehat{\Xi}_{23}^{ij} + \widehat{\Theta}_{23}(\bar{h}) & \bar{h} \widehat{X}_3 \\ * & * & \widehat{\Xi}_{33}^{ij} + \widehat{\Theta}_{33}(\bar{h}) & \widehat{\Xi}_{34}^{ij} + 2\lambda \bar{h} \widehat{X}_4 \\ * & * & * & \widehat{\Xi}_{44}^{ij} + \Theta_{44}(\bar{h}) \end{bmatrix}, \\ \Gamma_3^{ij} &= [\Gamma_2^{ijT} \ 0 \ 0 \ 0 \ 0]^T, \quad \Gamma_4^{ij} = [\tau_1 B_i^T \ \tau_2 B_i^T \ 0 \ 0 \ 0 \ 0 \ I \ I]^T. \end{aligned} \quad (50)$$

From lemma, for any $\gamma_5 > 0$,

$$\overline{\Phi}_1^{ij}(\bar{h}) + \gamma_5^{-1} \Gamma_3^{ij} \Gamma_3^{ijT} + \sigma^2 \gamma_5 \Gamma_4^{ij} \Gamma_4^{ijT} < 0. \quad (51)$$

Using Schur complement again, we get (42). Before and after multiplying (25) by $\text{diag}\{\widehat{G}, \widehat{G}, \widehat{G}, \widehat{G}\}$ and

$\text{diag}\{\widehat{G}^T, \widehat{G}^T, \widehat{G}^T, \widehat{G}^T\}$, by using the above similar method, we can also get (43). This completes the proof. $\square$

Remark 9. Theorem 8 provides a design method of the sampled-data controller with quantization for nonlinear systems. The conditions in Theorem 8 are formulated in LMIs and thus can be effectively solved by using LMI toolbox in MATLAB.

4. Examples

Consider the following nonlinear mass-spring system [42]:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -0.01x_1(t) - 0.67x_1^3(t) + u(t),\end{aligned}\quad (52)$$

where $x_1 \in [-1, 1]$, $x_2 \in [-0.8, 0.8]$.

The system matrices of the system in (1) are

$$F(x) = \begin{bmatrix} 0 & 1 \\ -0.01 - 0.67x_1^2 & 0 \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (53)$$

Choose the open covering as $U_1 = (-1.1, 1.1) \times (-0.9, 0.2)$ and $U_2 = (-1.1, 1.1) \times (0.1, 0.9)$. The corresponding sample data are chosen as $(0.5, -0.5)$ and $(-0.3, 0.6)$, respectively.

Calculating the coefficient matrices in (17), we have

$$\begin{aligned}A_1 &= \begin{bmatrix} 0 & 1 \\ -0.1775 & 0 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & 1 \\ -0.2512 & 0 \end{bmatrix}, \\ B_1 &= B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.\end{aligned}\quad (54)$$

The modeling errors are computed as follows:

$$\begin{aligned}\|(e_{ij}(x))_{n \times n}\| &= \left\| F(x) - \sum_{k=1}^N \alpha_k(x) A_k \right\| \\ &\leq \max_{1 \leq k \leq 2} \{\|F_k(x) - A_k\|\} = 0.6432,\end{aligned}\quad (55)$$

$$\|(\bar{e}_{ij}(x))_{n \times n}\| = \left\| G(x) - \sum_{k=1}^N \alpha_k(x) B_k \right\| = 0,$$

where $F_k(x)$ denotes $F(x)$ on U_k .

In this example, we choose the initial condition $x(0) = [1 \ 0.5]^T$, the minimum sampling period $h_1 = 0.01$, the maximum sampling period $h_2 = 0.03$, and the convergence rate $\lambda = 0.2$. In addition, the parameter for the quantizer is assumed to be $\rho = 0.9$. By solving the LMIs in Theorem 8, we can get the controller gains from (45) that

$$K_1 = [-1.3047 \ -4.0082], \quad K_2 = [-0.5914 \ -4.9775]. \quad (56)$$

The model in (52) has been studied by T-S fuzzy systems (T-S) method in [42]. To illustrate the effectiveness of partition of unity (PoU) method, the comparison of simulation results is shown in Figure 1. It can be shown from Figure 1 that the method used in our paper gives faster convergence rate. The reason is that the fuzzy method ignores the modeling error, while partition of unity method yields an equivalent model with modeling error, better representing the characteristics of the origin nonlinear system. Therefore, the results will hold if the modeling error exists by using fuzzy control method.

FIGURE 1: State response using T-S fuzzy and partition of unity methods, respectively.

5. Conclusion

In this paper, the problem of quantized sampled-data control for a class of nonlinear systems has been investigated. Based on partition of unity method, the system has been converted into a linear-like system with modeling error. By introducing a Lyapunov functional, sufficient conditions for the existence of desired controller have been derived. An example has been presented to show the effectiveness and benefits of proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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