Research Article

Asymptotic Optimality of Combined Double Sequential Weighted Probability Ratio Test for Three Composite Hypotheses

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We propose the weighted expected sample size (WESS) to evaluate the overall performance on the indifference-zones for three composite hypotheses' testing problem. Based on minimizing the WESS to control the expected sample sizes, a new sequential test is developed by utilizing two double sequential weighted probability ratio tests (2-SWPRTs) simultaneously. It is proven that the proposed test has a finite stopping time and is asymptotically optimal in the sense of asymptotically minimizing not only the expected sample size but also any positive moment of the stopping time on the indifference-zones under some mild conditions. Simulation studies illustrate that the proposed test has the smallest WESS and relative mean index (RMI) compared with Sobel-Wald and Whitehead-Brunier tests.

1. Introduction

Let $X_1, X_2, \dots$ be independent and identically distributed (i.i.d.) random variables whose common density function $f(x, \theta)$ (with respect to some nondegenerate measure ν) belongs to the exponential family

$$f(x, \theta) = \exp \{x\theta - \psi(\theta)\}, \quad \theta \in \Theta = (\underline{\theta}, \bar{\theta}), \quad (1)$$

where $\psi(\cdot)$ is a convex function and Θ is the natural parameter space with $-\infty \leq \underline{\theta} < \bar{\theta} \leq \infty$. The problem of interest is the following three composite hypotheses' testing problem:

$$\begin{aligned} H_1 : \theta \leq \theta_1 \text{ versus} \\ H_2 : \theta_2 \leq \theta \leq \theta_3 \text{ versus} \\ H_3 : \theta \geq \theta_4 \end{aligned} \quad (2)$$

$$(\theta_1 < \theta_2 < \theta_3 < \theta_4),$$

where $\theta_1, \theta_2, \theta_3, \theta_4 \in \Theta$. For example, in clinical trial applications, in order to compare the effects of two drugs (Goeman et al. [1]), the equivalence trial $H_0 : |\Delta| > \Delta_1$ versus $H_1 : |\Delta| < \Delta_0$ ($0 < \Delta_0 < \Delta_1$) would be more realistically

stated as $H_1 : \Delta < -\Delta_1$ (inferiority), $H_2 : -\Delta_0 < \Delta < \Delta_0$ (equivalence), and $H_3 : \Delta > \Delta_1$ (superiority), where Δ is the difference of effect between two drugs. The sequential testing of three or more hypotheses has been applied to a variety of engineering problems such as pattern recognition (Fu [2]; McMillen and Holmes [3]), multiple-resolution radar detection (Bussgang [4]), products comparisons (Anderson [5]), and others (Li et al. [6]). The intervals of $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$ are usually called indifference-zones and denoted by $\Theta = [\theta_1, \theta_2] \cup [\theta_3, \theta_4]$.

Published work on this problem has taken two main approaches. Pavlov [7], Baum and Veeravalli [8], and Dragalin et al. [9, 10] studied the class of tests motivated by the Bayesian framework. The second approach has focused on extending the sequential probability ratio test (SPRT) and double sequential probability ratio test (2-SPRT) to incorporate more than two hypotheses, such as Sobel and Wald [11], Armitage [12], Simons [13], Lorden [14], Whitehead and Brunier [15], and Li and Pu [16, 17]. Dragalin and Novikov [18] studied the problem of testing several composite hypotheses with an indifference-zone for an unknown parameter. Lai [19] considered the multihypothesis testing problem where some or all of these hypotheses are composite.

Among others, the tests proposed by Sobel and Wald [11] and Whitehead and Brunier [15] are usually used in practice for problem (2). Specifically, Sobel and Wald [11] proposed carrying out simultaneous SPRTs of H_1 versus H_2 and H_2 versus H_3 . However, when the true parameter is in the indifference-zones, the expected sample size of the Sobel-Wald test can be considerably larger than that of a fixed-sample-size test plan. Moreover, it is untruncated such that the number of observations required can not be predetermined, an undesirable property in many practical situations such as medical trial. To reduce the maximum expected sample size, Whitehead and Brunier [15] applied two 2-SPRTs instead of two SPRTs for the component tests, at the cost of larger expected sample sizes when the true parameter does not belong to the indifference-zones.

For one-sided composite hypotheses, in order to control the expected sample sizes, Wang et al. [20] proposed the double sequential weighted probability ratio test (2-SWPRT) based on mixture likelihood ratio statistics and showed that the 2-SWPRT is an asymptotically overall optimal test in the sense of asymptotically minimizing the expected sample sizes on the indifference-zone. Motivated by the attractive properties of the 2-SWPRT, we extend the existing work on problem (2) from pointwise optimality to overall performance optimality when there are different concerns of interest on different θ s. In particular, we propose an optimality criterion to evaluate the overall performance of sequential test plans on the indifference-zones for three composite hypotheses and correspondingly develop a new sequential test for problem (2) by utilizing two 2-SWPRTs as the component tests to reduce the expected sample sizes. We show the proposed test has a finite stopping time and is asymptotically optimal in the sense of asymptotically minimizing not only the expected sample size but also any positive moment of the stopping time on the indifference-zones. Simulation studies show that the proposed test not only has the smallest WESS compared with Sobel-Wald and Whitehead-Brunier tests, but also is superior to the Whitehead-Brunier test and comparable with the Sobel-Wald test when the true parameter does not belong to the indifference-zones. Moreover, the RMI also shows the proposed test is an efficient method to improve the overall performance.

