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Research Article

Finite-Time Consensus Algorithm for Multiple Nonholonomic Disturbed Systems with Its Application

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This paper deals with the problem of finite-time consensus of multiple nonholonomic disturbed systems. To accomplish this problem, the multiple nonholonomic systems are transformed into two multiple subsystems, and these two multiple subsystems are studied, respectively. For these two multiple subsystems, the terminal sliding mode (TSM) algorithms are designed, respectively, which achieve the finite-time reaching of sliding surface. Next, a switching control strategy is proposed to guarantee the finite-time consensus of all the states for multiple nonholonomic systems with disturbances. Finally, we demonstrate the effectiveness of the proposed consensus algorithms with application to multiple nonholonomic mobile robots.

1. Introduction

Over the last ten years, there has been increasing interest in the control problem of multiagent systems. Driven by its application to various scientific communities, such as the control of formations, optimization, and task assignment, many results have been given in recent years (see, e.g., [1–3] and the references therein).

As one of the most fundamental and important issues, consensus problem means that all the states of agents are required to agree with a common desired value. Since the exchange of information can only occur between the agent and its neighbours, the state consensus control of multiagent systems becomes difficult and challenging. We can classify the multiagent systems into leaderless and leader-follower systems. Many consensus protocols have been proposed for the leaderless and leader-follower multiagent systems in [4–7] and the references therein. Since many practical systems can be transformed into the first-order or second-order dynamics, it is significant and necessary to study the consensus algorithms for the first-order and second-order multiagent systems. The first-order multiagent systems with fixed or switching interconnection topology have been studied deeply in [8, 9] and the references therein. However, for the second-order multiagent systems, the problem becomes complicated and many existing consensus algorithms of the first-order multiagent systems can not be extended to it. In the last few years, lots of consensus algorithms have been proposed to solve the consensus problem of second-order multiagent systems (e.g., [10–12]).

Most existing consensus algorithms mentioned above for multiagent systems are usually asymptotic consensus algorithms. Since the finite-time stability appeared in [13], finite-time control has received compelling attention for its faster convergence and better robustness (e.g., [14–20]). To further investigate the control problems of multiagent systems with the property of finite-time convergence, some results have been given for the first-order and second-order multiagent systems (e.g., [21–27]). The work in [28] proposed a finite-time consensus via binary control protocols for the first-order systems which only requires sign information of the relative state measurements. In [29], the terminal sliding mode technique was used to design finite-time consensus algorithms for second-order systems. The work in [30] proposed a finite-time consensus protocol for leaderless and leader-follower second-order multiagent systems.

Most of the existing consensus algorithms were designed only for the simple single or double integrators. Since

many existing nonholonomic systems with nonholonomic constraints can not be transformed as simple single or double integrators, such as mobile robots (see, e.g., [31]), these existing finite-time consensus algorithms are generally not applicable. There have been several results of finite-time consensus algorithms for the multiple nonholonomic systems in recent years. The paper [32] discusses the cooperative control of multiple nonholonomic agents under different communication scenarios. In [33, 34], finite-time consensus protocols are presented for nonholonomic mobile agents with and without communication delay, respectively. In paper [35], the finite-time leader-following consensus problem for multiple nonholonomic mobile agents is discussed.

It is worth noting that these papers do not consider the problem of finite-time consensus of multiple nonholonomic uncertain systems. In this paper, the main contribution is that the finite-time consensus problem for a class of multiple nonholonomic uncertain systems is solved. To address this problem, we first proposed a finite-time consensus protocol for the second-order integral systems with uncertainties, which can be seen as a more general case of that in [30]. In addition, a finite-time consensus protocol is obtained for the first-order multiple integral systems in the presence of disturbances. Then, a switched strategy is designed to guarantee that the state consensus for nonholonomic multiagent systems with disturbances can be achieved in finite time.

The rest of this paper is arranged as follows. Section 2 presents the control objective and some preliminary results. Section 3 gives the finite-time consensus protocols design for multiple nonholonomic systems with disturbances, which are the main results of the paper. Simulation results of application to multiple nonholonomic mobile robots are given in Section 4 and finally conclusion is drawn in Section 5.

2. Problem and Preliminaries

In this section, a basic concept about graph theory, some useful lemmas, and control objective are introduced.

