Research Article

The Mean Stability Criteria in terms of Two Measures for Stochastic Differential Equations with Coefficient's Uncertainty

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This paper extends the stochastic stability criteria of two measures to the mean stability and proves the stability criteria for a kind of stochastic Itô's systems. Moreover, by applying optimal control approaches, the mean stability criteria in terms of two measures are also obtained for the stochastic systems with coefficient's uncertainty.

1. Introduction

Lyapunov's method, which makes an essential use of auxiliary functions (also called Lyapunov functions), is an important approach to study the stability of differential systems including ordinary differential equations (ODEs) and stochastic differential equations (SDEs). This method started in Lyapunov's original work in 1892 [1] for demonstrating stability of ODEs. In the 1960s, Movchan [2] studied the stability with two measures; such works were also developed and can be seen in [3]. In the past decades, Lyapunov's method is modified to the study of stability of Markovian processes [4], stochastic differential systems based on Brownian motions [5], semimartingales [6], or Lévy processes [7] and is also developed with the form of exponential stability [8] or LaSalle theorem [9], and so forth. Recently, the stability for systems with unknown parameters is also discussed [10], and the theorems of stability are widely applied in aerospace [11], state-feedback control [12], automatic control [13], neural networks [14], and other fields.

In this paper, we will discuss the following stochastic Itô's systems:

$$dx = f(t, x) dt + \sigma(t, x) dB_t, \quad (1)$$

$$x(t_0) = x_0, \quad t \geq t_0 \in \mathbb{R}_+,$$

where f, σ satisfy the usual Lipschitzian conditions and B is d -dimensional standard Brownian motion. It is well known that, for a stochastic process $x(t)$ and a given positive function $h(t, x)$, $\lim_{t \rightarrow \infty} h(t, x(\omega, t)) = 0$ almost surely discussed in [15] does not imply that $\lim_{t \rightarrow \infty} \mathbb{E}h(t, x(t)) \rightarrow 0$. So, we extend Lyapunov's methods used by [3] for ODEs to the stochastic cases and study the mean stability criteria in terms of two measures for system (1).

This paper is organized as follows: In Section 2, we first introduce Lyapunov's derivatives for (1) and deduce the basic comparison results in terms of Lyapunov's function. In Section 3, we prove the stochastic two-measure stability criteria for Itô systems, which can be seen as the extension of that of the ODEs. As described in [16], stability, robustness, and optimality can be considered systematically and simultaneously. In Section 4, the optimal control approach is extended to the stochastic systems with coefficient's uncertainty.

2. Basic Comparison Results for Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, P)$ be a given completed probability space, and $\{B_t\}_{t \geq 0}$ is a standard Brownian motion with filtration:

$$\mathcal{F}_t = \sigma(B_s, s \leq t). \quad (2)$$

Let $f \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}^n)$, $\sigma \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}^n \times \mathcal{R}^d)$ be deterministic functions and satisfy the following Lipschitz condition and linear growth condition: there exists $L > 0$, for every $x, y \in \mathcal{R}^n$, such that

$$\begin{aligned} \|f(t, x) - f(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| &\leq L \|x - y\|, \\ \|f(t, x)\| &\leq L(1 + \|x\|). \end{aligned} \quad (3)$$

For a given function $V \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R})$, $(t, x) \in \mathcal{R}_+ \times \mathcal{R}^n$, we denote

$$\begin{aligned} \mathcal{D}^+ V(t, x)_{(1)} &= \limsup_{h \rightarrow 0^+} \frac{1}{h} \\ &\cdot \mathbb{E} [V(t + h, x + hf(t, x) + \sqrt{h}\sigma(t, x)\xi) \\ &- V(t, x)], \end{aligned} \quad (4)$$

where ξ represents a d -dimensional random variable with standard normal distribution; that is, $\xi \sim N(0, I)$ and I is a d -order identity matrix.

Remark 1. We use the notation $\mathcal{D}^+ V(t, x)_{(1)}$ to emphasize the definition with respect to system (1). For convenience, we use the shortened form $\mathcal{D}^+ V$ to substitute $\mathcal{D}^+ V(t, x)_{(1)}$.