The rest of this paper is organized as follows. In Section 2, we review the Sobel-Wald and Whitehead-Brunier tests. The combined double sequential weighted probability ratio test (denoted by combined 2-SWPRT) is proposed and its properties are given in Section 3. Simulation results are provided in Section 4 and some conclusions are in Section 5. All technical details are given in Appendix.

2. Methodology Review

For one-sided composite hypotheses H_1 versus H_2 , the SPRT is optimal in the sense that it minimizes the expected sample sizes at θ_1 and θ_2 , and the 2-SPRT has (approximately) minimal maximum expected sample size over (θ_1, θ_2) among all sequential and nonsequential tests with the same error probabilities. Given the well-known optimality properties of the SPRT and 2-SPRT, it is natural to use the SPRTs and

2-SPRTs as the component tests to construct the sequential tests for problem (2), respectively. In this section, we briefly review the Sobel-Wald and Whitehead-Brunier tests.

For testing problem (2), the generalization of errors of types I and II is expressible in terms of a 3×3 error matrix $E = (\alpha_{jk})$, where $\alpha_{jk} = P[\text{accepting } H_j \mid H_k \text{ is true}]$ for $j, k = 1, 2, 3$. However, under some mild conditions, Sobel and Wald [11, pages 504-505] and Armitage [12, pages 142-143] showed that α_{31} and α_{13} are zero, which can be verified by the simulation results in Section 4. It becomes apparent that in the general case we have at most four "degrees of freedom" in choosing an error matrix. Without loss of generality, we consider $\Delta = (\tau, d)$ as a sequential test for problem (2), where τ is the stopping rule and d is the decision rule ($d = k$ means accepting H_k , $k = 1, 2, 3$). Set $\Theta_1 = [\theta, \theta_1]$, $\Theta_2 = [\theta_2, \theta_3]$ and $\Theta_3 = [\theta_4, \bar{\theta}]$. Given positive vectors $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ ($\alpha_i + \beta_i < 1$, $i = 1, 2$),

$$\Upsilon(\alpha, \beta) = \left\{ \Delta = (\tau, d) : \sup_{\theta \in \Theta_i} P(d = i + 1) \leq \alpha_i, \right. \\ \left. \sup_{\theta \in \Theta_{i+1}} P(d = i) \leq \beta_i, i = 1, 2 \right\} \quad (3)$$

is the set of all sequential tests with error probabilities controlled by α and β .

(1) *Sobel-Wald Test.* Since the hypotheses H_1 , H_2 , and H_3 are ordered, the sequential testing of problem (2) can be constructed by combining the following two one-sided composite hypotheses S_1 and S_2 :

$$S_1 : \theta \leq \theta_1 \quad \text{versus} \quad \theta_2 \leq \theta \leq \theta_3 \\ S_2 : \theta_2 \leq \theta \leq \theta_3 \quad \text{versus} \quad \theta \geq \theta_4. \quad (4)$$

Sobel and Wald [11] proposed operating S_1 and S_2 by the SPRTs simultaneously. For all $\theta, \rho \in \Theta$, define $r_n(\theta, \rho) = \prod_{i=1}^n f(x_i, \theta) / f(x_i, \rho)$. The stopping and decision rules of S_i ($i = 1, 2$) determined by the SRPT are

$$\tau_i^s = \inf \{n \geq 1 : r_n(\theta_{2i}, \theta_{2i-1}) \notin (A_i^s, B_i^s)\}, \quad i = 1, 2, \\ d_i^s = I(\tau_i^s < \infty, r_{\tau_i^s}(\theta_{2i}, \theta_{2i-1}) \geq B_i^s) + i, \quad (5)$$

where $I(\cdot)$ is the indicator function and A_i^s and B_i^s ($i = 1, 2$) are the boundary parameters ($0 < A_i^s < 1 < B_i^s < \infty$), which are usually set as

$$A_i^s = \frac{\beta_i}{1 - \alpha_i}, \quad B_i^s = \frac{1 - \beta_i}{\alpha_i}, \quad i = 1, 2, \quad (6)$$

to meet requirements on the error probabilities. When $B_1^s/B_2^s \leq 1$ and $A_1^s/A_2^s \leq 1$, Sobel and Wald [11] showed

the event $\{d_1^s = 1, d_2^s = 3\}$ is impossible. The stopping and decision rules of the Sobel-Wald test are defined as

$$\begin{aligned} \tau^s &= \max(\tau_1^s, \tau_2^s), \\ d^s &= \begin{cases} 1, & d_1^s = 1, d_2^s = 2; \\ 2, & d_1^s = 2, d_2^s = 2; \\ 3, & d_1^s = 2, d_2^s = 3. \end{cases} \end{aligned} \quad (7)$$