2.1. Graph Theory. Consider a leaderless multiagent system. Suppose that there are n nodes whose information can be exchanged with the single agent undirected neighbors. Define an undirected graph $G = \{V, E, A\}$, where $V = \{1, \ldots, n\}$ is the set of nodes, E is the set of edges, and $A = [a_{ij}] \in R^{n\times n}$ is the weighted adjacency matrix of G with nonnegative elements. Compared to the directed graph, the edge (i, j) denotes that ith and jth node can obtain information from each other. If there is an edge between the ith agent and the jth agent, then $a_{ij} = a_{ji} > 1$. We define specially $a_{ii} = 0$ for all $i = 1, \ldots, n$. Let $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ as a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij} = \sum_{j\in N_i} a_{ij}$. Then, the Laplacian matrix of digraph G is defined as $L = D - A$. Similar to the directed graph, the undirected graph G is connected if there is an undirected path between any two agents.

2.2. Some Lemmas

Lemma 1 (see [36]). *If L is the Laplacian matrix of a connected undirected graph G, one has*

$$x^T L x = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\left(x_i - x_j\right)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i} a_{ij}\left(x_i - x_j\right)^2, \quad (1)$$

for any $x = [x_1, \ldots, x_n]^T$. It is easy to conclude that 0 is a simple eigenvalue of L whose associated eigenvector is 1. Assume that $\lambda_1 = 0 \le \lambda_2 \le \cdots \le \lambda_n$ is the eigenvalues of L. Then one has $\lambda_2 > 0$. If $1^T x = 0$, then $x^T L x \ge \lambda_2 x^T x$.

The lemma below presents a Lyapunov-like theorem for finite-time stability of nonlinear systems.

Lemma 2 (see [14]). *Consider the system*

$$\begin{gathered} \dot{x} = F(x), \\ F(0) = 0, \\ x \in R^n. \end{gathered} \quad (2)$$

Suppose there is C^1 function $V(x)$ defined on a neighborhood $\widehat{U} \subset R^n$ of the origin. If the function $V(x)$ satisfies the following conditions where $c > 0$ and $0 < \alpha < 1$ are real numbers,

(1) *$V(x)$ is positive definite on $\widehat{U}$,*

(2) *$\dot{V}(x) + cV^{\alpha}(x) \le 0$, $\forall x \in \widehat{U}$,*

then, the origin of system is locally finite-time stable. The settling time, depending on the initial state $x(0) = x_0$, satisfies

$$T_x(x_0) \le \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)}. \quad (3)$$

If $V(x)$ is also radially unbounded and $\widehat{U} = R^n$, the system is globally finite-time stable.

In order to construct the finite-time controller, the following three *lemmas* about some useful inequalities are needed.

Lemma 3 (see [17]). *For any $a \in R$, $b \in R$, the following inequalities hold:*

$$\begin{gathered} |a+b|^q \le 2^{q-1}\left|a^q + b^q\right|, \\ (|a| + |b|)^{1/q} \le |a|^{1/q} + |b|^{1/q}, \end{gathered} \quad (4)$$

when $q \ge 1$ is a constant. If $q \ge 1$ is odd, then

$$\begin{gathered} |a-b|^q \le 2^{q-1}\left|a^q - b^q\right|, \\ \left(|a|^q - |b|^q\right) \le q|a-b|\left|a^{q-1} + b^{q-1}\right|. \end{gathered} \quad (5)$$

Lemma 4 (see [17]). *n and m are two positive real numbers, and $a \ge 0$, $b \ge 0$, and $\pi \ge 0$ are continuous functions. Then, for any constant $c_0 > 0$,*

$$\begin{aligned} a^n b^m \pi &\le c_0 a^{n+m} \\ &\quad + \frac{m}{n+m}\left[\frac{n}{c_0(n+m)}\right]^{n/m} b^{n+m}\pi^{(n+m)/m}. \end{aligned} \quad (6)$$

2.3. Problem Formulation. For the leaderless systems of multiple nonholonomic agents, the graph G is connected and the dynamics of the ith agent is presented as

$$\begin{aligned} \dot{x}_{i1} &= u_{i1} + d_{i1}, \\ \dot{x}_{i2} &= (u_{i1} + d_{i1}) x_{i3}, \\ \dot{x}_{i3} &= u_{i2} + d_{i2}, \end{aligned} \tag{7}$$

where x_{i1}, x_{i2}, and x_{i3} and u_{i1}, u_{i2} are the states and control inputs of the ith nonholonomic agent, respectively, and $d_{i1}(t)$, $d_{i2}(t)$ are disturbances.

In this paper, finite-time consensus protocols for leaderless multiple nonholonomic agents with disturbances are designed to achieve the state consensus in finite time. To achieve this objective, the following assumption is given first.

Assumption 5. The disturbances d_{i1}, d_{i2} are bounded with a known positive constant d_0.

