Research Article

Stability of Switched Server Systems with Constraints on Service-Time and Capacity of Buffers

Li Wang, Zhonghe He, and Chi Zhang

Beijing Key Lab of Urban Intelligent Traffic Control Technology, North China University of Technology, Beijing 100144, China

Correspondence should be addressed to Zhonghe He; zhonghehe@ncut.edu.cn

Received 6 February 2015; Revised 18 May 2015; Accepted 20 May 2015

Academic Editor: Sebastian Anita

Copyright © 2015 Li Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The execution of emptying policy ensures the convergence of any solution to the system to a unique periodic orbit, which does not impose constraints on service-time and capacity of buffers. Motivated by these problems, in this paper, the service-time-limited policy is first proposed based on the information resulted from the periodic orbit under emptying policy, which imposes lower and upper bounds on emptying time for the queue in each buffer, by introducing lower-limit and upper-limit service-time factors. Furthermore, the execution of service-time-limited policy in the case of finite buffer capacity is considered. Moreover, the notion of feasibility of states under service-time-limited policy is introduced and then the checking condition for feasibility of states is given; that is, the solution does not exceed the buffer capacity within the first cycle of the server. At last, a sufficient condition for determining upper-limit service-time factors ensuring that the given state is feasible is given.

1. Introduction

Switched server system is a class of mathematical models for queuing systems with finite number of conflicting queues alternately served by a single server. Moreover, there exists a nonzero setup time of the server whenever the server switches from serving one queue to another one, and assume that the jobs arrive at and leave the buffer at constant rates in this paper. The evolution of the system involves continuous changes of queues in buffers and discrete switching of the server, and thus switched server system is a special class of hybrid systems [1, 2], with extensive applications in practical problems, such as manufacturing systems [3, 4] and traffic signal control systems [5–7], and more applications of this field can be referred to [8].

Fundamental synthesis problem for switched server systems is to design the scheduling policy of the server. The emptying policy (i.e., the server alternately empties queues in buffers with any fixed cyclic sequence) was proposed in [9], under which any solution to the system asymptotically converges to a unique periodic orbit analytically determined by system parameters [6]. However, the emptying policy does not impose constraints on queue-emptying time in

converging process of the solution. In practical applications, the server with emptying policy must take longer time to empty buffers with larger queues, and thus other buffers have to wait longer time for service. Thus, in order to ensure fairness for all buffers, the upper bound for emptying time of each buffer based on emptying policy was considered in [10], and a conjecture about stability of the policy was given, which was further proved in [11]. Also, [12] considered distributed execution of emptying policy with upper bounds for queue-emptying time of buffers in the network with multiple servers. In most of literatures, a scheduling policy is first proposed, and then dynamic behaviors of the system are analyzed, as in [9]. In [13–16], a different idea for controlling the network was presented; that is, the steady state (a periodic orbit) of the system is first given, and then corresponding scheduling policy is derived ensuring the convergence of any solution to the steady state. However, the policies in [13–16] resulting from the given periodic orbit do not impose constraints on service time of buffers.

The problems about designs of the scheduling policies with constraints on queue serving process mainly result from practical applications. For example, in traffic intersection, the signal control for signalized intersections was modeled as

switched server systems in [5, 6], and emptying policy was applied, where signal light in a signalized intersection is seen as the server; incoming links to the signalized intersection are seen as buffers, which can accommodate queues of vehicles; the lost time between phase switching is seen as the nonzero setup time of the server; and signal control law is seen as the scheduling policy of the server. However, in traffic control [17], the shortest and longest green-time constraints on each of traffic phases are necessarily imposed for feasible signal control plans, with the purpose of ensuring traffic safety for drivers and pedestrians, and controlling total delay of signalized intersections, respectively. Thus, inspired by traffic control, the emptying policy is further extended in this paper, based on which the service-time-limited policy is proposed, with lower and upper bounds on queue-emptying time of each buffer by introducing lower-limit and upper-limit service-time factors, respectively. Furthermore, the buffer capacity is finite for most of real-world problems. For example, in a signalized intersection, incoming links with finite length only accommodates finite number of vehicles. Thus, the execution of service-time-limited policy in the case of finite buffer capacities is considered, and moreover the notion of feasibility of states under service-time-limited policy is introduced, that is, the state originating in which the solution asymptotically converges to the steady state (the periodic orbit) and does not exceed buffer capacities in the converging process. Moreover, the checking condition for feasibility of states is given; that is, the solution does not exceed buffer capacities within the first cycle of the server, and a sufficient condition for determining upper-limit service-time factors ensuring that the given state is feasible is given.

The paper is organized as follows. After descriptions for the model of switched server systems in Section 2, we introduce emptying and service-time-limited policies in Section 3. Feasibility of states and checking conditions under service-time-limited policy are considered in Section 4. Conclusions and future research topics are given in Section 5.

2. Descriptions of Switched Server Systems

A switched server system (see Figure 1 for illustration) consists of n ($n \geq 2$) buffers and a single server, where the server alternately serves buffers in terms of the scheduling policy and only one buffer each time. Let $x_i(t) \geq 0$ denote the queue of jobs in the buffer i at the moment $t \geq 0$. Because of nonnegative constraints on the queue of jobs in each buffer, the state space X of the system is defined as $X \triangleq \{[x_1, \dots, x_n]^T \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$. Assume that the jobs arrive at the buffer i at a constant rate $q_i > 0$ [lots/s]. Whenever the buffer i , in which there are accumulative queues, that is, $x_i(t) > 0$, is served by the server, the jobs leave the buffer i at a constant rate $s_i > 0$ [lots/s]; and whenever the buffer i , in which there are no accumulative queues, that is, $x_i(t) = 0$, is served by the server, the jobs leave the buffer i at the constant rate q_i . Both q_i and s_i are called arriving rate and service rate of jobs in the buffer i , respectively, and $q_i s_i^{-1}$ is called the load of the buffer i . Whenever the server switches from serving the buffer i to

FIGURE 1: A switched server system with n buffers.

the buffer j , there exists a nonzero setup time $l_{ij} > 0$, $i, j = 1, \dots, n$, $i \neq j$ [s], during which the server is in idle.

In terms of above descriptions for switched server systems, the dynamics of the queues of jobs in buffers can be described by the following.

Whenever the buffer i with $x_i(t) > 0$ is served by the server,

$$\dot{x}(t) = q - s_i e_i. \quad (1)$$

Whenever the buffer i with $x_i(t) = 0$ is served by the server,

$$\dot{x}(t) = q - q_i e_i. \quad (2)$$

Whenever the server switches from serving one buffer to another one,

$$\dot{x}(t) = q, \quad (3)$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in X$, $q = [q_1, \dots, q_n]^T$, and $e_i \in \mathbb{R}^n$ is n -dimensional unit vector; that is, the i th element of e_i equals one and other elements of e_i are zero.

