Four-Node Generalized Conforming Membrane Elements with Drilling DOFs Using Quadrilateral Area Coordinate Methods

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Two 4-node generalized conforming quadrilateral membrane elements with drilling DOF, named QAC4θ and QAC4θM, were successfully developed. Two kinds of quadrilateral area coordinates are used together in the assumed displacement fields of the new elements, so that the related formulations are quite straightforward and will keep the order of the Cartesian coordinates unchangeable while the mesh is distorted. The drilling DOF is defined as the additional rigid rotation at the element nodes to avoid improper constraint. Both elements can pass the strict patch test and exhibit better performance than other similar models. In particular, they are both free of trapezoidal locking in MacNeal’s beam test and insensitive to various mesh distortions.

1. Introduction

It is well known that adding the drilling degree of freedom (DOF) at each node of a plane membrane element can enhance the element performance without increasing the number of the element nodes. Furthermore, a plane membrane element with drilling DOFs can be combined with a plate bending element to form a flat-shell element, which can avoid the problem of singular coefficients associated with the DOFs in the direction normal to the plane of the shell element.

The research on drilling DOF can be traced to the 1960s. Olson and Bearden [1] proposed the first valuable triangular membrane element with drilling DOF. However, since the rotations of two adjacent edges are assumed to be equal, this element may not converge to the correct solution. Another model, a hybrid displacement triangular element with drilling DOF, was then proposed by Mohr [2], but its variational theory is not sufficient. The definition of the drilling DOFs proposed by Allman [3, 4] can be treated as a milestone for this topic, in which a quadratic displacement approximation was introduced to supplement the drilling DOFs at element nodes. Based on Allman’s work, numerous researches on plane elements with drilling degrees of freedom have been accomplished, such as the models proposed by Bergan and Felippa [5], Cook [6, 7], MacNeal and Harder [8], Yunus et al. [9], Hughes and Brezzi [10], Ibrahimbegovic et al. [11], Iura and Atluri [12], Cazzani and Atluri [13], Piltner and Taylor [14], Geyer and Groenwold [15], Pimpinelli [16], Groenwold et al. [17], Choi et al. [18], Choo et al. [19], Zhang and Kuang [20], Kugler et al. [21], and Cen et al. [22]. Long et al. [23–25] presented a new definition on the drilling DOFs. They treated these DOFs as the additional rigid rotations at the element nodes, so that the change of the angle between two adjacent edges along with the element deformation is allowed, and the rotation of the element edge has definite relation with the nodal drilling DOFs. Based on this assumption, the triangular element GT9 series and the quadrilateral element GQ12 series were developed. These elements all exhibit good performance [26], and among these models the quadrilateral element GQ12M8 is the best one.

It is also well known that most quadrilateral elements use the isoparametric coordinates ($\xi$, $\eta$) to express their formu-
lations. Lee and Bathe [27] have studied the influence of mesh distortions on the isoparametric membrane elements and showed that the serendipity family is quite sensitive to the mesh distortions. They concluded that the nonlinear transformation between the isoparametric (local) and the Cartesian (global) coordinates leads to such problem. Although the assumed displacement fields may contain high-order terms of \( \xi \) and \( \eta \), their complete order in Cartesian coordinates \( x \) and \( y \) will degrade significantly once the meshes are distorted, which will lead to low accuracy. In order to make the isoparametric displacement fields satisfy second order completeness in Cartesian coordinates, even fourth order isoparametric terms should be introduced, such as the abovementioned element GQ12M8. This makes the element formulations quite complicated.

For overcoming this inherent defect of the isoparametric coordinates, Long et al. [28, 29] developed the first kind of quadrilateral area coordinate method QACM-I. The QACM-I possesses an important character: the transformation between the area and the Cartesian coordinate systems is always linear. Thus, the disadvantage of the isoparametric coordinate system is avoided from the outset. Then, this new natural coordinate system was successfully applied to develop new finite element models. Soh et al. [30] constructed two 8-node plane quadrilateral elements AGQ6-I and AGQ6-II, which exhibit excellent performance in high-order benchmark examples; particularly, both can perfectly pass MacNeal's thin beam test [32]. These two 4-node elements aroused the interests in further studies on the QACM-I. Cen et al. [33] derived out the analytical element stiffness matrix of AGQ6-I and developed a family of the quadrilateral plane membrane elements [34]. Du and Cen [35] extended the element AGQ6-I to geometrically nonlinear analysis. Cardoso et al. [36–38] introduced the element AGQ6-I to develop distortion-immune shell elements for linear, nonlinear, and dynamic fracture analyses. Wang and Sun [39] used an arbitrary point

\[
\begin{align*}
L_1 = & 0 \quad L_2 = 0 \\
L_3 = 0 \quad L_4 = 0
\end{align*}
\]

as the zero coordinate axes to define the two independent coordinate components \( T_1 \) and \( T_2 \). All the three kinds of area quadrilateral coordinate can be used simultaneously in one element, which will make the formulations quite simple and straightforward.