3. Main Results

To facilitate our design, the dynamics of the ith agent is divided into two cascaded subsystems, which is composed of a first-order subsystem

$$\dot{x}_{i1} = u_{i1} + d_{i1} \tag{8}$$

and a second-order subsystem

$$\begin{aligned} \dot{x}_{i2} &= (u_{i1} + d_{i1}) x_{i3}, \\ \dot{x}_{i3} &= u_{i2} + d_{i2}. \end{aligned} \tag{9}$$

We define that

$$\begin{aligned} x_1 &= (x_{11}, \ldots, x_{n1}), \\ e_{i1} &= \sum_{j \in N_i} a_{ij} (x_{i1} - x_{j1}), \\ x_2 &= (x_{12}, \ldots, x_{n1}), \\ e_{i2} &= \sum_{j \in N_i} a_{ij} (x_{i2} - x_{j2}), \\ x_3 &= (x_{13}, \ldots, x_{n3}). \end{aligned} \tag{10}$$

3.1. The Design of Finite-Time Consensus Protocol for Multiple Subsystems (9). To design the finite-time consensus algorithm for the second-order multiple subsystems (9), we first choose the control u_{i1} as

$$u_{i1} = k_1, \tag{11}$$

where $k_1 > d_0$ is a positive constant. In this case, the multiple subsystems (9) reduce to

$$\begin{aligned} \dot{x}_{i2} &= (k_1 + d_{i1}) x_{i3}, \\ \dot{x}_{i3} &= u_{i2} + d_{i2}. \end{aligned} \tag{12}$$

Remark 6. In [30], when $d_{i1} = d_{i2} = 0$, a finite-time consensus algorithm is designed for multiagent systems (12). If the condition $d_{i1} = 0$, $|d_{i2}| < d_0$ is considered, the algorithm in [30] can only guarantee that the steady-state errors of any two agents will converge to a region Q about d_0. In the following theorem, when d_{i1} and d_{i2} are all bounded by a known positive constant d_0, we extend the result in [30] to achieve the finite-time consensus for multiple subsystems (12).

Now, we are ready to state our main result in this subsection.

Theorem 7. *Based on Assumption 5, for the multiple subsystems (12), the finite-time consensus protocol based on nonsingular TSM control is designed as*

$$\begin{aligned} u_{i2} = &- k_4 \operatorname{sgn} S_{i2} \\ &- k_2 \left[x_{i3}^p + k_3^p \left(\sum_{j \in N_i} a_{ij} (x_{i2} - x_{j2}) \right) \right]^{(2-p)/p} \end{aligned} \tag{13}$$

with

$$\begin{aligned} S_{i2} = x_{i3} + \int_0^\tau k_2 \Bigg[& x_{i3}^p \\ & + k_3^p \left(\sum_{j \in N_i} a_{ij} (x_{i2} - x_{j2}) \right) \Bigg]^{(2-p)/p} d\tau, \end{aligned} \tag{14}$$

where $1 < p = p_1/p_2 < 2$, p_1, p_2 are positive odd integers, $k_4 > d_0$ is a positive constant, and k_2, k_3 are two designed positive constants to be determined later. Under the finite-time consensus protocol (13), the states of all subsystems (12) can reach a consensus in a finite time.

Proof. Firstly, we prove that the manifold $S_{i2} \equiv 0$ can be established in a finite-time t_1 with protocol (13). Then, we show that, once the ideal sliding mode $S_{i2} \equiv 0$ is obtained, the state consensus for multiple subsystems (12) is achieved in a finite-time t_2.

Defining $U_1(t) = (1/2) \sum_{i=1}^n S_{i2}^2$, with u_{i2} and S_{i2} given in Theorem 7, the derivative of $U_1(t)$ is

$$\begin{aligned} \dot{U}_1(t) &= \sum_{i=1}^n S_{i2} (-k_4 \operatorname{sgn} S_{i2} + d_{i2}) \\ &\le -(k_4 - d_0) \sum_{i=1}^n |S_{i2}|. \end{aligned} \tag{15}$$

By Lemma 3, one gets

$$U_1^{1/2}(t) \le \frac{1}{\sqrt{2}} \sum_{i=1}^n |S_{i2}|. \tag{16}$$

For $\forall \alpha_1 \in (0, (k_4 - d_0)\sqrt{2})$, one can verify that

$$\dot{U}_1 + \alpha_1 U_1^{1/2} = \left(\frac{\alpha_1}{\sqrt{2}} - (k_4 - d_0) \right) \sum_{i=1}^n |S_{i2}| \le 0. \tag{17}$$