In the subsequent parts, we assume that the total load of buffers satisfies

$$\sum_{j=1}^n q_j s_j^{-1} < 1. \quad (4)$$

Obviously, there is no equilibrium in the system described by (1), (2), and (3), and the periodic orbit depending on the scheduling policy is the steady-state of the system, which attracts other trajectories of the system. It was proved in [15] that the inequality (4) is the sufficient and necessary condition for the existence of stable scheduling policy for the system.

3. Stability of Scheduling Policy

In this section, stability analysis of two scheduling policies, that is, emptying and service-time-limited policies, is presented, where the service-time-limited policy admits service-time constraints on buffers based on emptying policy.

3.1. Emptying Policy. The emptying policy is described as follows:

- (1) The buffers are served by the server in terms of any cyclic sequence, for example, $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$.
- (2) Whenever the server switches from serving the buffer i to the buffer $i + 1$ ($i = 1, \dots, n - 1$), there exists a nonzero setup time $l_{i,i+1} > 0$; and whenever the server switches from serving the buffer n to the buffer 1, the setup time is $l_{n1} > 0$.
- (3) When the buffer $i \in \{1, \dots, n\}$ is being served, the service-time $\tilde{g}_i(k)$ for the queue is given by

$$\tilde{g}_i(k) = \frac{x_i(t_k^i)}{s_i - q_i}, \quad (5)$$

where t_k^i , $k = 1, 2, \dots$, denotes the moment the server starts serving the buffer i within the k th cycle of the server, $x_i(t_k^i)$ denotes the queue of jobs in the buffer i at the moment t_k^i , and then $x_i(t_k^i)(s_i - q_i)^{-1}$ is the service-time for emptying the queue $x_i(t_k^i)$ in the buffer i .

From the statements in emptying policy, the server, with nonzero setup times, empties queues in buffers in terms of cyclic sequence. The following results hold.

Theorem 1 (see [6]). *Consider the switched server system described by (1), (2), and (3) under emptying policy. Assume that the total load of buffers satisfies (4). Then, the following statements hold:*

- (1) *There exists a unique periodic orbit $x^p(t) = [x_1^p(t), \dots, x_n^p(t)]^T$ to the system, which is globally asymptotically stable with respect to the state space X .*
- (2) *The period C of the periodic orbit $x^p(t)$ is given by*

$$C = \frac{L}{1 - \sum_{j=1}^n q_j s_j^{-1}}, \quad (6)$$

where $L \triangleq l_{12} + \dots + l_{n-1,n} + l_{n1}$ is the total idle time within one cycle of the server.

- (3) *For the periodic orbit $x^p(t)$, the service-time g_i for the queue in the buffer i is given by*

$$g_i = q_i s_i^{-1} C. \quad (7)$$

Remark 2. The periodic orbit in Theorem 1 is denoted by $x^p(t)$ in the succeeding parts. It is derived from (6) and (7) in Theorem 1, that the periodic orbit $x^p(t)$ can be uniquely determined by given system parameters, and satisfy $C = \sum_{j=1}^n g_j + L$. Importantly, from (7) in Theorem 1, the significance of the periodic orbit $x^p(t)$ is that within the period C and the total number of jobs arriving at the system is exactly equal to the total number of jobs leaving the system at service rates. Specifically, if the signalized intersection is modeled as a switched server system, inequality (4) is

the undersaturated condition for signalized intersections, and the period C is the minimum signal cycle (refer to detailed discussions in [6]). Moreover, the consensus problems (i.e., states of the system can converge to a common value by local protocol) have become fundamental investigations in coordinated control of multiagent systems, due to extensive applications in engineering fields (e.g., refer to [18, 19]). In the sense of traffic control, the saturation level of some direction is defined as the ratio of total number of vehicles arriving at and leaving the intersection. From the significance of the periodic orbit $x^p(t)$, saturation levels are equal in different directions. Then, the emptying policy can realize the consensus of saturation levels in traffic control, implying the balance of traffic loads in different directions. Thus, the periodic orbit $x^p(t)$ has practical meanings in applications to traffic control.

3.2. Service-Time-Limited Policy. The emptying policy does not restrict service-time for buffers. However, the problem of constraints on service-time of buffers is of importance in practical applications, as stated in Introduction. In this subsection, the service-time-limited policy is presented based on emptying policy, which can be described by the following.

The first two terms (1) and (2) are the same as those in descriptions of emptying policy; and (3) in emptying policy is replaced by the following:

- (3') When the buffer $i \in \{1, \dots, n\}$ is being served, the service-time $\tilde{g}_i(k)$ for the queue is given by

$$\tilde{g}_i(k) = \begin{cases} \underline{g}_i, & \text{If } \frac{x_i(t_k^i)}{s_i - q_i} < \underline{g}_i \\ \frac{x_i(t_k^i)}{s_i - q_i}, & \text{If } \underline{g}_i \leq \frac{x_i(t_k^i)}{s_i - q_i} \leq \bar{g}_i \\ \bar{g}_i, & \text{If } \frac{x_i(t_k^i)}{s_i - q_i} > \bar{g}_i, \end{cases} \quad (8)$$

where $\underline{g}_i \triangleq g_i - q_i s_i^{-1} \Gamma_{i,\min}$ and $\bar{g}_i \triangleq g_i + q_i s_i^{-1} \Gamma_{i,\max}$ are, respectively, the shortest and longest service-time assigned to the buffer i , where g_i is given by (7), and both $\Gamma_{i,\min}$ and $\Gamma_{i,\max}$ are, respectively, called service-time lower-limit and upper-limit factors, satisfying $0 < \Gamma_{i,\min} < C$ and $\Gamma_{i,\max} > 0$.

The information resulted from C and g_i of the periodic orbit $x^p(t)$ determined in Theorem 1 is utilized for the design of service-time-limited policy. From (8), the service-time $\tilde{g}_i(k)$ of the buffer i within the k th cycle is, respectively, restricted by the shortest service-time $\underline{g}_i$ and longest service-time $\bar{g}_i$. If the queue-emptying time $x_i(t_k^i)(s_i - q_i)^{-1}$ is less than $\underline{g}_i$ assigned to the buffer i , then $\tilde{g}_i(k) = \underline{g}_i$. In this case, the serving process of the buffer i is as follows: the queue in the buffer i is first served at the service-rate s_i until the queue is emptied (refer to dynamics in (1)) and then the buffer i is served at the arriving-rate q_i until the shortest service-time $\underline{g}_i$ ends (refer to dynamics in (2)). If the queue-emptying time $x_i(t_k^i)(s_i - q_i)^{-1}$ is more than $\bar{g}_i$ assigned to the buffer i ,

then $\bar{g}_i(k) = \bar{g}_i$. Otherwise, the queue $x_i(t_k^i)$ in the buffer i is emptied, and the server switches to the next buffer.