Remark 2. If $V \in C^{1,2}(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R})$, $\mathcal{D}^+ V(t, x)$ is Lyapunov's operator $\mathcal{L}$ associated with (1); that is,

$$\mathcal{D}^+ V_{(1)} = \mathcal{L}V := V_t + \langle V_x, f \rangle + \frac{1}{2} \langle \sigma \sigma^T, V_{xx} \rangle, \quad (5)$$

where V_t is the partial derivative for t , V_x is the gradient of V for x , and V_{xx} is the Hessian matrix of V for x .

Since (4) is dependent on expectation calculating, it is not easy to check whether $\mathcal{D}^+ V$ exists or not. The following lemma gives a condition for the existence of $\mathcal{D}^+ V$.

Lemma 3. Let $V \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R})$ be continuous one-order differentiable for t and also satisfy the following condition:

$$\begin{aligned} |V(t, x + y) - V(t, x) - \langle \alpha(t, x), y \rangle| &\leq K(t) \|y\|^2, \\ \forall x, y \in \mathcal{R}^n, \end{aligned} \quad (6)$$

where $\alpha(t, x)$ is continuous on $\mathcal{R}_+ \times \mathcal{R}^n$ and $K(t) \geq 0$ is locally bounded for t . Then $\mathcal{D}^+ V(t, x)$ exists at (t, x) .

Proof. Denote $y = hf(t, x) + \sqrt{h}\sigma(t, x)\xi$; then, for fixed t, x ,

$$\begin{aligned} &\frac{1}{h} \mathbb{E} [V(t + h, x + y) - V(t, x)] \\ &= \frac{1}{h} \mathbb{E} [V(t + h, x + y) - V(t + h, x)] \\ &\quad + \frac{1}{h} [V(t + h, x) - V(t, x)]. \end{aligned} \quad (7)$$

By (6) and distribution of ξ , we have the first item of the right side of (7) that is bounded:

$$\begin{aligned} &\langle \alpha(t, x), f(t, x) \rangle \\ &- K(t, x, y) \mathbb{E} \|\sqrt{h}f(t, x) + \sigma(t, x)\xi\|^2 \\ &\leq \frac{1}{h} \mathbb{E} [V(t + h, x + y) - V(t + h, x)] \\ &\leq \langle \alpha(t, x), f(t, x) \rangle \\ &- K(t) \mathbb{E} \|\sqrt{h}f(t, x) + \sigma(t, x)\xi\|^2. \end{aligned} \quad (8)$$

Since V is differentiable for t , so the last item of the right side of (7) is also bounded. Therefore, the supremum limit of (7) exists; that is, $\mathcal{D}^+ V(t, x)$ exist exactly. $\square$

The following lemmas will be used later.

Lemma 4 (see Theorem 7.1.2 in [6], or Theorems 4.3 and 4.4 in [17]). Suppose f, σ satisfy (3). Then, stochastic differential equation (1) admits a unique strong solution $\{x(t), t \geq 0\}$ such that, for any $h > 0$ ($[t, t + h] \subset [0, T]$, $l \geq 2$), there exists $(K_T$ is a constant dependent only on T, l , and L):

$$\mathbb{E} \sup_{t \leq s \leq t+h} \|x(s)\|^l \leq K_T (1 + \mathbb{E} \|x(t)\|^l), \quad (9)$$

$$\mathbb{E} \|x(r) - x(s)\|^l \leq K_T (1 + \mathbb{E} \|x(s)\|^l) |r - s|^{l/2}. \quad (10)$$

Lemma 5. Suppose V satisfies (6), and $x(t)$ is the solution of (1); let $m(t) = \mathbb{E} V(t, x(t))$; then

$$D^+ m(t) \leq \mathbb{E} \mathcal{D}^+ V(t, x(t)), \quad (11)$$

where $\mathcal{D}^+ V(t, x(t)) = \mathcal{D}^+ V(t, x)|_{x=x(t)}$ and $D^+ m(t)$ is the usual right upper Dini derivative defined by

$$D^+ m(t) = \limsup_{h \rightarrow 0^+} \frac{m(t + h) - m(t)}{h}. \quad (12)$$

Proof. For small $h > 0$, we have

$$\begin{aligned} m(t + h) - m(t) &= \mathbb{E} [V(t + h, x(t + h)) - V(t \\ &\quad + h, x(t) + hf(t, x(t)) + \sigma(t, x(t))(B_{t+h} - B_t)) \\ &\quad + V(t + h, x(t) + hf(t, x(t)) \\ &\quad + \sigma(t, x(t))(B_{t+h} - B_t)) - V(t, x(t))]. \end{aligned} \quad (13)$$