By Lemma 2, we can conclude that U_1 and so the manifold S_{i2} will reach zero in a finite-time t_1 satisfying

$$t_1 \leq \frac{2U_1^{1/2}(0)}{\alpha_1}. \tag{18}$$

Once $t \geq t_1$, we have $S_{i2} = 0$. Then system (12) will behave in an identical fashion; namely,

$$\begin{aligned} \dot{x}_{i2} &= (k_1 + d_{i1}) x_{i3}, \\ \dot{x}_{i3} &= -k_2 \left[x_{i3}^p + k_3^p \left(\sum_{j \in N_i} a_{ij} (x_{i2} - x_{j2}) \right) \right]^{(2-p)/p}. \end{aligned} \tag{19}$$

Next, we prove that there exist positive constants k_2, k_3 such that the states of multiple systems (19) can reach a consensus in a finite-time t_2. The stability analysis of system (19) is based on that of [30], but the perturbations are not considered there. The proof is divided into two steps.

Step 1. Define a Lyapunov function V_0 as

$$V_0(t) = \frac{1}{2} x_2^T L x_2 = \frac{1}{4} \sum_{i=1}^{n} \sum_{j \in N_i} a_{ij} (x_{i2} - x_{j2})^2. \tag{20}$$

With Lemma 1, the derivation of V_0 along system (19) satisfies

$$\begin{aligned} \dot{V}_0(t) &= \dot{x}_2^T L x_2 \\ &= (k_1 + d_{i1}) \sum_{i=1}^{n} x_{i3} \left(\sum_{j \in N_i} a_{ij} (x_{i2} - x_{j2}) \right). \end{aligned} \tag{21}$$

The virtual controller $x_{i3}^* = -k_3 e_{i2}^{1/p}$ renders

$$\begin{aligned} \dot{V}_0 \leq & -(k_1 - d_0) k_3 \sum_{i=1}^{n} e_{i2}^{(1+p)/p} \\ & + (k_1 + d_{i1}) \sum_{i=1}^{n} e_{i2} (x_{i3} - x_{i3}^*), \end{aligned} \tag{22}$$

where k_3 is a positive constant to be determined later.

Step 2. To continue the proof, we construct another Lyapunov function as

$$V(t) = V_0(t) + \sum_{i=1}^{n} V_i(t) \tag{23}$$

with

$$V_i(t) = \int_{x_{i3}^*}^{x_{i3}} \left(s^p - x_{i3}^{*p} \right)^{2-1/p} ds. \tag{24}$$

Taking the derivative of $V(t)$, we have

$$\dot{V}(t) = \dot{V}_0 + \sum_{i=1}^{n} \dot{V}_i(t). \tag{25}$$

In the following, the terms in the right hand of (25) are discussed, respectively.

Firstly, $\sum_{i=1}^{n} \dot{V}_i(t)$ is estimated. Define $\xi_i = x_{i3}^p - x_{i3}^{*p}$; the derivative of $V_i(t)$ is

$$\begin{aligned} \dot{V}_i(t) = & -\left(2 - \frac{1}{p}\right) \frac{dx_{i3}^{*p}}{dt} \int_{x_{i3}^*}^{x_{i3}} \left(s^p - x_{i3}^{*p} \right)^{1-1/p} ds \\ & + \xi_i^{2-1/p} \dot{x}_{i3}. \end{aligned} \tag{26}$$

To estimate the terms in the right hand of (26), we introduce the following proposition, whose proof is given in the appendix. □

Proposition 8. *There exists a positive constant c_1 such that*

$$\begin{aligned} \frac{dx_{i3}^{*p}}{dt} &\leq k_3^p c_1 \sum_{m=1}^{n} |x_{m3}|, \\ \left| \int_{x_{i3}^*}^{x_{i3}} \left(s^p - x_{i3}^{*p} \right)^{1-1/p} ds \right| &\leq |x_{i3} - x_{i3}^*| \, |\xi_i|^{1-1/p}. \end{aligned} \tag{27}$$

Putting (27) into (26), one has

$$\begin{aligned} & \dot{V}_i(t) \\ & \leq -\left(2 - \frac{1}{p}\right) c_1 k_3^p \left(\sum_{m=1}^{n} |x_{m3}| \right) |x_{i3} - x_{i3}^*| \, |\xi_i|^{1-1/p} \\ & \quad + \xi_i^{2-1/p} \dot{x}_{i3}. \end{aligned} \tag{28}$$

Let $d = 1 + 1/p$; to continue the proof, the following proposition whose proof is given in the appendix is presented.