Consider the following inequality:

$$\sum_{j=1}^n q_j s_j^{-1} < \min \left\{ \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j, \min}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j, \min}\}}, \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j, \max}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j, \max}\}} \right\}. \quad (9)$$

If $\Gamma_{i, \min}$ and $\Gamma_{i, \max}$ satisfy $0 < \Gamma_{1, \min} = \dots = \Gamma_{n, \min} < C$ and $\Gamma_{1, \max} = \dots = \Gamma_{n, \max} > 0$, then (9) is the same as (4).

The following results hold for switched server systems under service-time-limited policy.

Theorem 3. Consider the switched server system described by (1), (2), and (3) under service-time-limited policy. Assume that the total load of buffers satisfies (9). Then, any solution to the system asymptotically converges to the periodic orbit $x^P(t)$.

The proof of Theorem 3 can be referred to the appendix. Furthermore, consider the following two special cases for service-time-limited policy:

(C1) $\Gamma_{1, \min} = \dots = \Gamma_{n, \min} = 0$, $\Gamma_{i, \max} > 0$, $i = 1, \dots, n$; that is, $\underline{g}_i = g_i$, $\bar{g}_i > g_i$, $i = 1, \dots, n$.

(C2) $\Gamma_{1, \min} = \dots = \Gamma_{n, \min} = C$, $\Gamma_{i, \max} > 0$, $i = 1, \dots, n$; that is, $\underline{g}_i = 0$, $\bar{g}_i > g_i$, $i = 1, \dots, n$.

Consider the following inequality:

$$\sum_{j=1}^n q_j s_j^{-1} < \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j, \max}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j, \max}\}}. \quad (10)$$

If $\Gamma_{1, \max} = \dots = \Gamma_{n, \max} > 0$ is satisfied, then (10) is the same as (4).

Theorem 4. Consider the switched server system described by (1), (2), and (3) under service-time-limited policy with factors satisfying (C1) or (C2). Assume that the total load of buffers satisfies (10). Then, any solution to the system asymptotically converges to the periodic orbit $x^P(t)$.

Proof. When applying service-time-limited policy with factors satisfying (C1), the statements in Theorem 4 can be derived by setting $\Gamma_{i, \min} = 0$, $i = 1, \dots, n$, in the proof of Theorem 3; and when applying service-time-limited policy with factors satisfying (C2), the statements in Theorem 4 can be derived by Cases 1 and 3 in the proof of Theorem 3. In above two cases, (9) in the proof of Theorem 3 is changed to (10). $\square$

4. Feasibility of Service-Time-Limited Policy

Based on emptying policy, service-time-limited policy admits service-time constraints on buffers by introducing service-time lower-limit and upper-limit factors $\Gamma_{i, \min}$ and $\Gamma_{i, \max}$, $i = 1, \dots, n$, but does not bring constraints on the buffer capacity.

However, the buffer capacity is finite for most of practical problems. Thus, we furthermore consider the execution of service-time-limited policy in case of finite buffer capacity.

Let $x_i^{\max} > 0$, $i = 1, \dots, n$, be the capacity of the buffer i , defined as the maximum queue of jobs that the buffer i can accommodate. Then, the *admissible region* $M \subset X$ of the system is denoted as $M \triangleq [0, x_1^{\max}] \times \dots \times [0, x_n^{\max}]$.

It is derived, from the significance of the periodic orbit $x^P(t)$, that the maximum queue of jobs in the buffer i is given by $q_i(C - g_i)$ within the period C . Assume that the periodic orbit $x^P(t)$ lies inside the admissible region M ; that is,

$$x_i^{\max} > q_i(C - g_i), \quad i = 1, \dots, n. \quad (11)$$

Definition 5. Consider the switched server system described by (1), (2), and (3) under service-time-limited policy. The state $x_0 \in M$ is called *feasible* if for the given service-time lower-limit factors $\Gamma_{i, \min}^f$, $i = 1, \dots, n$, there exist service-time upper-limit factors $\Gamma_{i, \max}^c > 0$, $i = 1, \dots, n$, such that the solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ originating in x_0 asymptotically converges to the periodic orbit $x^P(t)$ and moreover satisfies $x(t) \in M$, $\forall t \geq 0$.

Furthermore, it is deduced from (11) that there must exist service-time upper-limit factors $\Gamma_{i, \max} > 0$, $i = 1, \dots, n$, satisfying the following inequalities:

$$\begin{aligned} x_i^{\max} &\geq q_i \left[(C - g_i) + \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j, \max} \right] \\ &= q_i \left[\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j, \max}) + L \right], \quad i = 1, \dots, n. \end{aligned} \quad (12)$$

Inequalities (12) indicate that when the queue of jobs in the buffer i is emptied, the queue of jobs in the buffer i does not exceed the buffer capacity after one cycle of the server. The factors $\Gamma_{i, \max}^c$, $i = 1, \dots, n$, satisfying (12) are noted as $\Gamma_{i, \max}^c$, $i = 1, \dots, n$, in the following parts.

Theorem 6. Consider the switched server system described by (1), (2), and (3) under service-time-limited policy with given $\Gamma_{i, \min}^f$ and $\Gamma_{i, \max}^c > 0$, $i = 1, \dots, n$. Assume that the total load of buffers satisfies (9) (or (10) if all $\Gamma_{i, \min}^f$, $i = 1, \dots, n$ satisfy (C1) or (C2)), and the state $x_0 \in M$ has the property that the solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ originating in the state x_0 satisfies the condition $[x_1(t_1^1), \dots, x_n(t_1^n)]^T \in M$, where t_1^i is the moment the server starts serving the buffer i within the first cycle of the server. Then, the state $x_0 \in M$ is feasible with respect to $\Gamma_{i, \max}^c > 0$, $i = 1, \dots, n$.

The proof of Theorem 6 can be referred to the appendix. It is derived from Theorem 6 that the checking condition for feasibility of the state is that the corresponding solution does not exceed the buffer capacity within the first cycle of the server with given $\Gamma_{i, \min}^f$ and $\Gamma_{i, \max}^c$, $i = 1, \dots, n$. Accordingly, the feasible region $X_0[\Gamma_{i, \min}^f, \Gamma_{i, \max}^c] \subseteq M$, that is, all of feasible states with respect to $\Gamma_{i, \min}^f$ and $\Gamma_{i, \max}^c$, $i =$

$1, \dots, n$, can be obtained from the checking condition for feasibility of the state. Specifically, analytic expression of feasible region $X_0[\Gamma_{i,\min}^f, \Gamma_{i,\max}^c]$ for switched server systems with two buffers can be easily determined as follows:

- (1) If $0 \leq \Gamma_{i,\min}^f < C$, $i = 1, 2$, then $X_0[\Gamma_{i,\min}^f, \Gamma_{i,\max}^c] = X_0^1 \cup X_0^2 \cup X_0^3$, where X_0^i , $i = 1, 2, 3$, are, respectively, given by

$$\begin{aligned} X_0^1 &= \left\{ x_0 \in M : \right. \\ &\quad \left. \begin{aligned} &x_1(0)(s_1 - q_1)^{-1} < g_1 - q_1 s_1^{-1} \Gamma_{1,\min}^f, \\ &x_2(0) + q_2 \left[(g_1 - q_1 s_1^{-1} \Gamma_{1,\min}^f) + l_{12} \right] \leq x_2^{\max} \end{aligned} \right\}; \\ X_0^2 &= \left\{ x_0 \in M : \right. \\ &\quad \left. \begin{aligned} &g_1 - q_1 s_1^{-1} \Gamma_{1,\min}^f \leq x_1(0)(s_1 - q_1)^{-1} \leq g_1 + q_1 s_1^{-1} \Gamma_{1,\max}^c, \\ &x_2(0) + q_2 \left[x_1(0)(s_1 - q_1)^{-1} + l_{12} \right] \leq x_2^{\max} \end{aligned} \right\}; \\ X_0^3 &= \left\{ x_0 \in M : \right. \\ &\quad \left. \begin{aligned} &x_1(0)(s_1 - q_1)^{-1} > g_1 + q_1 s_1^{-1} \Gamma_{1,\max}^c, \\ &x_2(0) + q_2 \left[(g_1 + q_1 s_1^{-1} \Gamma_{1,\max}^c) + l_{12} \right] \leq x_2^{\max} \end{aligned} \right\}. \end{aligned} \quad (13)$$

- (2) If $\Gamma_{i,\min}^f = C$, $i = 1, 2$, then $X_0[\Gamma_{i,\min}^f, \Gamma_{i,\max}^c] = X_0^1 \cup X_0^2$, where X_0^i , $i = 1, 2$, are respectively given by

$$\begin{aligned} X_0^1 &= \left\{ x_0 \in M : \right. \\ &\quad \left. \begin{aligned} &x_1(0)(s_1 - q_1)^{-1} \leq g_1 + q_1 s_1^{-1} \Gamma_{1,\max}^c, \\ &x_2(0) + q_2 \left[x_1(0)(s_1 - q_1)^{-1} + l_{12} \right] \leq x_2^{\max} \end{aligned} \right\}; \\ X_0^2 &= \left\{ x_0 \in M : \right. \\ &\quad \left. \begin{aligned} &x_1(0)(s_1 - q_1)^{-1} > g_1 + q_1 s_1^{-1} \Gamma_{1,\max}^c, \\ &x_2(0) + q_2 \left[(g_1 + q_1 s_1^{-1} \Gamma_{1,\max}^c) + l_{12} \right] \leq x_2^{\max} \end{aligned} \right\}. \end{aligned} \quad (14)$$

From Theorem 6, feasibility of the state depends on choices of factors $\Gamma_{i,\min}^f$ and $\Gamma_{i,\max}^c$, $i = 1, \dots, n$. However, infeasibility of the state with respect to some given factors $\Gamma_{i,\min}^f$ and $\Gamma_{i,\max}^c$, $i = 1, \dots, n$, does not imply inexistence of factors ensuring the state is feasible. Furthermore, we consider the problem of how to solve factors $\Gamma_{i,\max}^c$, $i = 1, \dots, n$, such that the given state is feasible with given $\Gamma_{i,\min}^f$, $i = 1, \dots, n$.

If service-time-limited policy is applied with given $\Gamma_{i,\min}^f = 0$ or $0 < \Gamma_{i,\min}^f < C$, $i = 1, \dots, n$, in terms of the check-

ing condition for feasibility of states in Theorem 6, the given state $x_0 = [x_1(0), \dots, x_n(0)]^T \in M$ is infeasible if at least one of the following inequalities holds:

$$\begin{aligned} x_i(0) &> x_i^{\max} \\ &- q_i \left[\sum_{j=1}^{i-1} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + \sum_{j=1}^{i-1} l_{j,j+1} \right], \end{aligned} \quad (15)$$

$i = 2, \dots, n.$

The significance of (15) is that the queue in the buffer $i \geq 2$ exceeds the buffer capacity even if all of buffers j , $j = 1, \dots, i-1$ are served with the shortest service-time within the first cycle of the server. Furthermore, a sufficient condition is given for determining $\Gamma_{i,\max}^c$, $i = 1, \dots, n$, ensuring the given state $x_0 \in M$ is feasible.

Proposition 7. Consider the switched server system described by (1), (2), and (3) under service-time-limited policy with given $\Gamma_{i,\min}^f$, $i = 1, \dots, n$. For the given state $x_0 \in M$, if the set $\Omega(x_0)$ of factors $[\Gamma_{1,\max}, \dots, \Gamma_{n,\max}]^T$ given by (16) is nonempty and there exists $[\Gamma_{1,\max}^c, \dots, \Gamma_{n,\max}^c]^T \in \Omega(x_0)$, such that the total load of buffers satisfies (9) (or (10) if all $\Gamma_{i,\min}^f$, $i = 1, \dots, n$, satisfy (C1) or (C2)), then the given state $x_0 \in M$ is feasible with respect to $[\Gamma_{1,\max}^c, \dots, \Gamma_{n,\max}^c]^T$:

$$\begin{aligned} (a) \quad &\begin{cases} \sum_{j \neq 1} q_j s_j^{-1} \Gamma_{j,\max} \leq \frac{x_1^{\max}}{q_1} - \sum_{j \neq 1} g_j - L \\ \vdots \\ \sum_{j \neq n} q_j s_j^{-1} \Gamma_{j,\max} \leq \frac{x_n^{\max}}{q_n} - \sum_{j \neq n} g_j - L. \end{cases} \\ (b) \quad &\begin{cases} q_1 s_1^{-1} \Gamma_{1,\max} \leq \frac{[x_2^{\max} - x_2(0)]}{q_2} - g_1 - l_{12} \\ \vdots \\ \sum_{j=1}^{n-1} q_j s_j^{-1} \Gamma_{j,\max} \leq \frac{[x_n^{\max} - x_n(0)]}{q_n} - \sum_{j=1}^{n-1} g_j - \sum_{j=1}^{n-1} l_{j,j+1} \end{cases} \\ (c) \quad &\begin{cases} 0 < \Gamma_{1,\max} \leq \Gamma_{1,\max}^* \\ \vdots \\ 0 < \Gamma_{n,\max} \leq \Gamma_{n,\max}^*. \end{cases} \end{aligned} \quad (16)$$

Proof. Inequalities (a) in (16) imply that (12) holds; and we can derive from (b) in (16) that

$$\begin{aligned} x_2(t_1^2) &\leq x_2(0) + q_2 \left[(g_1 + q_1 s_1^{-1} \Gamma_{1,\max}) + l_{12} \right] \\ &\leq x_2^{\max} \\ &\vdots \\ x_n(t_1^n) &\leq x_n(0) \end{aligned} \quad (17)$$

$$+ q_n \left[\sum_{j=1}^{n-1} (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + \sum_{j=1}^{n-1} l_{j,j+1} \right] \leq x_n^{\max},$$

which indicates that the solution does not exceed the buffer capacity within the first cycle of the server. Furthermore, $\Gamma_{i,\max}^*$, $i = 1, \dots, n$, in (c) are the maximum allowable service-time upper-limit factors. Thus, from Theorem 6, the given state $x_0 \in M$ is feasible. $\square$

5. Conclusions

For most of real-world problems about queuing systems, service-times and queues of buffers must be constrained. In this paper, inspired by practical problems in traffic control, the service-time-limited policy is proposed, which is the extension to emptying policy. Moreover, the execution of service-time-limited policy in the case of finite buffer capacities is considered, and the notion of feasibility of states under service-time-limited policy is presented. Furthermore, based on the checking condition for feasibility of states (i.e., the solution does not exceed buffer capacities within the first cycle of the server), a sufficient condition for determining feasibility of states is given.