In this paper, by combination with the definition of drilling DOFs proposed by Long et al. [23–25], a new plane membrane element with drilling DOFs, denoted by QAC4\( \theta \), was firstly developed by using the QACM-III. Then, by introducing a generalized bubble displacement field in terms of QACM-II into the element QAC4\( \theta \), a more accurate and robust element, denoted by QAC4\( \theta \)M, was constructed. Both elements can pass the strict patch test and exhibit better performance than other similar models. It is demonstrated again that the quadrilateral area coordinate methods are effective tools for developing high-performance quadrilateral finite element models.

2. Brief Reviews on the Quadrilateral Area Coordinate Methods

2.1. QACM-I [28, 29]. As shown in Figure 1, the position of an arbitrary point \( P \) within a quadrilateral element \( T \) is specified by the area coordinates \( L_1, L_2, L_3, \) and \( L_4 \), which are defined as

\[
L_i = \frac{A_i}{A}, \quad (i = 1, 2, 3, 4),
\]

where \( A \) is the area of the quadrilateral element; \( A_i \) \((i = 1, 2, 3, 4)\) are the areas of the four triangles constructed by point \( P \) and four element sides \( T_3, T_4, T_1, \) and \( T_2 \), respectively. \( L_1, L_2, L_3, \) and \( L_4 \) can be expressed in terms of Cartesian coordinates \((x, y)\) as follows:

\[
L_i = \frac{1}{2A} (a_i + b_i x + c_i y), \quad (i = 1, 2, 3, 4)
\]

Figure 1: Definition of the quadrilateral area coordinates \( L_i \) of the QACM-I.

2.2. QACM-II [33]. The two 4-node quadrilateral membrane elements AGQ6-I and AGQ6-II, which exhibit excellent performance in high-order benchmark examples; particularly, both can perfectly pass MacNeal's thin beam test [32]. These two 4-node elements aroused the interests in further studies on the QACM-I. Cen et al. [33] derived out the analytical element stiffness matrices and developed a family of the quadrilateral plane membrane elements [34]. Du and Cen [35] extended the element AGQ6-I to a geometrically nonlinear analysis. Cardoso et al. [36–38] introduced the element AGQ6-I to develop distortion-immune shell elements for linear, nonlinear, and dynamic fracture analyses. Wang and Sun [39] used the element AGQ6-II to formulate a new corotational nonlinear shell element. Chen et al. [40] modified the element AGQ6-I to make it pass the strict patch test. Li [41] improved the formulations and generalized them to simulate coupled solid-deformation/fluid-flow for porous geomaterials. Cardoso and Yoon [42], Prathap and Senthilkumar [43], and Flais et al. [44] discussed the convergence for related AGQ6 models. Besides the above plane elements, the QACM-I has also been successfully employed to develop thin plate [45], Mindlin-Reissner plate [46], laminated composite plate [47], and shell models [48–51].

Since the QACM-I contains four area coordinate components \((L_1, L_2, L_3, \text{and} L_4)\), among which only two are independent, users may be confused on how to formulate a complete high-order polynomial. In view of this disadvantage, Chen et al. [52] proposed the second kind of quadrilateral coordinate method QACM-II. This QACM-II uses two midlines of opposite sides as the coordinate axes and defines only two independent coordinate components \( Z_1 \) and \( Z_2 \). The element formulations expressed by the QACM-II are quite simpler than those in terms of the QACM-I [52, 53]. In 2010,
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with
\[ g_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j, \]
\[ (i = 1, 2, 3, 4; \ j = 2, 3, 4, 1; \ k = 3, 4, 1, 2). \]  

(3)

Four dimensionless shape parameters \( g_1, g_2, g_3, \) and \( g_4 \) to each of the quadrangles, as shown in Figure 2, must be defined as
\[ g_1 = \frac{A'}{A}, \quad g_2 = \frac{A''}{A}, \quad g_3 = 1 - g_1, \quad g_4 = 1 - g_2, \]
\[ (0 \leq g_i \leq 1), \]

where \( A' \) and \( A'' \) are the areas of \( \Delta 124 \) and \( \Delta 123 \), respectively. Different values of these shape parameters mean different shapes of a quadrangle. And the area coordinates of four corner nodes can be obtained:

\[
\begin{align*}
\text{Node 1} & : (g_2, g_4, 0, 0) \\
\text{Node 2} & : (0, g_3, g_1, 0) \\
\text{Node 3} & : (0, 0, g_4, g_2) \\
\text{Node 4} & : (g_3, 0, 0, g_1). 
\end{align*}
\]

(5)

The relations between the area coordinates \( L_i \) and the isoparametric coordinates \( (\xi, \eta) \) are
\[ L_1 = \frac{1}{4} (1 - \xi) [g_2 (1 - \eta) + g_3 (1 + \eta)] \]
\[ L_2 = \frac{1}{4} (1 - \eta) [g_4 (1 - \xi) + g_3 (1 + \xi)] \]
\[ L_3 = \frac{1}{4} (1 + \xi) [g_1 (1 - \eta) + g_4 (1 + \eta)] \]
\[ L_4 = \frac{1}{4} (1 + \eta) [g_1 (1 - \xi) + g_2 (1 + \xi)]. \]

(6)