The Sobel-Wald test is optimal in the sense that it minimizes the expected sample sizes at θ_{2i-1} and θ_{2i} ($i = 1, 2$) among all sequential and nonsequential tests whose error probabilities satisfy $Y(\alpha, \beta)$. However, its expected sample sizes at other parameters over $\bar{\Theta}$ may be unsatisfactory.

(2) *Whitehead-Brunier Test*. In order to minimize the maximum expected sample size under constraints (3), Whitehead and Brunier [15] applied the 2-SPRT to operate S_1 and S_2 , instead of the SPRT. As in Lorden [21], let $K(\theta, \rho) = E_\theta \log[f(x, \theta)/f(x, \rho)]$ be the Kullback-Leibler (KL) information number. Define $\hat{\theta}_i \in (\theta_{2i-1}, \theta_{2i})$ and n_i^* ($i = 1, 2$) by

$$\frac{|\log \alpha_i|}{K(\hat{\theta}_i, \theta_{2i-1})} = \frac{|\log \beta_i|}{K(\hat{\theta}_i, \theta_{2i})} = n_i^*, \quad i = 1, 2. \quad (8)$$

Set c_i^* such that $\Phi(c_i^*) = -a_{2i}^*/(a_{2i-1}^* - a_{2i}^*)$, $i = 1, 2$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, $a_{2i-1}^* = (\hat{\theta}_i - \theta_{2i-1})/K(\hat{\theta}_i, \theta_{2i-1})$, and $a_{2i}^* = (\hat{\theta}_i - \theta_{2i})/K(\hat{\theta}_i, \theta_{2i})$, $i = 1, 2$. Let

$$\tau_i^* = \hat{\theta}_i + c_i^* [n_i^* \psi''(\hat{\theta}_i)]^{-1/2}, \quad i = 1, 2. \quad (9)$$

The stopping and decision rules of S_i ($i = 1, 2$) determined by the 2-SPRT are

$$\tau_i^w = \inf \{n \geq 1 : r_n(\theta_i^*, \theta_{2i-1}) \geq A_i^w \text{ or } r_n(\theta_i^*, \theta_{2i}) \geq B_i^w\}, \quad i = 1, 2,$$

$$d_i^w = I(\tau_i^w < \infty, r_{\tau_i^w}(\theta_i^*, \theta_{2i-1}) \geq A_i^w) + i, \quad (10)$$

where A_i^w and B_i^w ($i = 1, 2$) are the boundary parameters ($0 < A_i^w, B_i^w < \infty$). The conservative values of A_i^w and B_i^w are $1/\alpha_i$ and $1/\beta_i$, in the sense that the real error probabilities may be much smaller than α_i and β_i ($i = 1, 2$), respectively. The stopping and decision rules of the Whitehead-Brunier test are defined as

$$\begin{aligned} \tau^w &= \max(\tau_1^w, \tau_2^w), \\ d^w &= \begin{cases} 1, & d_1^w = 1, d_2^w = 2; \\ 2, & d_1^w = 2, d_2^w = 2; \\ 3, & d_1^w = 2, d_2^w = 3. \end{cases} \end{aligned} \quad (11)$$

3. Optimality Criterion and Combined 2-SWPRT

For testing problem (2), if $\theta < \theta_1$ we prefer to accept H_1 and this preference is the stronger the smaller θ . Similarly,

if $\theta > \theta_4$ we prefer to accept H_3 , and we prefer to accept H_2 if $\theta_2 < \theta < \theta_3$. However, we have no strong preference between H_1 and H_2 if $\theta \in [\theta_1, \theta_2]$, and we also have no strong preference between H_2 and H_3 if $\theta \in [\theta_3, \theta_4]$. In these cases, we need more observations for decision. Thus, when the error probabilities satisfy $Y(\alpha, \beta)$, we focus on reduction of the expected sample sizes over the indifference-zones $\bar{\Theta}$ in applications. Let $g(\theta)$ be a nonnegative weight function which is sectionally continuous on $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$, respectively, and satisfies $\int_{\bar{\Theta}} g(\theta) d\theta = 1$. We define the weighted expected sample size as

$$\text{WESS}(g) = \int_{\bar{\Theta}} E_\theta \tau \cdot g(\theta) d\theta \quad (12)$$

to evaluate the overall performance of sequential test plans on $\bar{\Theta}$. The choice of g should be chosen according to practical needs (Sobel and Wald [11]). For example, let $g(\theta)$ be uniform weights when there are no differences on $\bar{\Theta}$; let $g(\theta)$ be assigned more weights when we focus more on reducing the expected sample size on these parameter points. As an overall evaluation, the $\text{WESS}(g)$ integrates the performances on $\bar{\Theta}$ by weighting the expected sample sizes.