We now prove that the first two items of the right side are the higher infinitesimal of h . By (1), we know that

$$\begin{aligned} \hat{x} &:= x(t + h) \\ &- (x(t) + hf(t, x(t)) + \sigma(t, x(t))(B_{t+h} - B_t)) \\ &= \int_t^{t+h} [f(s, x(s)) - f(t, x(t))] ds \\ &\quad + \int_t^{t+h} [\sigma(s, x(s)) - \sigma(t, x(t))] dB_s. \end{aligned} \quad (14)$$

For convenience, we denote $\widehat{f}_s := f(s, x(s)) - f(t, x(t))$, similar meaning for $\widehat{\sigma}_s$. We have

$$\begin{aligned} \mathbb{E} \|\widehat{x}\|^2 &\leq 2 \left(\mathbb{E} \left\| \int_t^{t+h} \widehat{f}_s ds \right\|^2 + \mathbb{E} \left\| \int_t^{t+h} \widehat{\sigma}_s dB_s \right\|^2 \right) \\ &\leq 2 \left[h \int_t^{t+h} \mathbb{E} \|\widehat{f}_s\|^2 ds + \int_t^{t+h} \mathbb{E} \|\widehat{\sigma}_s\|^2 ds \right] \\ &\leq 2L \left[h \int_t^{t+h} \mathbb{E} \|x(s) - x(t)\|^2 ds \right. \\ &\quad \left. + \int_t^{t+h} \mathbb{E} \|x(s) - x(t)\|^2 ds \right]. \end{aligned} \quad (15)$$

By Lemma 4 and inequality (10), we have

$$\mathbb{E} \|\widehat{x}\|^2 \leq 2LK_T \left(1 + \mathbb{E} \|x(t)\|^2 \right) \left(\frac{1}{2} h^3 + h^2 \right). \quad (16)$$

Let $\widetilde{x}_t := x(t) + hf(t, x(t)) + \sigma(t, x(t))(B_{t+h} - B_t)$; then $(\alpha(x))$ replaces $\alpha(t, x)$ for shortening)

$$\begin{aligned} \mathbb{E} |V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t) - \langle \alpha(\widetilde{x}_t), \widehat{x} \rangle| \\ \leq C(h) h^2. \end{aligned} \quad (17)$$

Now we estimate the order of

$$\begin{aligned} \mathbb{E} \langle \alpha(\widetilde{x}_t), \widehat{x} \rangle &= \mathbb{E} \langle \alpha(x(t)), \widehat{x} \rangle \\ &\quad + \mathbb{E} \langle \alpha(\widetilde{x}_t) - \alpha(x(t)), \widehat{x} \rangle. \end{aligned} \quad (18)$$

Since $\mathbb{E} \langle \alpha(x(t)), \widehat{x} \rangle = \mathbb{E} \langle \alpha(x(t)), \mathbb{E}[\widehat{x} | \mathcal{F}_t] \rangle$ and $\mathbb{E}[\widehat{x} | \mathcal{F}_t] = \mathbb{E}[\int_t^{t+h} \widehat{f}_s ds | \mathcal{F}_t]$, so

$$\begin{aligned} |\mathbb{E} \langle \alpha(x(t)), \widehat{x} \rangle| &= \left| \mathbb{E} \left\langle \alpha(x(t)), \int_t^{t+h} \widehat{f}_s ds \right\rangle \right| \\ &\leq C_1(h) h^{3/2}, \\ |\mathbb{E} \langle \alpha(\widetilde{x}_t) - \alpha(x(t)), \widehat{x} \rangle| &\leq \|\alpha(\widetilde{x}_t) - \alpha(x(t))\| \|\widehat{x}\| \\ &\leq C_2(h) h^{3/2}, \end{aligned} \quad (19)$$

where $C_1(h), C_2(h)$ are continuous positive functions. By (19), we have

$$|\mathbb{E} \langle \alpha(\widetilde{x}_t), \widehat{x} \rangle| \leq C_3(h) h^{3/2}. \quad (20)$$