The scheduling policy proposed in this paper admits taking into consideration service-time and queue constraints on buffers by the introduction of the notion of feasibility of states, and service-time upper-limit factors for the feasible state can be solved by testing the nonempty set $\Omega(x_0)$. Thus, our results can be applied to traffic control as stated in the Introduction, especially in critical saturation case; for example, the length of queues of vehicles on incoming links may be larger, with lower traffic loads satisfying (4). Signal control of T-shape intersection is typical application of our results, which can be referred to [6] for details.

From views of traffic control, the server may serve multiple nonconflicting flows, which is our further research extension of results in the paper.

Appendix

Proof of Theorem 3. Assume that t_k^i and T_k^i , respectively, represent moments that the server starts and finishes serving the queue in the buffer i in terms of service-time-limited policy, within the k th cycle of the server, $\forall i \in \{1, \dots, n\}$, $k = 1, 2, \dots$. Then, t_1^i is the moment that the server starts serving the buffer i within the first cycle of the server. Consider the following three possible cases for any solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ to the system originating in the initial state $x_0 \in X$:

Case 1. $g_i - q_i s_i^{-1} \Gamma_{i,\min} \leq x_i(t_1^i)(s_i - q_i)^{-1} \leq g_i + q_i s_i^{-1} \Gamma_{i,\max}$.

Case 2. $x_i(t_1^i)(s_i - q_i)^{-1} < g_i - q_i s_i^{-1} \Gamma_{i,\min}$.

Case 3. $x_i(t_1^i)(s_i - q_i)^{-1} > g_i + q_i s_i^{-1} \Gamma_{i,\max}$.

We prove that the solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ asymptotically converges to the periodic orbit $x^p(t)$ in any case above.

Case 1. If the queue-emptying time $x_i(t_1^i)(s_i - q_i)^{-1}$ of the buffer i satisfies

$$\begin{aligned} g_i - q_i s_i^{-1} \Gamma_{i,\min} &\leq x_i(t_1^i)(s_i - q_i)^{-1} \\ &\leq g_i + q_i s_i^{-1} \Gamma_{i,\max}, \end{aligned} \quad (\text{A.1})$$

then, the queue-emptying time $x_i(t_m^i)(s_i - q_i)^{-1}$, $\forall m \geq 1$ of the buffer i within any cycle satisfies

$$\begin{aligned} g_i - q_i s_i^{-1} \Gamma_{i,\min} &\leq x_i(t_m^i)(s_i - q_i)^{-1} \\ &\leq g_i + q_i s_i^{-1} \Gamma_{i,\max}. \end{aligned} \quad (\text{A.2})$$

Proof of Case 1. We prove Case 1 by using mathematical induction. From (A.1), Case 1 holds with $m = 1$. Furthermore, assume that Case 1 holds with some $m \geq 1$, then, in terms of service-time-limited policy, we have that $x_i(T_m^i) = 0$ and

$$\begin{aligned} q_i \left[\sum_{j \neq i} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + L \right] \\ \leq x_i(t_{m+1}^i) \leq q_i \left[\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L \right]. \end{aligned} \quad (\text{A.3})$$

The emptying time for the queue $q_i [\sum_{j \neq i} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + L]$ satisfies

$$\begin{aligned} &\frac{q_i [\sum_{j \neq i} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + L]}{s_i - q_i} \\ &= \frac{q_i [\sum_{j \neq i} g_j + L] - q_i \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\min}}{s_i - q_i} \\ &= \frac{q_i [C - g_i] - q_i \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\min}}{s_i - q_i} \\ &= \frac{s_i g_i - q_i g_i}{s_i - q_i} - \frac{q_i s_i^{-1} \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\min}}{1 - q_i s_i^{-1}} \\ &\geq g_i - \frac{q_i s_i^{-1} \max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\} \sum_{j \neq i} q_j s_j^{-1}}{1 - q_i s_i^{-1}}. \end{aligned} \quad (\text{A.4})$$

From (9), we have that

$$\begin{aligned} &\sum_{j \neq i} q_j s_j^{-1} + q_i s_i^{-1} \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}} \\ &\leq \sum_{j=1}^n q_j s_j^{-1} < \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}}. \end{aligned} \quad (\text{A.5})$$

Then,

$$\begin{aligned} &\sum_{j \neq i} q_j s_j^{-1} < [1 - q_i s_i^{-1}] \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}} \\ &\leq [1 - q_i s_i^{-1}] \frac{\Gamma_{i,\min}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}}. \end{aligned} \quad (\text{A.6})$$

Substitute (A.6) into (A.4); we have that

$$\begin{aligned} \frac{x_i(t_{m+1}^i)}{s_i - q_i} &\geq \frac{q_i [\sum_{j \neq i} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + L]}{s_i - q_i} \\ &\geq g_i - q_i s_i^{-1} \Gamma_{i,\min}. \end{aligned} \quad (\text{A.7})$$

The emptying time for the queue $q_i [\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L]$ satisfies

$$\begin{aligned} &\frac{q_i [\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L]}{s_i - q_i} \\ &= \frac{q_i [\sum_{j \neq i} g_j + L] + q_i \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\max}}{s_i - q_i} \\ &= \frac{q_i [C - g_i] + q_i \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\max}}{s_i - q_i} \\ &= \frac{s_i g_i - q_i g_i}{s_i - q_i} + \frac{q_i s_i^{-1} \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\max}}{1 - q_i s_i^{-1}} \\ &\leq g_i + \frac{q_i s_i^{-1} \max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\} \sum_{j \neq i} q_j s_j^{-1}}{1 - q_i s_i^{-1}}. \end{aligned} \quad (\text{A.8})$$

From (9), we have that

$$\begin{aligned} &\sum_{j \neq i} q_j s_j^{-1} + q_i s_i^{-1} \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}} \\ &\leq \sum_{j=1}^n q_j s_j^{-1} < \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}}. \end{aligned} \quad (\text{A.9})$$

Then,

$$\begin{aligned} \sum_{j \neq i} q_j s_j^{-1} &< [1 - q_i s_i^{-1}] \frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}} \\ &\leq [1 - q_i s_i^{-1}] \frac{\Gamma_{i,\max}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\max}\}}. \end{aligned} \quad (\text{A.10})$$

Substitute (A.10) into (A.8); we have that

$$\begin{aligned} \frac{x_i(t_{m+1}^i)}{s_i - q_i} &\leq \frac{q_i [\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L]}{s_i - q_i} \\ &\leq g_i + q_i s_i^{-1} \Gamma_{i,\max}. \end{aligned} \quad (\text{A.11})$$

Thus, from (A.7) and (A.11), Case 1 holds with $m + 1$. Then, Case 1 holds by induction. Here the proof of Case 1 ends.