Motivated by Wang et al. [20], we propose operating S_1 and S_2 by the 2-SWPRT. Specifically, the stopping and decision rules of S_i ($i = 1, 2$) by the 2-SWPRT are

$$\begin{aligned} \tau_i^* &= \inf \{n \geq 1 : R_n^i \geq A_i \text{ or } \tilde{R}_n^i \geq B_i\}, \quad i = 1, 2, \\ d_i &= I(\tau_i^* < \infty, R_{\tau_i^*}^i \geq A_i) + i, \end{aligned} \quad (13)$$

where

$$\begin{aligned} R_n^i &= \int_{\theta_i^*}^{\theta_{2i}} r_n(\theta, \theta_{2i-1}) g(\theta) d\theta, \\ \tilde{R}_n^i &= \int_{\theta_{2i-1}}^{\theta_i^*} r_n(\theta, \theta_{2i}) g(\theta) d\theta, \end{aligned} \quad (14)$$

$i = 1, 2,$

where A_i and B_i ($i = 1, 2$) are the boundary parameters ($0 < A_i, B_i < \infty$). Hence, the stopping and decision rules of the combined 2-SWPRT are defined as

$$\begin{aligned} \tau &= \max(\tau_1^*, \tau_2^*), \\ d &= \begin{cases} 1, & d_1 = 1, d_2 = 2; \\ 2, & d_1 = 2, d_2 = 2; \\ 3, & d_1 = 2, d_2 = 3. \end{cases} \end{aligned} \quad (15)$$

Some features of the combined 2-SWPRT are provided in the following theorems, whose proofs are provided in appendices.

First, we show the error probabilities of the combined 2-SWPRT can be easily controlled and the stopping time is finite.

Theorem 1. *There exist boundaries A_i and B_i ($i = 1, 2$) such that (τ, d) in (15) belongs to $Y(\alpha, \beta)$.*

Theorem 2. For any given nonnegative sectionally continuous weight function $g(\theta)$, the stopping time of the combined 2-SWPRT is finite.

Second, we prove that the combined 2-SWPRT is asymptotically optimal on $\bar{\Theta}$.

Definition 3. $(\tau, d) \in Y(\alpha, \beta)$ is said to be asymptotically optimal on $\bar{\Theta}$, if

$$\lim_{\substack{\alpha_i + \beta_i \rightarrow 0 \\ \log(\alpha_i) \approx \log(\beta_i)}} \frac{E_\theta \tau}{-\log(\alpha_i + \beta_i)} = J_i(\theta), \quad \theta \in [\theta_{2i-1}, \theta_{2i}], \quad (16)$$

where $J_i(\theta) = \min(1/K(\theta, \theta_{2i-1}), 1/K(\theta, \theta_{2i}))$, $i = 1, 2$.

Theorem 4. When $A_i = \alpha_i^{-1}$ and $B_i = \beta_i^{-1}$, the (τ, d) defined by (15) is asymptotically optimal on $\bar{\Theta}$.

Third, we show that any positive moment of the stopping time is asymptotically optimal on the indifference-zones.

Theorem 5. Under the conditions of Theorem 4, for all $q \geq 1$ and $\theta \in [\theta_{2i-1}, \theta_{2i}]$, $i = 1, 2$,

$$\lim_{\substack{\alpha_i + \beta_i \rightarrow 0 \\ \log(\alpha_i) \approx \log(\beta_i)}} E_\theta \left[\left(\frac{\tau}{-\log(\alpha_i + \beta_i)} \right)^q \right] = \left(\frac{1}{J_i(\theta)} \right)^q, \quad i = 1, 2. \quad (17)$$

4. Simulation Studies

In this section, we conduct simulation studies to examine the performances of the combined 2-SWPRT, the Sobel-Wald test, and Whitehead-Brunier test based on the normal and Bernoulli distributions. In particular, we considered two weight functions for $g(\theta)$ as follows; (1) uniform weights: $g(\theta) = 0.5 \sum_{i=1}^2 I(\theta \in [\theta_{2i-1}, \theta_{2i}]) / (\theta_{2i} - \theta_{2i-1})$; (2) KL weights: $g(\theta) = 0.5 \sum_{i=1}^2 I(\theta \in [\theta_{2i-1}, \theta_{2i}]) M_i(\theta) / [\int_{\theta_{2i-1}}^{\theta_{2i}} M_i(\theta) d\theta]$, where $M_i(\theta) = \max(K(\theta, \theta_{2i-1}), K(\theta, \theta_{2i}))$. As in Wang et al. [20], the corresponding formulations of the statistics R_n^i and $\tilde{R}_n^i$ ($i = 1, 2$) can be obtained. The boundaries of the tests are determined through 10^6 Monte Carlo trials, which make the relative differences between the real error probabilities (α'_i and β'_i) and the required ones (α_i and β_i) within 1%; that is, $|\alpha'_i - \alpha_i|/\alpha_i < 1\%$ and $|\beta'_i - \beta_i|/\beta_i < 1\%$.