Since

$$\begin{aligned} &|\mathbb{E} [V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t)]| \\ &\leq \mathbb{E} |V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t) - \langle \alpha(\widetilde{x}_t), \widehat{x} \rangle| \\ &\quad + |\mathbb{E} \langle \alpha(\widetilde{x}_t), \widehat{x} \rangle|, \end{aligned} \quad (21)$$

by (17) and (20), we see that the first two items of right side are higher infinitesimal of t . So we have

$$\lim_{h \rightarrow 0^+} \frac{1}{h} |\mathbb{E} [V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t)]| = 0. \quad (22)$$

For the last two items of (13), since $x(t)$ is independent of $B_{t+h} - B_t$ with normal distribution $N(0, h)$, so we have

$$\begin{aligned} &\limsup_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [V(t+h, x(t) + hf(t, x(t))) \\ &\quad + \sigma(t, x(t))(B_{t+h} - B_t)) - V(t, x(t))] \\ &\leq \mathbb{E} \left\{ \limsup_{h \rightarrow 0} \frac{1}{h} \right. \\ &\quad \cdot \mathbb{E} [V(t+h, y + hf(t, y) + \sigma(t, y)(B_{t+h} - B_t)) \\ &\quad \left. - V(t, y)]_{y=x(t)} \right\} = \mathbb{E} [\mathcal{D}^+ V(t, y)]_{y=x(t)}. \end{aligned} \quad (23)$$

This proves (11). $\square$

The following lemma will be used in the proof of Theorem 7.

Lemma 6 (see Theorem 1.5.2 in [3]). Let $g \in C(\mathcal{R}_+ \times \mathcal{R}_+, \mathcal{R})$ and $r(t)$ be the maximal solution of

$$\begin{aligned} \dot{u} &= g(t, u), \\ u(t_0) &= u_0, \end{aligned} \quad (24)$$

existing on $[t_0, \infty)$. Suppose $m \in C(\mathcal{R}_+, \mathcal{R}_+)$ and $Dm(t) \leq g(t, m(t))$, $t \geq t_0$, where D is any fixed Dini derivative. Then $m(t_0) \leq u_0$ implies $m(t) \leq r(t)$, $t \geq t_0$.

Now we formulate the basic comparison results in terms of Lyapunov function V .

Theorem 7. Assume $\mathcal{D}^+ V(t, x)$ satisfies

$$\mathcal{D}^+ V(t, x) \leq g(t, V(t, x)), \quad (t, x) \in \mathcal{R}_+ \times \mathcal{R}^n, \quad (25)$$

where $g \in C(\mathcal{R}_+^2, \mathcal{R})$ is concave for u . Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of the differential equation

$$\begin{aligned} \dot{u} &= g(t, u), \\ u(t_0) &= u_0 \geq 0. \end{aligned} \quad (26)$$

Then, for every solution of (1) $x(t) = x(t, t_0, x_0)$, $V(t_0, x_0) \leq u_0$ implies

$$\mathbb{E} V(t, x(t)) \leq r(t), \quad t \geq t_0. \quad (27)$$

Proof. Denote $m(t) = \mathbb{E} V(t, x(t))$. By Lemma 5 and the concave of g we have

$$D^+ m(t) \leq g(t, m(t)), \quad m(t_0) \leq u_0. \quad (28)$$

By Lemma 6, we can obtain the result (27). $\square$

Remark 8. If $V \in C^{1,2}(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}_+)$, the inequality (25) became

$$\mathcal{L}V(t, x) \leq g(t, V(t, x)), \quad (t, x) \in \mathcal{R}_+ \times \mathcal{R}^n. \quad (29)$$

3. Stability Criteria in terms of Two Measures

Now we discuss the two-measure stability criteria for the stochastic differential system (1). We assume $f(t, 0) = \sigma(t, 0) = 0$ for all $t \geq t_0$. Firstly, we give some definitions for stochastic stability.