Case 2. If the queue-emptying time $x_i(t_1^i)(s_i - q_i)^{-1}$ of the buffer i satisfies

$$x_i(t_1^i)(s_i - q_i)^{-1} < g_i - q_i s_i^{-1} \Gamma_{i,\min}, \quad (\text{A.12})$$

then, there must exist $k_0 > 1$ such that

$$\begin{aligned} g_i - q_i s_i^{-1} \Gamma_{i,\min} &\leq x_i(t_{k_0+m}^i)(s_i - q_i)^{-1} \\ &\leq g_i + q_i s_i^{-1} \Gamma_{i,\max}, \quad \forall m \geq 0. \end{aligned} \quad (\text{A.13})$$

Proof of Case 2. After one cycle of the server from time t_1^i , we have that

$$x_i(t_2^i) = x_i(t_1^i) + A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i), \quad (\text{A.14})$$

where $A_i(t_1^i, t_2^i) > 0$ and $D_i(t_1^i, t_2^i) > 0$ are total amounts of jobs arriving at and leaving the buffer i within one cycle, respectively.

From (A.12), $A_i(t_1^i, t_2^i)$ and $D_i(t_1^i, t_2^i)$ in (A.14), respectively, satisfy

$$A_i(t_1^i, t_2^i) \geq q_i \left[\sum_{j=1}^n (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + L \right], \quad (\text{A.15})$$

$$D_i(t_1^i, t_2^i) < s_i (g_i - q_i s_i^{-1} \Gamma_{i,\min}).$$

Then, the increment $A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i)$ in the buffer i satisfies

$$\begin{aligned} &A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i) \\ &> q_i \left[\sum_{j=1}^n (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + L \right] \\ &\quad - s_i (g_i - q_i s_i^{-1} \Gamma_{i,\min}) \\ &= \left[q_i \left(\sum_{j=1}^n g_j + L \right) - s_i g_i \right] \\ &\quad + q_i \left[\Gamma_{i,\min} - \sum_{j=1}^n q_j s_j^{-1} \Gamma_{j,\min} \right] = [q_i C - s_i g_i] \\ &\quad + q_i \left[\Gamma_{i,\min} - \sum_{j=1}^n q_j s_j^{-1} \Gamma_{j,\min} \right] \\ &= q_i \left[\Gamma_{i,\min} - \sum_{j=1}^n q_j s_j^{-1} \Gamma_{j,\min} \right] \\ &\geq q_i \max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\} \\ &\quad \cdot \left[\frac{\min_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}}{\max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\}} - \sum_{j=1}^n q_j s_j^{-1} \right]. \end{aligned} \quad (\text{A.16})$$

Thus, from (9) and (A.14), we have that $A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i) > 0$ and $x_i(t_2^i)(s_i - q_i)^{-1} > x_i(t_1^i)(s_i - q_i)^{-1}$. From analogous procedures above, we can derive the following conclusions that if

$$x_i(t_p^i)(s_i - q_i)^{-1} < g_i - q_i s_i^{-1} \Gamma_{i,\min}, \quad p = 1, \dots, k, \quad (\text{A.17})$$

then, $\{x_i(t_p^i)(s_i - q_i)^{-1}\}_{p=1}^{k+1}$ is a strictly monotonic increasing sequence, which indicates that there must exist $k_0 \geq 2$, such that

$$x_i(t_{k_0-1}^i)(s_i - q_i)^{-1} < g_i - q_i s_i^{-1} \Gamma_{i,\min}, \quad (\text{A.18})$$

$$x_i(t_{k_0}^i)(s_i - q_i)^{-1} \geq g_i - q_i s_i^{-1} \Gamma_{i,\min}. \quad (\text{A.19})$$

In terms of service-time-limited policy, (A.18) and (A.8), we have that

$$\begin{aligned} x_i(t_{k_0}^i)(s_i - q_i)^{-1} &\leq \frac{q_i \left[\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L \right]}{s_i - q_i} \\ &\leq g_i + q_i s_i^{-1} \Gamma_{i,\max}. \end{aligned} \quad (\text{A.20})$$

Then, Case 2 can be obtained from (A.19), (A.20), and results in Case 1. Here the end of proof of Case 2.

Case 3. If the queue-emptying time $x_i(t_1^i)(s_i - q_i)^{-1}$ of the buffer i satisfies

$$x_i(t_1^i)(s_i - q_i)^{-1} > g_i + q_i s_i^{-1} \Gamma_{i,\max}, \quad (\text{A.21})$$

Then, there must exist $k_0 > 1$ such that

$$\begin{aligned} g_i - q_i s_i^{-1} \Gamma_{i,\min} &\leq x_i(t_{k_0+m}^i)(s_i - q_i)^{-1} \\ &\leq g_i + q_i s_i^{-1} \Gamma_{i,\max}, \quad \forall m \geq 0. \end{aligned} \quad (\text{A.22})$$

Proof of Case 3. After one cycle of the server from time t_1^i , in terms of (A.21) and service-time-limited policy, $A_i(t_1^i, t_2^i)$ and $D_i(t_1^i, t_2^i)$ in (A.14), respectively, satisfy

$$A_i(t_1^i, t_2^i) \leq q_i \left[\sum_{j=1}^n (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L \right], \quad (\text{A.23})$$

$$D_i(t_1^i, t_2^i) = s_i (g_i + q_i s_i^{-1} \Gamma_{i,\max}).$$

Then, the increment $A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i)$ in the buffer i satisfies

$$\begin{aligned} A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i) &\leq q_i \left[\sum_{j=1}^n (g_j + q_j s_j^{-1} \Gamma_{j,\max}) + L \right] \\ &\quad - s_i (g_i + q_i s_i^{-1} \Gamma_{i,\max}) = [q_i C - s_i g_i] \end{aligned}$$

$$\begin{aligned} &+ q_i \left[\sum_{j=1}^n q_j s_j^{-1} \Gamma_{j,\max} - \Gamma_{i,\max} \right] \\ &= q_i \left[\sum_{j=1}^n q_j s_j^{-1} \Gamma_{j,\max} - \Gamma_{i,\max} \right] \\ &\leq q_i \max_{j \in \{1, \dots, n\}} \{ \Gamma_{j,\max} \} \\ &\quad \cdot \left[\sum_{j=1}^n q_j s_j^{-1} - \frac{\min_{j \in \{1, \dots, n\}} \{ \Gamma_{j,\max} \}}{\max_{j \in \{1, \dots, n\}} \{ \Gamma_{j,\max} \}} \right]. \end{aligned} \quad (\text{A.24})$$