Proposition 9. *There exists a positive constant c_2, c_3 such that*

$$\begin{aligned} & |x_{m3}| \, |x_{i3} - x_{i3}^*| \, |\xi_i|^{1-1/p} \\ & \leq (k_3 + 1) |\xi_i|^d + c_2 |\xi_m|^d + k_3 c_3 |e_{m2}|^d. \end{aligned} \tag{29}$$

Substituting (29) into (28), one has

$$\begin{aligned} \dot{V}_i(t) \leq & \frac{n(2-p)(k_3+1) k_3^p}{p} |\xi_i|^d \\ & + \frac{c_2 (2-p) k_3^p}{p} \sum_{m=1}^{n} |\xi_m|^d \\ & + \frac{c_3 (2-p) k_3^{p+1}}{p} \sum_{m=1}^{n} |e_{m2}|^d + \xi_i^{2-1/p} \dot{x}_{i3}. \end{aligned} \tag{30}$$

By (19) and the definition of ξ_i, one has $\dot{x}_{i3} = -k_2\xi_i^{(2-p)/p}$. Then, with (30), one gets

$$\sum_{i=1}^{n}\dot{V}_i(t) \le \left(\frac{n(2-p)(k_3+1)k_3^p}{p} + \frac{nc_2(2-p)k_3^p}{p} - k_2\right) \cdot \sum_{i=1}^{n}|\xi_i|^d + \frac{nc_3(2-p)k_3^{p+1}}{p}\sum_{i=1}^{n}|e_{i2}|^d. \tag{31}$$

Secondly, $\dot{V}_0$ is estimated. From (22), one obtains

$$\dot{V}_0 \le -(k_1-d_0)k_3\sum_{i=1}^{n}e_{i2}^d + (k_1+d_0)\left|\sum_{i=1}^{n}e_{i2}(x_{i3}-x_{i3}^*)\right|. \tag{32}$$

To estimate the last term of (32), one introduces the following proposition whose proof is given in the appendix.

Proposition 10. *There exists a positive constant c_4 such that*

$$\left|\sum_{i=1}^{n}e_{i2}(x_{i3}-x_{i3}^*)\right| \le c_4\sum_{i=1}^{n}\left(|e_{i2}|^d + |\xi_i|^d\right). \tag{33}$$

Using Proposition 11, one has

$$\dot{V}_0(t) \le ((k_1+d_0)c_4 - (k_1-d_0)k_3)\sum_{i=1}^{n}e_{i2}^d + (k_1+d_0)c_4\sum_{i=1}^{n}|\xi_i|^d. \tag{34}$$

Putting (31) and (34) into (25) yields

$$\dot{V}(t) \le \left(\frac{nc_3(2-p)k_3^{p+1}}{p} + (k_1+d_0)c_4 - (k_1-d_0)k_3\right)\sum_{i=1}^{n}e_{i2}^d + \left(\frac{n(2-p)(k_3+1)k_3^p}{p} + \frac{nc_2(2-p)k_3^p}{p} + (k_1+d_0)c_4 - k_2\right)\sum_{i=1}^{n}\xi_i^d. \tag{35}$$

Now, the parameters k_2, k_3 are determined in turn. Let η be as any given positive constant; one first selects the parameter k_3. Define

$$f(k_3) = \frac{nc_3(2-p)k_3^{p+1}}{p} + (k_1+d_0)c_4 - (k_1-d_0)k_3 \triangleq a_1k_3^{p+1} - a_2k_3 + a_3. \tag{36}$$

It is easy to conclude that, for $\forall a_1, a_2, a_3 \in (0,\infty)$ and $p \in (1,2)$, one can select k_3 large enough to satisfy

$$f(k_3) = \frac{nc_3(2-p)k_3^{p+1}}{p} + (k_1+d_0)c_4 - (k_1-d_0)k_3 > \eta. \tag{37}$$

Once the parameter k_3 is fixed, one can determine the parameter k_2 satisfying

$$k_2 > \eta + \frac{n(2-p)(k_3+1)k_3^p}{p} + \frac{nc_2(2-p)k_3^p}{p} + (k_1+d_0)c_4. \tag{38}$$

With k_2, k_3 satisfying (38) and (37), it is easy to verify that

$$\dot{V}(t) \le -\eta\sum_{i=1}^{n}e_{i2}^d - \eta\sum_{i=1}^{n}\xi_i^d. \tag{39}$$

To estimate $V(t)$, two useful propositions are introduced in the following.

Proposition 11. *With λ_2 given in Lemma 1, the following inequality holds:*

$$\sum_{i=1}^{n}e_{i2}^2 \ge 2\lambda_2V_0. \tag{40}$$

The proof of Proposition 11 is straightforward by Lemma 1 and thus is omitted here.