Given the boundaries, we obtained the simulated $\text{WESS}(g) = \sum_{\theta \in S} E_\theta \tau \cdot g(\theta)$ to approximate integral (12) as follows. Let $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$ be discrete as the finite sets of parameters $S_1 = [\theta_1, \theta_1 + \Delta, \theta_1 + 2\Delta, \dots, \theta_1^*, \dots, \theta_2 - \Delta, \theta_2]$ and $S_2 = [\theta_3, \theta_3 + \Delta, \theta_3 + 2\Delta, \dots, \theta_2^*, \dots, \theta_4 - \Delta, \theta_4]$ with increase Δ , respectively. Denote $S = S_1 \cup S_2 \subset \bar{\Theta}$ and the weight function $g(\theta)$ is calculated based on $\theta \in S$; that is, $g(\theta) = 0.5 \sum_{i=1}^2 I(\theta \in S_i) M_i(\theta) / [\sum_{\theta \in S_i} M_i(\theta) d\theta]$ for KL weights. We also compute the RMI to assess the relative

efficiency between different test plans. According to Wang et al. [20], we define

$$\text{RMI}(g) = \sum_{\theta \in S} \frac{E_\theta \tau - SE_\theta \tau}{SE_\theta \tau} \cdot g(\theta), \quad (18)$$

where $SE_\theta \tau$ is the smallest $E_\theta \tau$ among the compared tests, that is, the Sobel-Wald test, the Whitehead-Brunier test, and the combined 2-SWPRT. A test plan with a smaller $\text{RMI}(g)$ value is considered better in its overall performance.

4.1. Test for the Normal Mean with Known Variance. Suppose $X_1, X_2, \dots$ are i.i.d. from $N(\theta, 1)$, $-\theta_1 = \theta_4 = 1.5$, $-\theta_2 = \theta_3 = 0.5$, and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.01$. According to Lorden [21], we have $\theta_1^* = -1$ and $\theta_2^* = 1$. The stopping boundaries are obtained as follows:

- (1) for the Sobel-Wald test, $A_i^s = 0.018$ and $B_i^s = 55.73$;
- (2) for the Whitehead-Brunier test, $A_i^w = B_i^w = 37.36$;
- (3) for the combined 2-SWPRT, $A_i = B_i = 26.92$ for the uniform weights and $A_i = B_i = 13.78$ for the KL weights, respectively.

As expected, we found that α_{13} and α_{31} of these three tests are equal to 0. Set $\Delta = 0.05$. Through another simulation study with 10^5 replications, the $\text{WESS}(g)$ and $\text{RMI}(g)$ are presented in Table 1. Similarly, the expected sample sizes for $\theta \in [-2, 2]$ are illustrated in Figure 1.

It is clear that the combined 2-SWPRTs have the smallest $\text{WESS}(g)$ in all cases. In fact, compared with the Sobel-Wald and Whitehead-Brunier tests, the $\text{WESS}(g)$ of the combined 2-SWPRT has been reduced by 11.36% and 5.86% for the uniform weights, and 8.13% and 7.57% for the KL weights. Meanwhile, in terms of the $\text{RMI}(g)$, the combined 2-SWPRT also performs best overall.

From Figure 1, it also can be seen that the expected sample size of the combined 2-SWPRT is slightly larger than the Whitehead-Brunier test when the true parameter is close to θ_i^* ($i = 1, 2$) and almost the same as the Sobel-Wald test when the true parameter is close to θ_{2i-1} or θ_{2i} ($i = 1, 2$). When the true parameter belongs to Θ_k ($k = 1, 2, 3$), the combined 2-SWPRT performs better than the Whitehead-Brunier test and is comparable with the Sobel-Wald test.

4.2. Test for the True Proportion of a Bernoulli Distribution. Suppose $X_1, X_2, \dots$ are i.i.d. random variables from the Bernoulli distribution and $P(X_1 = 1) = p = 1 - P(X_1 = 0)$ ($0 < p < 1$). The three composite hypotheses' testing problem is

$$\begin{aligned} H_0 : p &\leq p_1 \text{ versus} \\ H_1 : p_2 &\leq p \leq p_3 \text{ versus} \\ H_3 : p &\geq p_4, \end{aligned} \quad (19)$$

where $0 < p_1 < p_2 < p_3 < p_4 < 1$. Let $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.01$, $p_1 = 0.1$, $p_2 = 0.3$, $p_3 = 0.4$, and $p_4 = 0.7$. According to (9), we have

$$p_i^* = \frac{\log((1 - p_{2i-1}) / (1 - p_{2i}))}{[\log(p_{2i} / p_{2i-1}) + \log((1 - p_{2i-1}) / (1 - p_{2i}))]}, \quad (20)$$