Thus, from (9) and (A.14), we have that $A_i(t_1^i, t_2^i) - D_i(t_1^i, t_2^i) < 0$ and $x_i(t_2^i)(s_i - q_i)^{-1} < x_i(t_1^i)(s_i - q_i)^{-1}$. From analogous procedures above, we can derive the following conclusions that if

$$x_i(t_p^i)(s_i - q_i)^{-1} > g_i + q_i s_i^{-1} \Gamma_{i,\max}, \quad p = 1, \dots, k, \quad (\text{A.25})$$

then, $\{x_i(t_p^i)(s_i - q_i)^{-1}\}_{p=1}^{k+1}$ is a strictly monotonic decreasing sequence, which indicates that there must exist $k_0 \geq 2$, such that

$$x_i(t_{k_0-1}^i)(s_i - q_i)^{-1} > g_i + q_i s_i^{-1} \Gamma_{i,\max}, \quad (\text{A.26})$$

$$x_i(t_{k_0}^i)(s_i - q_i)^{-1} \leq g_i + q_i s_i^{-1} \Gamma_{i,\max}. \quad (\text{A.27})$$

After one cycle of the server from time $t_{k_0-1}^i$, we have that

$$\begin{aligned} x_i(t_{k_0}^i) &= x_i(t_{k_0-1}^i) + A_i(t_{k_0-1}^i, t_{k_0}^i) \\ &\quad - D_i(t_{k_0-1}^i, t_{k_0}^i), \end{aligned} \quad (\text{A.28})$$

where from (A.26), $A_i(t_{k_0-1}^i, t_{k_0}^i)$ and $D_i(t_{k_0-1}^i, t_{k_0}^i)$, respectively, satisfy

$$\begin{aligned} A_i(t_{k_0-1}^i, t_{k_0}^i) &\geq q_i \left[\sum_{j \neq i} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) \right. \\ &\quad \left. + (g_i + q_i s_i^{-1} \Gamma_{i,\max}) + L \right], \\ D_i(t_{k_0-1}^i, t_{k_0}^i) &= s_i (g_i + q_i s_i^{-1} \Gamma_{i,\max}). \end{aligned} \quad (\text{A.29})$$

Then, the increment $A_i(t_{k_0-1}^i, t_{k_0}^i) - D_i(t_{k_0-1}^i, t_{k_0}^i)$ in the buffer i satisfies

$$\begin{aligned} & A_i(t_{k_0-1}^i, t_{k_0}^i) - D_i(t_{k_0-1}^i, t_{k_0}^i) \\ & \geq q_i \left[\sum_{j \neq i} (g_j - q_j s_j^{-1} \Gamma_{j,\min}) + (g_i + q_i s_i^{-1} \Gamma_{i,\max}) \right. \\ & \quad \left. + L \right] - s_i (g_i + q_i s_i^{-1} \Gamma_{i,\max}) = q_i \Gamma_{i,\max} [q_i s_i^{-1} \\ & \quad - 1] - q_i \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\min}. \end{aligned} \quad (\text{A.30})$$

Then,

$$\begin{aligned} & \frac{A_i(t_{k_0-1}^i, t_{k_0}^i) - D_i(t_{k_0-1}^i, t_{k_0}^i)}{s_i - q_i} \\ & \geq \frac{q_i \Gamma_{i,\max} [q_i s_i^{-1} - 1] - q_i \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\min}}{s_i - q_i} \\ & = \frac{q_i s_i^{-1} \Gamma_{i,\max}}{1 - q_i s_i^{-1}} [q_i s_i^{-1} - 1] \\ & \quad - \frac{q_i s_i^{-1}}{1 - q_i s_i^{-1}} \sum_{j \neq i} q_j s_j^{-1} \Gamma_{j,\min} \\ & \geq -q_i s_i^{-1} \Gamma_{i,\max} \\ & \quad - \frac{q_i s_i^{-1}}{1 - q_i s_i^{-1}} \max_{j \in \{1, \dots, n\}} \{\Gamma_{j,\min}\} \sum_{j \neq i} q_j s_j^{-1}. \end{aligned} \quad (\text{A.31})$$

Substitute (A.6) into (A.31), we have that

$$\begin{aligned} & \frac{A_i(t_{k_0-1}^i, t_{k_0}^i) - D_i(t_{k_0-1}^i, t_{k_0}^i)}{s_i - q_i} \\ & \geq -q_i s_i^{-1} \Gamma_{i,\max} - q_i s_i^{-1} \Gamma_{i,\min}. \end{aligned} \quad (\text{A.32})$$

From (A.28), (A.26), and (A.32),

$$x_i(t_{k_0}^i)(s_i - q_i)^{-1} \geq g_i - q_i s_i^{-1} \Gamma_{i,\min}. \quad (\text{A.33})$$

Then, Case 3 can be obtained from (A.27), (A.33), and results in Case 1. Here ends the proof of Case 3.

In conclusion, for any one of three possible cases, the service-time-limited policy converges to emptying policy. Thus, from results in Theorem 1, the solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ asymptotically converges to the periodic orbit $x^P(t)$. $\square$

Proof of Theorem 6. Consider switched server systems under service-time-limited policy with $0 < \Gamma_{i,\min}^f < C, \Gamma_{i,\max}^c > 0$, $i = 1, \dots, n$. We first prove the following statement.

Statement 1. If the state $x_0 \in M$ has the property stated in Theorem 6, then, the condition $[x_1(t_k^1), \dots, x_n(t_k^n)]^T \in M$, $\forall k \geq 1$, holds.

Proof of Statement 1. We prove the results in Statement 1 by using mathematical induction. In the case of $k = 1$, Statement 1 holds because of the property of the state $x_0 \in M$. Furthermore, assume that Statement 1 holds for some $k = k_0$, $k_0 \geq 1$; that is, $[x_1(t_{k_0}^1), \dots, x_n(t_{k_0}^n)]^T \in M$. Consider three possible cases for any buffer $i \in \{1, \dots, n\}$.

Case 1. If the queue-emptying time $x_i(t_{k_0}^i)(s_i - q_i)^{-1}$ satisfies $x_i(t_{k_0}^i)(s_i - q_i)^{-1} < g_i - q_i s_i^{-1} \Gamma_{i,\min}^f$, then, in terms of service-time-limited policy, we have that

$$x_i(t_{k_0+1}^i) \leq q_i \left[\sum_{j \neq i} (g_j + q_j s_j^{-1} \Gamma_{j,\max}^c) + L \right]. \quad (\text{A.34})$$

It is derived, from (12) and (A.34), that $x_i(t_{k_0+1}^i) \leq x_i^{\max}$.