2.2. QACM-II [52]. As shown in Figure 3, \( M_i \) (\( i = 1, 2, 3, 4 \)) are the midside points of element sides \( 23, 34, 41, \) and \( 12 \), respectively. Thus, the position of an arbitrary point \( P \) within the quadrilateral element \( 1234 \) can be uniquely specified by the two-component area coordinates \( Z_1 \) and \( Z_2 \) (QACM-II), which are defined as
\[ Z_1 = 4 \frac{\Omega_1}{A}, \quad Z_2 = 4 \frac{\Omega_2}{A}, \]
\[ (7) \]

where \( \Omega_1 \) and \( \Omega_2 \) are the generalized areas of \( \Delta PM_2 M_4 \) and \( \Delta PM_4 M_1 \), respectively. It must be noted here that the values of generalized areas \( \Omega_1 \) and \( \Omega_2 \) can be both positive and negative: for \( \Delta PM_2 M_4 \) (or \( \Delta PM_4 M_1 \)), if the permutation order of points \( P, M_2, \) and \( M_4 \) (or \( P, M_4, \) and \( M_1 \)) is anticlockwise, a positive \( \Omega_1 \) (or \( \Omega_2 \)) should be taken; otherwise, \( \Omega_1 \) (or \( \Omega_2 \)) should be negative.

Thus, the local coordinates of the corner nodes and midside points can be obtained:

\[
\begin{align*}
\text{Node 1} & : (-g_1 - g_2, -g_1 - g_4) \\
\text{Node 2} & : (g_1 + g_2, -g_1 - g_3) \\
\text{Node 3} & : (g_3 + g_4, g_2 + g_3) \\
\text{Node 4} & : (-g_3 - g_4, g_1 + g_4) \\
M_1 & : (1, 0) \quad M_2 : (0, 1) \quad M_3 : (-1, 0) \quad M_4 : (0, -1). 
\end{align*}
\]

(8)

The relations between the QACM-II and the QACM-I are
\[ Z_1 = \frac{A}{A} [(a_3 - a_1) + (b_5 - b_1) x + (c_5 - c_1) y] + \bar{g}_1 \]
\[ Z_2 = \frac{A}{A} [(a_4 - a_2) + (b_4 - b_2) x + (c_4 - c_2) y] + \bar{g}_2 \]
\[ \text{with} \quad \bar{g}_1 = g_2 - g_1, \quad \bar{g}_2 = g_3 - g_2. \]

(9)

And \( Z_1 \) and \( Z_2 \) can also be expressed in terms of \( \xi \) and \( \eta \) as follows:
\[ Z_1 = \xi + \bar{g}_1 \xi \eta \]
\[ Z_2 = \eta + \bar{g}_2 \xi \eta. \]

(10)

It can be seen that the new area coordinates \( Z_1 \) and \( Z_2 \) will degenerate to the isoparametric coordinates \( \xi \) and \( \eta \) for rectangular element cases.
2.3. QACM-III [54]. As shown in Figure 4, \( T_{13} \) and \( T_{24} \) are the two diagonals of the quadrilateral \( 1234 \). Then, the position of an arbitrary point \( P \) within or outside the quadrilateral \( 1234 \) can be uniquely specified by the two-component area coordinates \( T_1 \) and \( T_2 \) (QACM-III), which are defined as

\[
T_1 = \frac{S_1}{A}, \quad T_2 = \frac{S_2}{A},
\]

(11)

where \( S_1 \) and \( S_2 \) are the generalized areas of \( \Delta P42 \) and \( \Delta P13 \), respectively. The values of generalized areas \( S_1 \) and \( S_2 \) can be both positive and negative: for \( \Delta P42 \) (or \( \Delta P13 \)), if the permutation order of the points \( P, 4, \) and \( 2 \) (or \( P, 1, \) and \( 3 \)) is anticlockwise, a positive \( S_1 \) (or \( S_2 \)) should be taken; otherwise, \( S_1 \) (or \( S_2 \)) should be negative.

Then, the local coordinates of the corner nodes can be written as

\[
\begin{align*}
\text{node 1} & \quad (-g_1, 0) \\
\text{node 2} & \quad (0, -g_2) \\
\text{node 3} & \quad (g_3, 0) \\
\text{node 4} & \quad (0, g_4).
\end{align*}
\]

(12)

The relations between the QACM-III and the QACM-I are

\[
T_1 = g_3 - L_1 - L_2 = L_3 + L_4 - g_1,
\]

\[
T_2 = g_4 - L_2 - L_3 = L_4 + L_1 - g_2.
\]

(13)

And \( T_1 \) and \( T_2 \) can also be expressed in terms of \( \xi \) and \( \eta \) as follows:

\[
T_1 = \frac{1}{4} \left[ \xi + \eta + (g_3 - g_1)(1 + \xi \eta) \right],
\]

\[
T_2 = \frac{1}{4} \left[ -\xi + \eta + (g_4 - g_2)(1 - \xi \eta) \right].
\]

(14)

3. Definition of the Drilling DOFs

As shown in Figure 5, Long et al. [23–25] defined the drilling DOFs \( \theta_i \) as the additional rigid rotations at the element nodes.

The characteristics of this definition are as follows.

(1) The change of the angle between two adjacent sides along with the element deformation is allowed.

(2) The rotation \( \theta \) of the element side has definite relation with the nodal drilling freedom \( \theta \).