TABLE 1: WESS(g) and RMI(g) for testing normal mean.

g	Uniform weights			KL weights		
	Sobel-Wald	Whitehead-Brunier	Combined 2-SWPRT	Sobel-Wald	Whitehead-Brunier	Combined 2-SWPRT
WESS	15.231	14.342	13.511	13.818	13.734	12.694
RMI	0.268	0.1652	0.0183	0.1629	0.2117	0.0178

TABLE 2: WESS(g) and RMI(g) for testing proportion in Bernoulli distribution.

g	Uniform weights			KL weights		
	Sobel-Wald	Whitehead-Brunier	Combined 2-SWPRT	Sobel-Wald	Whitehead-Brunier	Combined 2-SWPRT
WESS	48.978	44.163	43.253	43.387	41.262	40.035
RMI	0.2508	0.0726	0.0163	0.1615	0.0985	0.0215

FIGURE 1: Expected sample sizes for testing normal mean, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.01$, $-\theta_1 = \theta_4 = 1.5$, and $-\theta_2 = \theta_3 = 0.5$. FIGURE 2: Expected sample sizes for testing proportion in Bernoulli distribution, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.01$, $p_2 = 0.1$, $p_2 = 0.3$, $p_3 = 0.4$, and $p_4 = 0.7$.

such that $p_1^* = 0.186$ and $p_2^* = 0.553$ in the Whitehead-Brunier test and combined 2-SWPRT. The stopping boundaries are obtained as follows:

- (1) for the Sobel-Wald test, $A_1^s = 0.012$, $B_1^s = 66.52$, $A_2^s = 0.014$, and $B_2^s = 77.23$;
- (2) for the Whitehead-Brunier test, $A_1^w = 39.78$, $B_1^w = 46.51$, $A_2^w = 44.33$, and $B_2^w = 43.27$;
- (3) for the combined 2-SWPRT, $A_1 = 13.06$, $B_1 = 20.32$, $A_2 = 16.96$, and $B_2 = 16.27$ for the uniform weights and $A_1 = 19.62$, $B_1 = 14.35$, $A_2 = 17.18$, and $B_2 = 13.27$ for the KL weights.

In this case, the values of $\alpha_{13} = \alpha_{31} = 0$. Set $\Delta = 0.0625$. Through another simulation study with 10^5 replications, the WESS(g) and RMI(g) are presented in Table 2. Similarly, the expected sample sizes for $p \in (0, 1)$ are illustrated in Figure 2.

It can be seen from Table 2 that the combined 2-SWPRT still has the smallest WESS(g) and RMI(g) for the Bernoulli distribution. Meanwhile, from Figure 2, we have similar

conclusions as those in the normal distribution cases in Section 4.1.

5. Summary

In this paper, we propose the WESS(g) to evaluate the overall performance on the indifference-zones for three composite hypotheses' testing problem. In order to minimize WESS(g) to control the expected sample sizes, we developed a new sequential test by utilizing two 2-SWPRTs simultaneously. We have shown the proposed test is an asymptotically optimal test in the sense of asymptotically minimizing the expected sample sizes on the indifferent-zones.

According to the simulation results, compared with the Sobel-Wald and Whitehead-Brunier tests, we conclude that the proposed test has the following merits: (1) it has the smallest WESS(g) and RMI(g); (2) when the true parameter is close to θ_i^* ($i = 1, 2$), the proposed test has comparable performance with Whitehead-Brunier test; when the true parameter is close to θ_{2i-1} or θ_{2i} ($i = 1, 2$), it has almost

the same results as the Sobel-Wald test; when the true parameter does not belong to Θ , the proposed test also performs better than the Whitehead-Brunier test and has comparable performance with the Sobel-Wald test; (3) the proposed test is easy to implement and can be extended to multihypothesis testing problems. Future work will be concerned with the method of determining the boundaries in an analytical way instead of the Monte Carlo method.

Appendix

We provide sketch proofs of Theorems 1, 2, 4, and 5.

Proof of Theorem 1. Let $\mathcal{F}_n = \sigma(x_1, \dots, x_n)$, $n = 1, 2, \dots$. Note that $(R_n^1, F_n, n \geq 1)$ is a supermartingale under P_θ , $\forall \theta \in \Theta_1$. Therefore, for all $\theta \in \Theta_1$,

$$\begin{aligned} P_\theta(d=2) &\leq P_\theta(\tau_1 < \infty) \\ &\leq \int_{\{\tau_1 < \infty\}} A_1^{-1} R_{\tau_1}^1 dP_\theta \\ &\leq E_\theta[A_1^{-1} R_1^1]. \end{aligned} \quad (\text{A.1})$$