Definition 9. The stochastic differential system (1) is said to be

- (S₁) mean (h_0, h) -equistable, if for each $\epsilon > 0$ and $t_0 \in \mathcal{R}_+$, there exists a function $\delta = \delta(t_0, \epsilon) > 0$ which is continuous in t_0 for each ϵ such that

$$\mathbb{E}h_0(t_0, x_0) < \delta \quad \text{implies} \quad \mathbb{E}h(t, x(t)) < \epsilon, \quad t \geq t_0, \quad (30)$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (1);

- (S₂) mean (h_0, h) -uniformly stable, if (S₁) holds with δ being independent of t_0 ;

- (S₃) mean (h_0, h) -quasiequiasymptotically stable, if for each $\epsilon > 0$ and $t_0 \in \mathcal{R}_+$, there exist positive number $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that

$$\mathbb{E}h_0(t_0, x_0) < \delta_0 \quad \text{implies} \quad \mathbb{E}h(t, x(t)) < \epsilon, \quad t \geq t_0 + T; \quad (31)$$

- (S₄) mean (h_0, h) -quasiuniform asymptotically stable if (S₃) holds with δ_0 and T being independent of t_0 ;

- (S₅) mean (h_0, h) -asymptotically stable if (S₁) holds and, given $t_0 \in \mathcal{R}_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that

$$\mathbb{E}h_0(t_0, x_0) < \delta_0 \quad \text{implies} \quad \lim_{t \rightarrow \infty} \mathbb{E}h(t, x(t)) = 0; \quad (32)$$

- (S₆) mean (h_0, h) -uniformly equiasymptotically stable, if (S₁) and (S₃) hold together;

- (S₇) mean (h_0, h) -uniformly asymptotically stable if (S₂) and (S₄) hold simultaneously;

- (S₈) mean (h_0, h) -unstable if (S₁) fails to hold.

The following classes of functions will be used in this paper:

$\mathcal{K} = \{a \in C(\mathcal{R}_+, \mathcal{R}_+) : a(u) \text{ is strictly increasing in } u \text{ and } a(0) = 0\},$

$\mathcal{CK} = \{a \in C(\mathcal{R}_+^2, \mathcal{R}_+) : a(t, s) \in \mathcal{K} \text{ for each } t\},$

$\Gamma = \{h \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}_+) : \inf h(t, x) = 0\}.$

Definition 10. Let $h_0, h \in \Gamma$. Then, we say that h_0 is finer than h if there exists a function $\phi \in \mathcal{CK}$ such that $h(t, x) \leq \phi(t, h_0(t, x))$. Furthermore, if ϕ is independent of t then we call h_0 uniformly finer than h .

Definition 11. Let $V \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}_+)$. If there exists a function convex $b \in \mathcal{K}$ such that $b(h(t, x)) \leq V(t, x)$, then we call V h -positive definite. If there exists a concave function $a \in \mathcal{K}$ such that $V(t, x) \leq a(h(t, x))$, then we call V h -decrecent.

Theorem 12. Assume that

(A₀) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h ,

(A₁) $V \in C(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}_+)$, $V(t, x)$ satisfies (6), and V is h -positive definite and h_0 -decrecent,

(A₂) $g \in C(\mathcal{R}_+^2, \mathcal{R})$ and $g(t, 0) \equiv 0$,

(A₃) $\mathcal{D}^+V(t, x) \leq g(t, V(t, x))$ for $(t, x) \in \mathcal{R}_+ \times \mathcal{R}^n$.

Then, the stability properties of the trivial solution of (26) imply the corresponding (h_0, h) -stability properties of (1).

Proof. Since V is h -positive definite, so there exists a convex $b \in \mathcal{K}$ such that

$$b(h(t, x)) \leq V(t, x), \quad (t, x) \in \mathcal{R}_+ \times \mathcal{R}^n. \quad (33)$$

Suppose the trivial solution of (26) is equistable and $r(t, t_0, r(t_0))$ ($t \geq t_0$) is its maximal solutions with initial time t_0 and initial value $r(t_0) \geq 0$, then, for every $\epsilon > 0$, there exists $\delta_1 > 0$, when $r(t_0) < \delta_1$,

$$r(t) = r(t, t_0, r(t_0)) < b(\epsilon), \quad t \geq t_0. \quad (34)$$

Let $r(t_0) = \mathbb{E}V(t_0, x(t_0))$; then, by Theorem 7, we have

$$\mathbb{E}V(t, x(t)) \leq r(t), \quad t \geq t_0. \quad (35)$$

Since V is h_0 -decrecent, there exists a concave $a \in \mathcal{K}$ such that

$$V(t, x) \leq a(h_0(t, x)), \quad (t, x) \in \mathcal{R}_+ \times \mathcal{R}^n. \quad (36)$$