In this definition, the displacement fields within the domain of an element are assumed to include two parts:

\[
u = u^0 + u^\theta = \left\{ \begin{array}{c} u^0 \\ \phi \end{array} \right\} + \left\{ \begin{array}{c} u^\theta \\ \phi \end{array} \right\},
\]

(15)

where \( u^0 \) are the displacement fields determined by the nodal translational displacements and \( u^\theta \) are the additional displacement fields only determined by the vertex rigid rotations.

The element nodal displacement vector \( \mathbf{q}^e \) is defined by

\[
\mathbf{q}^e = [u_1 \ v_1 \ \theta_1 \ u_2 \ v_2 \ \theta_2 \ u_3 \ v_3 \ \theta_3 \ u_4 \ v_4 \ \theta_4]^T.
\]

(16)

4. Formulations of the New Elements

QAC4\( \theta \) and QAC4\( \theta \)M

According to the definition of drilling DOFs, the element boundary displacement can be assumed as

\[
\mathbf{u}_{ij} = u_{ij}^0 + \mathbf{u}_{\theta ij},
\]

\[
\mathbf{v}_{ij} = v_{ij}^0 + \mathbf{v}_{\theta ij},
\]

(17)

(\( ij = 12, 23, 34, 41 \)),

where the translational displacements \( \overline{u}_{ij} \) and \( \overline{v}_{ij} \) can be interpolated by the nodal displacements

\[
\begin{align*}
\begin{bmatrix} \overline{u}_{12}^0 \\ \overline{v}_{12}^0 \end{bmatrix} & = -T_1 \left\{ \begin{array}{c} u_1 \\ \phi_1 \end{array} \right\} - T_2 \left\{ \begin{array}{c} u_2 \\ \phi_2 \end{array} \right\}, \\
\begin{bmatrix} \overline{u}_{23}^0 \\ \overline{v}_{23}^0 \end{bmatrix} & = -T_2 \left\{ \begin{array}{c} u_2 \\ \phi_2 \end{array} \right\} + T_1 \left\{ \begin{array}{c} u_3 \\ \phi_3 \end{array} \right\},
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} \overline{u}_{13}^0 \\ \overline{v}_{13}^0 \end{bmatrix} & = -T_1 \left\{ \begin{array}{c} u_1 \\ \phi_1 \end{array} \right\} + T_3 \left\{ \begin{array}{c} u_3 \\ \phi_3 \end{array} \right\}, \\
\begin{bmatrix} \overline{u}_{34}^0 \\ \overline{v}_{34}^0 \end{bmatrix} & = -T_3 \left\{ \begin{array}{c} u_3 \\ \phi_3 \end{array} \right\} - T_2 \left\{ \begin{array}{c} u_2 \\ \phi_2 \end{array} \right\}.
\end{align*}
\]
Figure 5: DOFs of a membrane element.

(a) Translational freedom

\[
\begin{bmatrix}
\tilde{u}_{11} \\
\tilde{u}_{12} \\
\tilde{u}_{21} \\
\tilde{u}_{22} \\
\tilde{u}_{31} \\
\tilde{u}_{32} \\
\tilde{u}_{41} \\
\tilde{u}_{42}
\end{bmatrix} = T_1 \begin{bmatrix} 1 & u_3 \\ g_3 & v_3 \end{bmatrix} + T_2 \begin{bmatrix} u_4 \\ g_4 & v_4 \end{bmatrix},
\]

(b) Drilling freedom

\[
\begin{bmatrix}
\tilde{\theta}_{12} \\
\tilde{\theta}_{23} \\
\tilde{\theta}_{34} \\
\tilde{\theta}_{41}
\end{bmatrix} = T_2 \begin{bmatrix} u_4 \\ g_4 & v_4 \end{bmatrix} - T_1 \begin{bmatrix} u_1 \\ g_1 & v_1 \end{bmatrix}.
\]

The element boundary displacements caused by the additional vertex rigid rotations can be assumed by using the QACM-III:

\[
\begin{bmatrix}
\tilde{u}_{12} \\
\tilde{u}_{21} \\
\tilde{u}_{34} \\
\tilde{u}_{43}
\end{bmatrix} = T_1 T_2 \begin{bmatrix} \frac{u_2}{g_2 g_1} \\ \frac{u_4}{g_4 g_1} \\ \frac{u_3}{g_3 g_1} \\ \frac{u_1}{g_1 g_1} \end{bmatrix},
\]

\[
\begin{bmatrix}
\tilde{\theta}_{12} \\
\tilde{\theta}_{23} \\
\tilde{\theta}_{34} \\
\tilde{\theta}_{41}
\end{bmatrix} = T_1 T_2 \begin{bmatrix} \frac{-T_2 (\theta_4)}{g_4 g_2} - \frac{T_1 (\theta_1)}{g_1 g_2} \\ \frac{T_1 (\theta_3)}{g_3 g_1} - \frac{T_2 (\theta_2)}{g_2 g_1} \\ \frac{T_1 (\theta_4)}{g_4 g_3} - \frac{T_2 (\theta_2)}{g_2 g_3} \\ \frac{T_1 (\theta_1)}{g_1 g_4} - \frac{T_2 (\theta_3)}{g_3 g_4} \end{bmatrix}.
\]