On the other hand, following Lemma 1 of Chen and Hickernell [22], for any positive integer m and $\theta \leq \theta_1 < \lambda$, we have

$$E_\theta[r_m(\lambda, \theta_1)] \leq 1. \quad (\text{A.2})$$

Thus,

$$\begin{aligned} E_\theta[A_1^{-1} R_1^1] &= A_1^{-1} \int_{\theta_1^*}^{\theta_2} E_\theta[r_1(t, \theta_1)] g(t) dt \\ &\leq A_1^{-1} \int_{\theta_1^*}^{\theta_2} g(t) dt. \end{aligned} \quad (\text{A.3})$$

Combining (A.1) and (A.3), we have

$$P_\theta(d=2) \leq A_1^{-1} \int_{\theta_1^*}^{\theta_2} g(t) dt, \quad \forall \theta \in \Theta_1. \quad (\text{A.4})$$

In particular, setting

$$A_1 = \alpha_1^{-1} \int_{\theta_1^*}^{\theta_2} g(t) dt, \quad (\text{A.5})$$

we have $\sup_{\theta \in \Theta_1} P_\theta(d=2) \leq \alpha_1$. Similarly, we can prove that $\sup_{\theta \in \Theta_2} P_\theta(d=1) \leq \beta_1$, $\sup_{\theta \in \Theta_2} P_\theta(d=3) \leq \alpha_2$, and $\sup_{\theta \in \Theta_3} P_\theta(d=2) \leq \beta_2$ with $B_1 = \beta_1^{-1} \int_{\theta_1^*}^{\theta_2^*} g(t) dt$, $A_2 = \alpha_2^{-1} \int_{\theta_2^*}^{\theta_4} g(t) dt$, and $B_2 = \beta_2^{-1} \int_{\theta_3}^{\theta_2^*} g(t) dt$, respectively. $\square$

Proof of Theorem 2. If $g(\theta)$ is a sectionally continuous function, according to Theorem 3.2 of Wang et al. [20], we know that (1) for all $A_1 > 0$ and $s_1 > (\psi(\theta_1^*) - \psi(\theta_1))/(\theta_1^* - \theta_1)$, there exists $T_{A_1}(s_1) < \infty$, such that $R_n^1 \geq A_1$ when $n \geq T_{A_1}(s_1)$ and $S_n \geq ns_1$; (2) for all $B_1 > 0$ and $s_1 < (\psi(\theta_2) - \psi(\theta_1^*)) / (\theta_2 - \theta_1^*)$, there exists $T_{B_1}(s_1) < \infty$, such that $\bar{R}_n^1 \geq B_1$ when $n \geq T_{B_1}(s_1)$ and $S_n \leq ns_1$, where $S_n = \sum_{l=1}^n x_l$.

Noting that $\psi(\theta)$ is convex, we have $(\psi(\theta_1^*) - \psi(\theta_1))/(\theta_1^* - \theta_1) < (\psi(\theta_2) - \psi(\theta_1^*)) / (\theta_2 - \theta_1^*)$. It is easy to choose s_1 such that

$$\frac{\psi(\theta_1^*) - \psi(\theta_1)}{\theta_1^* - \theta_1} < s_1 < \frac{\psi(\theta_2) - \psi(\theta_1^*)}{\theta_2 - \theta_1^*}. \quad (\text{A.6})$$

Let $T_{A_1 B_1} = \max(T_{A_1}(s_1), T_{B_1}(s_1))$. Then, we have $\tau_1^* \leq T_{A_1 B_1}$. Similarly, for all $A_2 > 0$ and $B_2 > 0$, we can prove that there exist $T_{A_2}(s_2) < \infty$ and $T_{B_2}(s_2) < \infty$ such that $\tau_2^* \leq T_{A_2 B_2}$. Thus, we have

$$\tau \leq \max(T_{A_1 B_1}, T_{A_2 B_2}). \quad (\text{A.7})$$

$\square$

Proof of Theorem 4. Using Hoeffding inequality (see Hoeffding [23]), we know

$$\liminf_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} \frac{E_\theta \tau}{-\log(\alpha_1 + \beta_1)} \geq J_1(\theta), \quad \theta \in [\theta_1, \theta_2], \quad (\text{A.8})$$

so it suffices to show

$$\limsup_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} \frac{E_\theta \tau}{-\log(\alpha_1 + \beta_1)} \leq J_1(\theta), \quad \theta \in [\theta_1, \theta_2]. \quad (\text{A.9})$$

According to Theorem 3.3 of Wang et al. [20], for all $\theta \in [\theta_1^*, \theta_2]$,

$$\lim_{\alpha_1 \rightarrow 0} \frac{\tau_1}{\log(A_1)} = \frac{1}{K(\theta, \theta_1)} \quad (\text{a.s. } P_\theta). \quad (\text{A.10})$$