So

$$r(t_0) \leq \mathbb{E}a(h_0(t_0, x(t_0))) \leq a(\mathbb{E}h_0(t_0, x(t_0))). \quad (37)$$

Since a is continuous and strictly increasing, so let $\delta = a^{-1}(\delta_1)$; then when $\mathbb{E}h_0(t_0, x(t_0)) < \delta$, inequality (34) holds. Combining (33), (34), and (35) and using the strictly increase of b , we can gain

$$\mathbb{E}h(t, t_0, x(t_0)) < \epsilon, \quad t \geq t_0, \quad (38)$$

which implies (1) (h_0, h) -equistability. $\square$

Remark 13. If $V \in C^{1,2}(\mathcal{R}_+ \times \mathcal{R}^n, \mathcal{R}_+)$, then condition (A₃) can be replaced by

$$\mathcal{L}V(t, x) \leq g(t, V(t, x)). \quad (39)$$

Remark 14. The stabilities of auxiliary ordinary differential equation (26) are defined by Definition 2.4.1 in [3].

Example 15. Consider the following 2-dimensional Itô's system:

$$\begin{aligned} dx_1(t) &= x_2(t) dt, \\ dx_2(t) &= -(bx_1(t) + ax_2(t)) dt \\ &\quad + (cx_2(t) + ex_1(t)) dB_t. \end{aligned} \quad (40)$$

Let $h(t, x) = h_0(t, x) = e^{\gamma t}(x_1^2 + x_2^2)$; suppose $V(t, x)$ has the form

$$V(t, x) = e^{\gamma t}(\alpha x_1^2 + 2\beta x_1 x_2 + x_2^2). \quad (41)$$

Then

$$\begin{aligned} \mathcal{L}V &= e^{\gamma s}[(\gamma\alpha - 2\beta b + e^2)x_1^2 \\ &+ (2\gamma\beta + 2\alpha - 2\beta a - 2b + 2ce)x_1(s)x_2 \\ &+ (\gamma + 2\beta - 2a + c^2)x_2^2]. \end{aligned} \quad (42)$$

In order to make $V(t, x)h$ -positive, we let

$$\begin{aligned} \alpha &> 0, \\ \alpha - \beta^2 &> 0. \end{aligned} \quad (43)$$

Let $g(t, u) = -\theta u$, $\theta > 0$; in order to find $\theta > 0$ to satisfy

$$\mathcal{L}V \leq g(t, V), \quad (44)$$

we set

$$\begin{aligned} 2\beta\gamma + 2\alpha - 2\beta a - 2b + 2ce &= 0, \\ \gamma\alpha - 2\beta b + e^2 &< 0, \\ \gamma + 2\beta - 2a + c^2 &< 0. \end{aligned} \quad (45)$$

When $\gamma < a$, combining (43) and (45), we have that, when

$$\begin{aligned} \max \left\{ \frac{1}{2} \left[a - \gamma - \sqrt{(\gamma - a)^2 + 4(b - ce)} \right], \right. \\ \left. \frac{\gamma b - \gamma ce + e^2}{\gamma^2 + 2b - \gamma a} \right\} < \beta < \min \left\{ \frac{b - ce}{\gamma - a}, a \right. \\ \left. - \frac{1}{2}(\gamma + c^2), \right. \\ \left. \frac{1}{2} \left[a - \gamma + \sqrt{(\gamma - a)^2 + 4(b - ce)} \right] \right\}, \end{aligned} \quad (46)$$

there exists $\theta > 0$ which satisfies (44). Moreover, the trivial solution of

$$\dot{u} = -\theta u \quad (47)$$

is uniformly asymptotically stable; by Theorem 12, the stochastic differential is mean- (h_0, h) -asymptotically stable.