Proposition 12. *The following inequality always holds:*

$$\sum_{i=1}^{n}V_i(t) \le \frac{1}{(2-1/p)k_3^{1+p}}\sum_{i=1}^{n}|\xi_i|^2. \tag{41}$$

With Propositions 11 and 12, one gets

$$V(t) \le c_5\left(\sum_{i=1}^{n}e_{i2}^2 + \sum_{i=1}^{n}\xi_i^2\right) \tag{42}$$

with

$$c_5 = \max\left\{\frac{1}{2\lambda_2}, \frac{1}{(2-1/p)k_3^{1+p}}\right\}. \tag{43}$$

For $0 < d/2 < 1$, using Lemma 3, one has

$$V^{d/2}(t) \le c_5^{d/2}\left(\sum_{i=1}^{n}e_{i2}^d + \sum_{i=1}^{n}\xi_i^d\right). \tag{44}$$

Putting (39) and (44) together, for $\forall\alpha_2 \in (0, \eta/c_5^{d/2})$, one has

$$\dot{V}(t) + \alpha_2V^{d/2}(t) \le \left(\alpha_2c_5^{d/2} - \eta\right)\sum_{i=1}^{n}\left(e_{i2}^d + \xi_i^d\right) \le 0. \tag{45}$$

By Lemma 2, one concludes that $V(t)$ will reach zero in a finite-time t_2 satisfying

$$t_2 \le \frac{V^{1-d/2}(t_1)}{\alpha_2(1-d/2)}. \tag{46}$$

Hence, under control (13) with parameters k_2, k_3 presented in (38) and (37), the state consensus for multiple subsystems (12) can be achieved in a finite-time $T_1 \ge t_1 + t_2$.

Remark 13. It is easy to know that $V(t) = 0$ means $V_0(t) = V_i(t) = 0$. Since the graph G is connected, we obtain that $x_{i2} = x_{j2}$, $x_{i3} = x_{i3}^*$, for $\forall i, j = 1, \ldots, n$. It is worth noting that $x_{i3}^* = 0$ once $x_{i2} = x_{j2}$. Thus, for $\forall i, j = 1, \ldots, n$, the following equations,

$$\begin{aligned} x_{i2} &= x_{j2} = x_2^*, \\ x_{i3} &= 0, \\ u_{i2} &= 0, \end{aligned} \tag{47}$$

will be established before T_1, where x_2^* is a constant. Once (47) is established, it is easy to conclude that, for $\forall t \ge T_1$, $i = 1, \ldots, n$, if $x_{i3} = 0$ is kept with the control u_{i2}, the state consensus of x_{i2}, x_{i3} is kept no matter what the control u_{i1} is taken as.

3.2. The Design of Finite-Time Consensus Protocol for Multiple Subsystems (8). Motivated by Theorem 7, for the multiple subsystems (8), the following theorem is obtained as the main result of this subsection.

Theorem 14. *Based on Assumption 5, for the multiple subsystems (8), the finite-time consensus protocol based on nonsingular TSM control is designed as*

$$u_{i1} = -k_6 \left[\sum_{j \in N_i} a_{ij} \left(x_{i1} - x_{j1} \right) \right]^{1/\tilde{p}} - k_7 \operatorname{sgn} S_{i1} \tag{48}$$

with

$$S_{i1} = x_{i1} + \int_0^\tau k_6 \left[\sum_{j \in N_i} a_{ij} \left(x_{i1} - x_{j1} \right) \right]^{1/\tilde{p}} d\tau, \tag{49}$$

where $1 < \tilde{p} = \tilde{p}_1/\tilde{p}_2 < 2$, $\tilde{p}_1, \tilde{p}_2$ are positive odd integers, and $k_6 > 0$, $k_7 > d_0$ are positive constants. Under the protocol, all the states of multiple systems (8) can reach a consensus in a finite time.

Proof. The proof is divided into two steps. At the first step, we prove that the manifold S_{i1} can reach zero in a finite time with protocol (48). In the second step, we show that, once the ideal sliding mode $S_{i1} = 0$ is established, the state consensus for multiagent subsystems (8) is achieved in a finite time.

Step 1. Define a Lyapunov function $U_2 = (1/2)\sum_{i=1}^n S_{i1}^2$; taking the derivative of U_2 yields

$$\dot{U}_2 \le \left(-k_7 \operatorname{sgn} S_{i1} + d_{i1}\right) S_{i1} \le -\left(k_7 - d_0\right) \left|S_{i1}\right|. \tag{50}$$

Following the same line to obtain (17), we have

$$\dot{U}_2 + \alpha_3 U_2^{1/2} \le -\left(\left(k_7 - d_0\right) - \frac{\alpha_3}{\sqrt{2}} \right) \sum_{i=1}^n \left|S_{i1}\right| \le 0 \tag{51}$$

for $\forall \alpha_3 \in (0, \sqrt{2}(k_7 - d_0))$. Then, by Lemma 2, U_2 and so the manifold S_{i1} can reach zero in a finite time.