Case 2. If the queue-emptying time $x_i(t_{k_0}^i)(s_i - q_i)^{-1}$ satisfies

$$g_i - q_i s_i^{-1} \Gamma_{i,\min}^f \leq x_i(t_{k_0}^i)(s_i - q_i)^{-1} \leq g_i + q_i s_i^{-1} \Gamma_{i,\max}^c, \quad (\text{A.35})$$

then, in terms of service-time-limited policy, (A.34) still holds. Thus, we have that $x_i(t_{k_0+1}^i) \leq x_i^{\max}$.

Case 3. If the queue-emptying time $x_i(t_{k_0}^i)(s_i - q_i)^{-1}$ satisfies $x_i(t_{k_0}^i)(s_i - q_i)^{-1} > g_i + q_i s_i^{-1} \Gamma_{i,\max}^c$, then, from proof of Case 3 in proof of Theorem 3, we have that $x_i(t_{k_0+1}^i) < x_i(t_{k_0}^i) \leq x_i^{\max}$.

In conclusion, we have that $[x_1(t_{k_0+1}^1), \dots, x_n(t_{k_0+1}^n)]^T \in M$, which indicates that Statement 1 holds for $k = k_0 + 1$. By mathematical induction, Statement 1 holds for $\forall k \geq 1$. Here ends the proof of Statement 1.

Statement 1 immediately implies $x(t) = [x_1(t), \dots, x_n(t)]^T \in M$, $\forall t \geq 0$. Moreover, from Theorem 3, the solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ asymptotically converges to the periodic orbit $x^P(t)$. Then, from Definition 5, the state $x_0 \in M$ is feasible.

Furthermore, Statement 1 still holds for $\Gamma_{i,\min}^f, i = 1, \dots, n$, satisfying (C1) or (C2), which implies $x(t) = [x_1(t), \dots, x_n(t)]^T \in M$, $\forall t \geq 0$. Moreover, from Theorem 4, the solution $x(t) = [x_1(t), \dots, x_n(t)]^T$ asymptotically converges to the periodic orbit $x^P(t)$. Then, from Definition 5, the state $x_0 \in M$ is feasible. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the referees for their constructive comments. This work is partially supported by National Natural Science Foundation of China (51308005 and 61374191) and Scientific Research Project of Beijing Education Committee (PXM2015_014212_000023, PXM2015_014212_000018, PXM2015_014212_000019, and PXM2015_014212_000021).

References

- [1] A. S. Matveev and A. V. Savkin, *Qualitative Theory of Hybrid Dynamical Systems*, Birkhäuser, Boston, Mass, USA, 2000.
- [2] W. P. Heemels, B. De Schutter, J. Lunze, and M. Lazar, "Stability analysis and controller synthesis for hybrid dynamical systems," *Philosophical Transactions of the Royal Society of London. Series A*, vol. 368, no. 1930, pp. 4937–4960, 2010.
- [3] J. R. Perkins and P. R. Kumar, "Stable, distributed, real-time scheduling of flexible manufacturing/assembly/disassembly systems," *IEEE Transactions on Automatic Control*, vol. 34, no. 2, pp. 139–148, 1989.
- [4] J. R. Perkins, C. Humes Jr., and P. R. Kumar, "Distributed scheduling of flexible manufacturing systems: stability and performance," *IEEE Transactions on Robotics and Automation*, vol. 10, no. 2, pp. 133–141, 1994.
- [5] Y.-Z. Chen, H.-F. Li, and J. Ni, "Modeling and analysis of cyclic linear differential automata for T-intersection signal timing," *Control Theory & Applications*, vol. 28, no. 12, pp. 1773–1778, 2011.
- [6] Z. H. He, Y. Z. Chen, J. J. Shi, X. G. Han, and X. Wu, "Steady-state control for signalized intersections modeled as switched server system," in *Proceedings of the American Control Conference (ACC '13)*, pp. 842–847, Washington, DC, USA, June 2013.
- [7] M. A. A. Boon, I. J. B. F. Adan, E. M. M. Winands, and D. G. Down, "Delays at signalized intersections with exhaustive traffic control," *Probability in the Engineering and Informational Sciences*, vol. 26, no. 3, pp. 337–373, 2012.
- [8] M. A. A. Boon, R. D. van der Mei, and E. M. M. Winands, "Applications of polling systems," *Surveys in Operations Research and Management Science*, vol. 16, no. 2, pp. 67–82, 2011.
- [9] A. V. Savkin and A. S. Matveev, "Cyclic linear differential automata: a simple class of hybrid dynamical systems," *Automatica*, vol. 36, no. 5, pp. 727–734, 2000.
- [10] Z. G. Li, Y. C. Soh, and C. Y. Wen, *Switched and Impulsive Systems: Analysis, Design, and Applications*, Springer, Berlin, Germany, 2005.
- [11] Z.-H. He, Y.-Z. Chen, and J.-J. Shi, "Stability of switched server system and signal timing of intersection," *Control Theory & Applications*, vol. 30, no. 2, pp. 194–200, 2013.
- [12] A. V. Savkin and J. Somlo, "Optimal distributed real-time scheduling of flexible manufacturing networks modeled as hybrid dynamical systems," *Robotics and Computer-Integrated Manufacturing*, vol. 25, no. 3, pp. 597–609, 2009.
- [13] E. Lefeber and J. E. Rooda, "Controller design for switched linear systems with setups," *Physica A: Statistical Mechanics and Its Applications*, vol. 363, no. 1, pp. 48–61, 2006.
- [14] E. Lefeber and J. E. Rooda, "Controller design for flow networks of switched servers with setup times: the Kumar-Seidman case as an illustrative example," *Asian Journal of Control*, vol. 10, no. 1, pp. 55–66, 2008.
- [15] V. Feoktistova, A. Matveev, E. Lefeber, and J. E. Rooda, "Designs of optimal switching feedback decentralized control policies for fluid queueing networks," *Mathematics of Control, Signals, and Systems*, vol. 24, no. 4, pp. 477–503, 2012.
- [16] J. A. W. M. van Eekelen, E. Lefeber, and J. E. Rooda, "Feedback control of 2-product server with setups and bounded buffers," in *Proceedings of the American Control Conference*, pp. 544–549, 2006.
- [17] C. Diakaki, M. Papageorgiou, and K. Aboudolas, "A multivariable regulator approach to traffic-responsive network-wide signal control," *Control Engineering Practice*, vol. 10, no. 2, pp. 183–195, 2002.
- [18] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
- [19] Y. R. Ge, Y. Z. Chen, Y. X. Zhang, and Z. H. He, "State consensus analysis and design for high-order discrete-time linear multi-agent systems," *Mathematical Problems in Engineering*, vol. 2013, Article ID 192351, 13 pages, 2013.