However, in practice, the coefficients maybe have some uncertainty properties; that is, we only know the range of the parameters a, b . Then how to discuss the stability of such systems with uncertainty is still a very interesting problem. The following section will introduce an optimal control approach to discuss the stability of such systems with uncertainty.

4. The Stability for Systems with Uncertainty

Consider the following stochastic differential equations with uncertainty:

$$dx = [f(x) + g(x)k(x)]dt + \sigma(x)dB_t, \quad (48)$$

where $k(x)$ is an uncertainty function, and $f(0) = k(0) = \sigma(0) = 0$, that is, the trivial solution, is 0 of (48). Furthermore, we also suppose there exists a function $k_{\max}(x) \geq 0$ such that

$$\|k(x)\| \leq k_{\max}(x). \quad (49)$$

Now we discuss how to determine the asymptotic stability of system (48) for all uncertainty function k . Similar to the methods applied by [18, 19], we can translate this stability problem into an optimal problem.

For the nominal system

$$dx = [f(x) + g(x)u]dt + \sigma(x)dB_t, \quad (50)$$

suppose we can find a state-feedback control $u = \bar{u}(x)$ that minimizes the cost functional

$$J(x(\cdot), u) = \mathbb{E}^y \int_0^\infty [f_{\max}^2(x) + \rho h_0(x) + 3u^T u]dt, \quad (51)$$

where $h_0 \in C(\mathcal{R}^n, \mathcal{R}_+)$, y is the initial value of (50), $\rho > 0$, and u is admissible on $[0, T]$ for each $T < \infty$, and satisfies

$$J(x(\cdot), u) < \infty. \quad (52)$$

Let the value function

$$V(y) = \min_u J(x(\cdot), u). \quad (53)$$

Theorem 16. Suppose $\bar{u}(x)$ is an optimal control of problem (53), and there exists $\bar{\rho} \leq \rho$ satisfying

$$\|\bar{u}\|^2 \leq \frac{\bar{\rho}}{2} h_0(x), \quad (54)$$

and the value function $V \in C^2(\mathcal{R}^n, \mathcal{R}_+)$, V is also h positive, and h_0 is decrescent, then system (48) is uniformly mean (h_0, h) uniformly asymptotically stable for all uncertainties $k(x)$.

Proof. The values function V satisfies the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \min_{u \in \mathcal{R}^p} \left(f_{\max}^2 + x^T x + u^T u + \langle V_x, f + gu \rangle \right. \\ \left. + \frac{1}{2} \langle \sigma \sigma^T, V_{xx} \rangle \right) = 0. \end{aligned} \quad (55)$$

So, the optimal control $\bar{u}$ satisfies

$$\begin{aligned} f_{\max}^2 + x^T x + \bar{u}^T \bar{u} + \langle V_x, f + g\bar{u} \rangle + \frac{1}{2} \langle \sigma \sigma^T, V_{xx} \rangle \\ = 0, \\ 2\bar{u}^T + V_x^T g = 0. \end{aligned} \quad (56)$$

Then, the Lyapunov generator of (48) for V is given as

$$\begin{aligned}\mathcal{L}V(x) &:= \langle V_x, f + gk \rangle + \frac{1}{2} \langle V_{xx}, \sigma \sigma^T \rangle \\ &= \langle V_x, gk \rangle - \langle V_x, g\bar{u} \rangle - h_0(x) - k_{\max}^2 - \bar{u}^T \bar{u} \\ &\leq \|k\|^2 - k_{\max}^2 - (\rho - \bar{\rho}) h_0(x) \\ &\leq -(\rho - \bar{\rho}) h_0(x).\end{aligned}\quad (57)$$

So, for the solutions $x(t)$ of (48) with uncertainty $k(x)$, applying Itô's formula to $V(x(t))$, we have, when $t \geq t_0$,

$$\begin{aligned}\mathbb{E}V(x(t)) - \mathbb{E}V(x(t_0)) &= \mathbb{E} \int_{t_0}^t \mathcal{L}V(x(s)) ds \\ &\leq -(\rho - \bar{\rho}) \mathbb{E} \int_{t_0}^t h_0(x) ds.\end{aligned}\quad (58)$$