Step 2. Once the ideal sliding mode $S_{i1} = 0$ is established, multiple systems (8) will behave in an identical fashion; namely,

$$\dot{x}_{i1} = -k_6 \left[\sum_{j \in N_i} a_{ij} \left(x_{i1} - x_{j1} \right) \right]^{1/\tilde{p}} \tag{52}$$

with $k_6 > 0$. In the following, we will prove that the finite-time consensus for multiple systems (52) can be achieved.

Define a Lyapunov function $U_3(t)$ as

$$U_3(t) = \frac{1}{2} x_1^T L x_1 = \frac{1}{4} \sum_{i=1}^n \sum_{j \in N_i} a_{ij} \left(x_{i1} - x_{j1} \right)^2. \tag{53}$$

With Lemma 1 and the definition of e_{i1}, the derivation of U_3 along system (52) satisfies

$$\begin{aligned} \dot{U}_3(t) &= \dot{x}_1^T L x_1 = \sum_{i=1}^n \dot{x}_{i1} \left(\sum_{j \in N_i} a_{ij} \left(x_{i1} - x_{j1} \right) \right) \\ &= -k_6 \sum_{i=1}^n e_{i1}^{\tilde{d}}, \end{aligned} \tag{54}$$

where $\tilde{d} = 1 + 1/\tilde{p}$. From Lemma 3, one has

$$\dot{U}_3(t) \le -k_6 \left(\sum_{i=1}^n e_{i1}^2 \right)^{\tilde{d}/2}. \tag{55}$$

Following the same line to obtain Proposition 8, by Lemma 1, we get

$$\sum_{i=1}^n e_{i1}^2 \ge 2\lambda_2 U_3(t). \tag{56}$$

Putting (55) and (56) together, we obtain that

$$\dot{U}_3(t) \le -k_6 \left(2\lambda_2\right)^{\tilde{d}/2} U_3^{\tilde{d}/2}(t). \tag{57}$$

Using Lemma 2, we conclude that U_3 converges to zero with a finite time; that is, the states of multiple systems (52) can reach a consensus in a finite time.

Hence, the proof of Theorem 14 is completed with the two steps above. □

3.3. The Overall Design of Finite-Time Consensus Protocol for Multiple Systems (7). Based on Theorems 7 and 14, we are ready to state our main result of this subsection for multiple systems (7).

Theorem 15. *Based on Assumption 5, for the leaderless multiagent systems (7), the finite-time consensus protocol based on a switched strategy is designed as*

$$u_{i1} = k_1, \quad t \le T_1,$$

$$u_{i1} = -k_6 \left[\sum_{j\in N_i} a_{ij}\left(x_{i1} - x_{j1}\right)\right]^{1/\tilde{p}} - k_7 \operatorname{sgn} S_{i1}, \quad t > T_1, \tag{58}$$

$$u_{i2} = -k_4 \operatorname{sgn} S_{i2} - k_2 \left[x_{i3}^p + k_3^p \left(\sum_{j\in N_i} a_{ij}\left(x_{i2} - x_{j2}\right)\right)\right]^{(2-p)/p}, \quad t \le T_1, \tag{59}$$

$$u_{i2} = -k_5 \operatorname{sgn} x_{i3}, \quad t > T_1,$$

where $k_1, k_4, k_5, k_7 > d_0$, $k_6 > 0$, $1 < p = p_1/p_2 < 2$, $1 < \tilde{p} = \tilde{p}_1/\tilde{p}_2 < 2$, $p_1, p_2, \tilde{p}_1, \tilde{p}_2$ are positive odd integers, k_2, k_3 are designed parameters satisfying (38) and (37), $T_1 \ge t_1 + t_2$ are switched time, and S_{i1}, S_{i2} are ideal sliding surface selected as (49) and (14). Under this protocol, all the states of multiple systems (7) can reach a consensus in a finite time.

Proof. From Theorems 7 and 14, it is easy to conclude that the consensus of x_{i2}, x_{i3} can be achieved with a finite-time T_1, and when $t \ge T_1$, the consensus of x_{i1} can be achieved in finite time. Therefore, here we only need to prove that the consensus of x_{i2}, x_{i3} achieved within T_1 can be kept after $t \ge T_1$.