(18)

The element displacement fields can be assumed in QACM-III as follows:

\[
u = \alpha_1 + \alpha_2 T_1 + \alpha_3 T_2 + \alpha_4 T_1 T_2 + \alpha_5 T_1^2 + \alpha_6 T_2^2,
\]

\[
\nu = \beta_1 + \beta_2 T_1 + \beta_3 T_2 + \beta_4 T_1 T_2 + \beta_5 T_1^2 + \beta_6 T_2^2.
\]

(21)

In order to determine the constant \( \alpha_i \), six generalized conforming conditions are introduced:

\[
\sum_{i=1}^{4} (u - \bar{u})_i = 0,
\]

\[
\sum_{i=1}^{4} \xi_i \eta_i (u - \bar{u})_i = 0,
\]

\[
\int_{s_{ij}} (u - \bar{u}) d\bar{s} = 0 \quad (ij = 23, 34, 41, 12).
\]

(22)

The conforming conditions for \( \nu \) are similar to those for \( u \).

Then, the constants \( \alpha_i \) and \( \beta_i \) in (21) can be solved:

\[
\alpha = L^{-1} R_n q_n,
\]

\[
\beta = L^{-1} R_v q_v.
\]

(23)
where

\[
\begin{align*}
q_u &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^T \\
q_v &= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^T \\
L &= \begin{bmatrix}
4g_3 - g_1 & g_4 - g_2 & 0 & g_1^2 + g_3^2 & g_2^2 + g_4^2 \\
0 & g_3 - g_1 & g_4 - g_2 & 0 & g_1^2 + g_3^2 & -g_2^2 - g_4^2 \\
1 & \frac{g_3}{2} & -\frac{g_2}{2} & \frac{g_2g_3}{6} & \frac{g_1^2}{3} & \frac{g_3^2}{3} \\
1 & \frac{g_3}{2} & \frac{g_4}{2} & \frac{g_2g_3}{6} & \frac{g_1^2}{3} & \frac{g_3^2}{3} \\
1 & -\frac{g_3}{2} & \frac{g_4}{2} & \frac{g_2g_3}{6} & \frac{g_1^2}{3} & \frac{g_3^2}{3} \\
1 & \frac{g_3}{2} & -\frac{g_2}{2} & \frac{g_2g_3}{6} & \frac{g_1^2}{3} & \frac{g_3^2}{3} \\
\end{bmatrix}
\end{align*}
\]

\[
R_u = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & b_1 & -b_1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & b_2 & -b_2 \\
1 & 0 & 0 & 1 & 0 & -b_3 & 0 & b_3 \\
1 & 1 & 2 & 0 & 0 & b_4 & -b_4 & 0 \\
\end{bmatrix}
\]

\[
R_v = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & c_1 & -c_1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & c_2 & -c_2 \\
1 & 0 & 0 & 1 & 0 & -c_3 & 0 & c_3 \\
1 & 1 & 2 & 0 & 0 & c_4 & -c_4 & 0 \\
\end{bmatrix}
\]

And the shape functions of additional displacement fields related to the vertex rigid rotations are

\[
N_{ad} = H_{u,3} + H_{u,3}T_1 + H_{u,3}T_2 + H_{u,5}T_1T_2
\]

\[
+ H_{u,6}T_1^2 + H_{u,6}T_2^2
\]

\[
(i = 1, 2, 3, 4)
\]

\[
N_{a\theta} = H_{v,3} + H_{v,3}T_1 + H_{v,3}T_2 + H_{v,4}T_1T_2
\]

\[
+ H_{v,5}T_1^2 + H_{v,6}T_2^2
\]

\[
i = 1, 2, 3, 4, \quad j = i + 4, \quad j > 4.
\]

The element strain fields are given by

\[
e = B_q q',
\]

where

\[
B_q = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix},
\]

\[
B_i = \begin{bmatrix}
\frac{\partial N_i^0}{\partial x} & 0 & \frac{\partial N_{ad}^i}{\partial x} & 0 \\
0 & \frac{\partial N_i^0}{\partial y} & \frac{\partial N_{ad}^i}{\partial y} & \frac{\partial N_{ad}^i}{\partial y + \partial x} \\
\frac{\partial N_i^0}{\partial y} & \frac{\partial N_i^0}{\partial x} & \frac{\partial N_{ad}^i}{\partial x + \partial y} & \frac{\partial N_{ad}^i}{\partial x} \\
\end{bmatrix}
\]

\[
(i = 1, 2, 3, 4)
\]

Let

\[
H_u = L^{-1} R_u, \quad H_v = L^{-1} R_v
\]