Hence

$$\begin{aligned}\mathbb{E}V(x(t)) &\leq \mathbb{E}V(x(t_0)) - \mathbb{E} \int_{t_0}^t h_0(x(s)) ds, \\ &\quad \forall t \geq t_0.\end{aligned}\quad (59)$$

So $m(t) = \mathbb{E}V(x(t))$ is decreasing on $[0, \infty]$. Now we show $m(t) \rightarrow 0$ when $t \rightarrow \infty$. Suppose $m(t) \rightarrow m_0 > 0$. Since V is h_0 decrescent, so there exists $c > 0$ and $t_0 > 0$, such that $\mathbb{E}h_0(x(t)) > c > 0$ ($t \geq t_0$) combining with (59); we have

$$\begin{aligned}\mathbb{E}V(x(t)) &\leq \mathbb{E}V(x(t_0)) - c(t - t_0) \rightarrow -\infty, \\ &\quad t \rightarrow \infty.\end{aligned}\quad (60)$$

This contradicts the fact that $V(x) \geq 0$. This implies that system (48) is uniformly mean (h_0, h) uniformly asymptotically stable. $\square$

Corollary 17. *Specially, let*

$$\begin{aligned}f(x) &= Ax, \\ g(x) &= G, \\ \sigma(x) &= Cx, \\ k_{\max}(x) &= M \|x\|.\end{aligned}\quad (61)$$

The value function can be given by $V(x) = x^T P x$ and the optimal control $\bar{u}(x) = -G^T P x$, where $P > 0$ satisfies the following Riccati equation:

$$(M^2 + 1)I + PA + A^T P + C^T P C - PGG^T P = 0. \quad (62)$$

Let $\lambda_{\max}$ be the maximal eigenvalue of $PGG^T P$. If $\lambda_{\max}^2 \leq \rho/2$, then the corresponding system with uncertainty is uniformly mean (h_0, h) -equistable for all uncertainties $k(x)$ (which is also mean square asymptotically stable for all uncertainties $k(x)$).

Example 18. Consider (41) with uncertainty coefficients. In (41), replace a, b by $a + \sin x_1$ and $b + \cos x_2$, respectively, and

a takes values in $[1.4, 1.6]$, b in $[0.9, 1.1]$, $c = -1$, and $e = 1$. Then the system with uncertainty is obtained:

$$\begin{aligned}dx_1(t) &= x_2(t) dt, \\ dx_2(t) &= -[(b + 0.1 \cos B_t)x_1(t) \\ &\quad + (a + 0.1 \sin B_t)x_2(t)] dt + (cx_2(t) + ex_1(t)) dB_t.\end{aligned}\quad (63)$$

Let

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 \\ -1 & -1.5 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 \\ 1.5 - b - 0.1 \cos B_t & 1 - a - 0.1 \sin B_t \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},\end{aligned}\quad (64)$$

And let G be the 2-order identical matrix. Let $h(x) = h_0(x) = \|x\|^2$; then

$$\begin{aligned}f(x) &= Ax, \\ g(x) &= G, \\ k(x) &= Bx, \\ \sigma(x) &= Cx.\end{aligned}\quad (65)$$

We can take $k_{\max}(x) = 0.3\sqrt{x_1^2 + x_2^2}$ and the auxiliary optimal problem is

$$dx = (Ax + Gu) dt + CxdB_t, \quad (66)$$

with the cost functional

$$J(x(\cdot), u) = \mathbb{E}^y \int_0^\infty [3x^T x + u^T u] dt, \quad (67)$$

$\rho = 2.91$, solving (62) with $M = 0.3$; we have

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}. \quad (68)$$

By Corollary 17, we can determine that the stochastic system (63) is uniformly mean (h_0, h) uniformly stable for all uncertainties.

5. Conclusion

In this paper, we extend the stability criteria of two measures to the mean stability situations for the stochastic systems with uncertainty. For the usual SDE, we give the results of mean stability criteria which are the basic criteria for such systems. As far as the systems with uncertainty, in order to resolve the difficulties coming from the coefficient uncertainty, we use the optimal control results as an auxiliary method to determine the mean stability. Furthermore, the stability criteria in terms of two measures for other stochastic systems, such as systems with Markovian jumps or Poisson jumps, are worth further studying.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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