Since $A_1 = \alpha_1^{-1}$, when $\alpha_1 + \beta_1 \rightarrow 0$ and $\log(\alpha_1) \approx \log(\beta_1)$, we have

$$-\log(\alpha_1 + \beta_1) \rightarrow -\log(\alpha_1) = \log(A_1). \quad (\text{A.11})$$

Therefore, for all $\theta \in [\theta_1^*, \theta_2]$,

$$\limsup_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} \frac{E_\theta \tau_1^*}{-\log(\alpha_1 + \beta_1)} \leq \lim_{\alpha_1 \rightarrow 0} \frac{E_\theta \tau_1}{\log(A_1)} = \frac{1}{K(\theta, \theta_1)}. \quad (\text{A.12})$$

Similarly, for all $\theta \in [\theta_1, \theta_1^*]$, we have

$$\limsup_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} \frac{E_\theta \tau_1^*}{-\log(\alpha_1 + \beta_1)} \leq \lim_{\beta_1 \rightarrow 0} \frac{E_\theta \tilde{\tau}_1}{\log(B_1)} = \frac{1}{K(\theta, \theta_2)}. \quad (\text{A.13})$$

Combining two inequalities (A.12) and (A.13), we have

$$\limsup_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} \frac{E_\theta \tau}{-\log(\alpha_1 + \beta_1)} \leq J_1(\theta), \quad \theta \in [\theta_1, \theta_2]. \quad (\text{A.14})$$

According to (A.8) and (A.14),

$$\lim_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} \frac{E_\theta \tau}{-\log(\alpha_1 + \beta_1)} = J_1(\theta), \quad \theta \in [\theta_1, \theta_2]. \quad (\text{A.15})$$

Similarly, we can prove that

$$\lim_{\substack{\alpha_2 + \beta_2 \rightarrow 0 \\ \log(\alpha_2) \approx \log(\beta_2)}} \frac{E_\theta \tau}{-\log(\alpha_2 + \beta_2)} = J_2(\theta), \quad \theta \in [\theta_3, \theta_4]. \quad (\text{A.16})$$

□

Proof of Theorem 5. Using Lemma 3.6 of Chen [24], for all $q \geq 1$, we know

$$\begin{aligned} \lim_{\alpha_1 \rightarrow 0} E_\theta \left[\left(\frac{\tau_1}{-\log(\alpha_1)} \right)^q \right] &= \frac{1}{[K(\theta, \theta_1)]^q}, \quad (\theta_1^* \leq \theta \leq \theta_2), \\ \lim_{\beta_1 \rightarrow 0} E_\theta \left[\left(\frac{\tilde{\tau}_1}{-\log(\beta_1)} \right)^q \right] &= \frac{1}{[K(\theta, \theta_2)]^q}, \quad (\theta_1 \leq \theta \leq \theta_1^*). \end{aligned} \quad (\text{A.17})$$

Similar to Theorem 4, for all $\theta \in [\theta_1^*, \theta_2]$, we have

$$\begin{aligned} \limsup_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} E_\theta \left[\left(\frac{\tau_1^*}{-\log(\alpha_1 + \beta_1)} \right)^q \right] \\ \leq \lim_{\alpha_1 \rightarrow 0} E_\theta \left[\left(\frac{\tau_1}{-\log(\alpha_1)} \right)^q \right] = \frac{1}{[K(\theta, \theta_1)]^q}. \end{aligned} \quad (\text{A.18})$$

For all $\theta \in [\theta_1, \theta_1^*]$, there is

$$\begin{aligned} \limsup_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} E_\theta \left[\left(\frac{\tau_1^*}{-\log(\alpha_1 + \beta_1)} \right)^q \right] \\ \leq \lim_{\beta_1 \rightarrow 0} E_\theta \left[\left(\frac{\tilde{\tau}_1}{-\log(\beta_1)} \right)^q \right] = \frac{1}{[K(\theta, \theta_2)]^q}. \end{aligned} \quad (\text{A.19})$$

According to (A.18), (A.19), and Hoeffding inequality, we have

$$\begin{aligned} \lim_{\substack{\alpha_1 + \beta_1 \rightarrow 0 \\ \log(\alpha_1) \approx \log(\beta_1)}} E_\theta \left[\left(\frac{\tau}{-\log(\alpha_1 + \beta_1)} \right)^q \right] \\ = \left(\frac{1}{J_1(\theta)} \right)^q, \quad \theta \in [\theta_1, \theta_2]. \end{aligned} \quad (\text{A.20})$$

Similarly, we can prove that

$$\begin{aligned} \lim_{\substack{\alpha_2 + \beta_2 \rightarrow 0 \\ \log(\alpha_2) \approx \log(\beta_2)}} E_\theta \left[\left(\frac{\tau}{-\log(\alpha_2 + \beta_2)} \right)^q \right] \\ = \left(\frac{1}{J_1(\theta)} \right)^q, \quad \theta \in [\theta_3, \theta_4]. \end{aligned} \quad (\text{A.21})$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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