Thus, the displacement fields can be written in the following:

\[
u = \sum_{i=1}^{4} N_i^0 u_i + \sum_{i=1}^{4} N_{ad}^i \theta_i, \quad v = \sum_{i=1}^{4} N_i^0 v_i + \sum_{i=1}^{4} N_{ad}^i \theta_i
\]

where the shape functions of translational displacements are

\[
N_i^0 = H_{u,li} + H_{u,2i}T_1 + H_{u,3i}T_2 + H_{u,4i}T_1T_2
\]

\[
+ H_{u,5i}T_1^2 + H_{u,6i}T_2^2
\]

\[
(i = 1, 2, 3, 4)
\]
\[ \frac{\partial N_{a1i}}{\partial x} = \frac{H_{a2j}}{2A} \left[ \frac{y_4 - y_2}{x_2 - x_4} \right] + \frac{H_{a3j}}{2A} \left[ \frac{y_1 - y_3}{x_3 - x_1} \right] \]

\[ + \frac{H_{a4j}}{A} \left[ \frac{(y_4 - y_2) T_2 + (y_1 - y_3) T_1}{(x_2 - x_4) T_2 + (x_3 - x_1) T_1} \right] \]

\[ + \frac{H_{a5j}}{A} \left[ \frac{(y_1 - y_3) T_1}{(x_3 - x_1) T_2} \right] \]

Then, the corresponding strain vector can be written as

\[ \epsilon_A = \begin{pmatrix} \epsilon_{ax} \\ \epsilon_{ay} \end{pmatrix} = B_A \begin{pmatrix} \lambda_1 \\ \lambda_1' \end{pmatrix} = B_A \lambda_1, \] (36)

where

\[ B_A = \begin{bmatrix} \frac{\partial N_{a1}}{\partial x} & 0 \\ 0 & \frac{\partial N_{a1}}{\partial y} \end{bmatrix}, \]

k^e = k_{yj} - k_{yj}^T k_{\lambda\lambda}^{-1} k_{\lambda j}, \] (38)

And \( \lambda'_i \) (i = 1 \sim 5) have similar relations. Thus, the shape functions of this additional displacement field can be obtained:

\[ N_{a1} = Z_1^2 + Z_2^2 + \frac{2(g_1 - g_2)}{3} Z_1 + \frac{2(g_1 - g_2)}{3} Z_2 \]

\[ + \frac{2(g_1 g_3 + g_2 g_4) - 5}{3} \lambda_1. \] (35)

5. Numerical Examples

Seven benchmark problems, which are listed in Table 1, have been used for evaluating the performance of the elements. The results solved by the other 14 element models listed in Table 2 are also given for comparison.

Example 1 (patch test). The constant strain/stress patch test using irregular mesh is shown in Figure 6. Let Young's modulus \( E = 1000 \), Poisson's ratio \( \mu = 0.25 \), and thickness of the patch \( t = 1 \). Both QAC4\( \theta \) and QAC4\( \theta M \) can present exact solutions.
8 Mathematical Problems in Engineering

1.50.52

\( E = 1000 \) \( \mu = 0.25 \)

Figure 6: Patch test of constant stress/strain state.

Table 1: List of benchmark problems.

<table>
<thead>
<tr>
<th>Number</th>
<th>Benchmark problems (figure number)</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Patch test (Figure 6)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Cook’s skew beam (Figure 7)</td>
<td>Table 3</td>
</tr>
<tr>
<td>3</td>
<td>Beam divided by five quadrilateral elements (Figure 8)</td>
<td>Table 4</td>
</tr>
<tr>
<td>4</td>
<td>Beam divided by four quadrilateral elements (Figure 9)</td>
<td>Table 5</td>
</tr>
<tr>
<td>5</td>
<td>MacNeal’s thin beam (Figure 10)</td>
<td>Table 6</td>
</tr>
<tr>
<td>6</td>
<td>Thin curving beam (Figure 11)</td>
<td>Table 7</td>
</tr>
<tr>
<td>7</td>
<td>Beam divided by two elements with distortion parameter (Figure 12)</td>
<td>Table 8</td>
</tr>
</tbody>
</table>

Table 2: List of element models for comparison.

<table>
<thead>
<tr>
<th>Number</th>
<th>Element model</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Q4</td>
<td>4-node isoparametric element</td>
</tr>
<tr>
<td>2</td>
<td>Q6</td>
<td>4-node isoparametric element with internal parameters Wilson et al. [55]</td>
</tr>
<tr>
<td>3</td>
<td>QM6</td>
<td>4-node isoparametric element with internal parameters Taylor et al. [56]</td>
</tr>
<tr>
<td>4</td>
<td>P-S</td>
<td>Hybrid stress element Pian and Sumihara [57]</td>
</tr>
<tr>
<td>5</td>
<td>QUAD4</td>
<td>4-node element in MSC/NASTRAN MacNeal and Harder [32]</td>
</tr>
<tr>
<td>6</td>
<td>Q4S</td>
<td>Membrane element with drilling DOFs MacNeal and Harder [8]</td>
</tr>
<tr>
<td>7</td>
<td>GQ12</td>
<td>Membrane element with drilling DOFs Long and Xu [23]</td>
</tr>
<tr>
<td>8</td>
<td>GQ12M8</td>
<td>Membrane element with drilling DOFs Long and Xu [23]</td>
</tr>
<tr>
<td>9</td>
<td>D-type</td>
<td>Membrane element with drilling DOFs Ibrahimbegovic et al. [11]</td>
</tr>
<tr>
<td>10</td>
<td>Groenwoldt995</td>
<td>Membrane element with drilling DOFs Groenwold and Stander [58]</td>
</tr>
<tr>
<td>11</td>
<td>AQR8</td>
<td>Refined hybrid element Aminpour [59]</td>
</tr>
<tr>
<td>12</td>
<td>RGD20</td>
<td>Membrane element with drilling DOFs Chen and Cheung [60]</td>
</tr>
<tr>
<td>13</td>
<td>Q8</td>
<td>8-node isoparametric element</td>
</tr>
</tbody>
</table>

Example 2 (Cook’s skew beam). This example, in which a skew cantilever with shear distributed load at the free edge, as shown in Figure 7, was proposed by Cook et al. [61]. The results of vertical deflection at point C, the maximum principal stress at point A, and the minimum principal stress at point B are listed in Table 3.

Example 3 (cantilever beam divided by five quadrilateral elements). The cantilever beam, as shown in Figure 8, is divided by five irregular quadrilateral elements. And two loading cases are considered: (a) pure bending under moment \( M \) and (b) linear bending under transverse force \( P \). Young’s modulus \( E = 1500 \), and Poisson’s ratio \( \nu = 0.25 \). The results of the vertical deflection \( v_A \) at point A and the stress \( \sigma_{xB} \) at point B are given in Table 4.

Example 4 (cantilever beam divided by five quadrilateral elements). As shown in Figure 9, the cantilever beam is
Table 3: Results of Cook’s beam.

<table>
<thead>
<tr>
<th>Element</th>
<th>(V_C)</th>
<th>(\sigma_{A_{\text{max}}})</th>
<th>(\sigma_{B_{\text{min}}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4</td>
<td>11.80</td>
<td>11.80</td>
<td>10.23</td>
</tr>
<tr>
<td>Q6</td>
<td>21.61</td>
<td>21.61</td>
<td>19.93</td>
</tr>
<tr>
<td>D-type</td>
<td>20.68</td>
<td>20.68</td>
<td>19.36</td>
</tr>
<tr>
<td>GQ12</td>
<td>20.89</td>
<td>20.89</td>
<td>19.19</td>
</tr>
<tr>
<td>GQ12M8</td>
<td>22.49</td>
<td>22.49</td>
<td>20.81</td>
</tr>
<tr>
<td>QAC4(\theta)</td>
<td>21.00</td>
<td>21.00</td>
<td>19.26</td>
</tr>
<tr>
<td>QAC4(\theta) M</td>
<td>22.25</td>
<td>22.25</td>
<td>21.57</td>
</tr>
<tr>
<td>Reference values</td>
<td>23.96</td>
<td>0.2362</td>
<td>-0.2023</td>
</tr>
</tbody>
</table>

Table 4: Results of cantilever beam with five elements.

<table>
<thead>
<tr>
<th>Elements</th>
<th>Load I</th>
<th>Load II</th>
</tr>
</thead>
<tbody>
<tr>
<td>(V_C)</td>
<td>(\sigma_{A_{\text{max}}})</td>
<td>(\sigma_{B_{\text{min}}})</td>
</tr>
<tr>
<td>Q4</td>
<td>45.7</td>
<td>-1761</td>
</tr>
<tr>
<td>Q6</td>
<td>98.4</td>
<td>-2428</td>
</tr>
<tr>
<td>GQ12</td>
<td>95.5</td>
<td>-2989</td>
</tr>
<tr>
<td>GQ12M8</td>
<td>100.0</td>
<td>-3000</td>
</tr>
<tr>
<td>QAC4(\theta)</td>
<td>100.0</td>
<td>-3000</td>
</tr>
<tr>
<td>QAC4(\theta) M</td>
<td>100.0</td>
<td>-3000</td>
</tr>
<tr>
<td>Exact</td>
<td>100.0</td>
<td>-3000</td>
</tr>
</tbody>
</table>

Example 5 (MacNeal’s beam). Consider the thin beams presented in Figure 10. Three different mesh shapes, rectangular, parallelogram, and trapezoidal, are adopted. This example, proposed by MacNeal and Harder [32], is a famous
Table 5: Results of cantilever with four elements.

<table>
<thead>
<tr>
<th>Elements</th>
<th>Tip deflections</th>
<th>Normalized values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Point A</td>
<td>Point B</td>
</tr>
<tr>
<td>D-type</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Q4S</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Groenwold1995</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>GQ12</td>
<td>0.3337</td>
<td>0.3324</td>
</tr>
<tr>
<td>GQ12M</td>
<td>0.3420</td>
<td>0.3404</td>
</tr>
<tr>
<td>QAC4θ</td>
<td>0.3523</td>
<td>0.3516</td>
</tr>
<tr>
<td>QAC4θM</td>
<td>0.3523</td>
<td>0.3516</td>
</tr>
<tr>
<td>Reference value</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6: The normalized results of the tip deflection for MacNeal’s beam.

<table>
<thead>
<tr>
<th>Elements</th>
<th>Load P Mesh (a)</th>
<th>Load P Mesh (b)</th>
<th>Load P Mesh (c)</th>
<th>Load M Mesh (a)</th>
<th>Load M Mesh (b)</th>
<th>Load M Mesh (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4</td>
<td>0.093</td>
<td>0.035</td>
<td>0.003</td>
<td>0.093</td>
<td>0.031</td>
<td>0.022</td>
</tr>
<tr>
<td>Q6</td>
<td>0.993</td>
<td>0.677</td>
<td>0.106</td>
<td>1.000</td>
<td>0.759</td>
<td>0.093</td>
</tr>
<tr>
<td>QM6</td>
<td>0.993</td>
<td>0.623</td>
<td>0.044</td>
<td>1.000</td>
<td>0.722</td>
<td>0.037</td>
</tr>
<tr>
<td>QUAD4</td>
<td>0.904</td>
<td>0.080</td>
<td>0.071</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>P-S</td>
<td>0.993</td>
<td>0.798</td>
<td>0.221</td>
<td>1.000</td>
<td>0.852</td>
<td>0.167</td>
</tr>
<tr>
<td>RGD20</td>
<td>0.981</td>
<td>0.625</td>
<td>0.047</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>AQR8</td>
<td>0.993</td>
<td>0.986</td>
<td>0.977</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Q4S</td>
<td>0.993</td>
<td>0.986</td>
<td>0.988</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>QAC4θ</td>
<td>0.904</td>
<td>0.867</td>
<td>0.906</td>
<td>0.910</td>
<td>0.8804</td>
<td>0.930</td>
</tr>
<tr>
<td>Exact</td>
<td>1.000 (−0.1081)</td>
<td></td>
<td></td>
<td>1.000 (−0.0054)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7: The tip deflection of a thin curving beam.

<table>
<thead>
<tr>
<th>h/R</th>
<th>Q4</th>
<th>QM6</th>
<th>QUAD4</th>
<th>GQ12</th>
<th>GQ12M</th>
<th>QAC4θ</th>
<th>QAC4θM</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>0.024</td>
<td>0.339</td>
<td>0.615</td>
<td>0.670</td>
<td>0.897</td>
<td>0.712</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.006</td>
<td>0.001</td>
<td>0.022</td>
<td>0.163</td>
<td>0.612</td>
<td>0.896</td>
<td>0.645</td>
<td>1.008</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 8: Results of the tip deflection of a cantilever beam with distorted parameter ε.

<table>
<thead>
<tr>
<th>ε</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>4.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4</td>
<td>28.0</td>
<td>21.0</td>
<td>14.1</td>
<td>9.7</td>
<td>8.3</td>
<td>7.2</td>
<td>6.2</td>
</tr>
<tr>
<td>Q8</td>
<td>100</td>
<td>100</td>
<td>99.3</td>
<td>89.3</td>
<td>59.7</td>
<td>31.6</td>
<td>19.0</td>
</tr>
<tr>
<td>QM6</td>
<td>100</td>
<td>80.9</td>
<td>62.7</td>
<td>54.4</td>
<td>53.6</td>
<td>51.2</td>
<td>46.8</td>
</tr>
<tr>
<td>P-S</td>
<td>100</td>
<td>81.0</td>
<td>62.9</td>
<td>55.0</td>
<td>54.7</td>
<td>53.1</td>
<td>49.8</td>
</tr>
<tr>
<td>GQ12</td>
<td>100</td>
<td>97.9</td>
<td>86.3</td>
<td>48.7</td>
<td>24.9</td>
<td>13.3</td>
<td>8.0</td>
</tr>
<tr>
<td>GQ12M</td>
<td>100</td>
<td>98.7</td>
<td>93.9</td>
<td>74.1</td>
<td>51.0</td>
<td>33.4</td>
<td>23.3</td>
</tr>
<tr>
<td>QAC4θ</td>
<td>100</td>
<td>99.9</td>
<td>98.9</td>
<td>99.8</td>
<td>102.0</td>
<td>102.2</td>
<td>100.3</td>
</tr>
<tr>
<td>QAC4θM</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Example 6 (thin curving beam). As shown in Figure 11, a cantilever thin curving beam is subjected to a transverse force at the tip. And it is also divided by five elements. Two thickness-radius ratios, (i) h/R = 0.03 and (ii) h/R = 0.006, are considered. The results of the tip displacement are listed in Table 7.

Example 7 (cantilever beam divided by two elements containing a parameter of distortion). The cantilever beam shown in Figure 12 is divided by two elements. The shape of the two elements varies with the distorted parameter ε. When ε = 0, both elements are rectangular. But with the increase of ε, the mesh will be distorted more and more seriously. This is another famous benchmark for testing the sensitivity to the mesh distortion. For pure bending problem, the results of the tip deflection at point A are listed in Table 8.
6. Conclusions

In this paper, two membrane elements with drilling DOFs, named QAC4θ and QAC4θM, are developed by using the quadrilateral area coordinate methods QACM-II and QACM-III. In their formulations, the additional rigid rotations at the element nodes are considered as the drilling DOFs, so that these two elements can allow the change of the angle between two adjacent sides along with the element deformations. Furthermore, since the quadrilateral area coordinates can keep the order of the Cartesian coordinates unchangeable while the mesh is distorted, the new elements exhibit better performance than other similar models and insensitivity to mesh distortion. It is demonstrated again that the quadrilateral area coordinate methods are effective tools for developing high-performance quadrilateral finite element models.

Conflict of Interests

The authors declare no conflict of interests regarding the publication of this paper.

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