From (58), when $t > T_1$, we have $u_{i2} = -k_5 \operatorname{sign} x_{i3}$. Define a Lyapunov function $U_4(t) = (1/2)\sum_{i=1}^n x_{i3}^2$; the derivation of $U_4(t)$ satisfies

$$\dot{U}_4(t) = \sum_{i=1}^n \left(-k_5\left|x_{i3}\right| + d_{i2}x_{i3}\right) \le -\left(k_5 - d_0\right)\sum_{i=1}^n \left|x_{i3}\right|. \tag{60}$$

From Lemma 3, we get that $\forall \alpha_4 \in (0, \sqrt{2}(k_5 - d_0))$

$$\dot{U}_4(t) + \alpha_4 U_4^{1/2}(t) \le -\left(\left(k_5 - d_0\right) - \frac{\alpha_4}{\sqrt{2}}\right)\sum_{i=1}^n \left|x_{i3}\right| \le 0. \tag{61}$$

By Lemma 2, we conclude that U_4 will converge to zero in a finite time $\bar{t} \le U_4^{1/2}(x_{i3}(T_1))/(\alpha_4/2)$. From Remark 13, we know that $x_{i3}(T_1) = 0$. Therefore, it can be concluded that the convergence time $\bar{t} = 0$; that is, $x_{i3} = 0$ is always kept after $t > T_1$.

Hence, we complete the proof of Theorem 15. □

4. Simulation Results

In this section, we consider the leaderless multiagent systems with 5 mobile robots whose schematic is shown in Figure 1. The ith robot is illustrated with two driven rear wheels and a front freewheel. The two rear wheels are driven, respectively, with fixed orientation. The generalized coordinates are $q = (x_i, y_i, \theta_i)$, where x_i, y_i are the Cartesian coordinates of the middle point of the rear wheel and θ_i measures the orientation of the car body with respect to x-axis. Under the assumption of no wheel slips, the kinetic model of the ith robot can be derived as

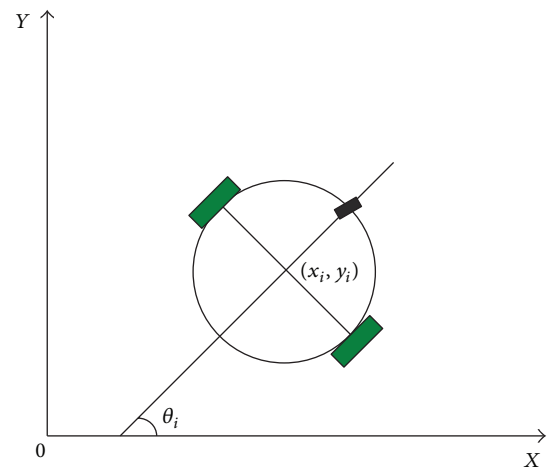

Figure 1: Generalized coordinates of the ith car-like robot.

$$\begin{aligned} \dot{x}_i &= \left(v_i + d_{i2}\right)\cos\theta_i, \\ \dot{y}_i &= \left(v_i + d_{i2}\right)\sin\theta_i, \\ \dot{\theta}_i &= w_i + d_{i1}, \end{aligned} \tag{62}$$

where v_i, w_i are the linear and angular velocities of the ith mobile robot and $d_{i1} = 0$, $d_{i2} = 0.9\sin(t)$ are disturbances. With the coordinate transformation

$$\begin{aligned} x_{i1} &= -\theta_i, \\ x_{i2} &= -x_i\sin\theta_i + y_i\cos\theta_i, \\ x_{i3} &= x_i\cos\theta_i + y_i\sin\theta_i, \end{aligned} \tag{63}$$

$$\begin{aligned} v_i &= x_{i3}u_{i1} + u_{i2} = \left(x_i\cos\theta_i + y_i\sin\theta_i\right)u_{i1} + u_{i2}, \\ w_i &= -u_{i1}, \end{aligned} \tag{64}$$

system (62) is transformed as

$$\begin{aligned} \dot{x}_{i1} &= u_{i1} - d_{i1}, \\ \dot{x}_{i2} &= \left(u_{i1} - d_{i1}\right)x_{i3}, \\ \dot{x}_{i3} &= u_{i2} + d_{i2}. \end{aligned} \tag{65}$$

Then, the consensus protocols presented in Theorem 15 can be applied.

The communication among the 5 agents is given in Figure 2. The initial states are given as $x(0) = (1, 1.1, 1.2, 1.3, 0.9)$,

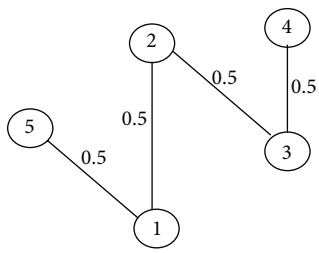

Figure 2: The communication among agents.

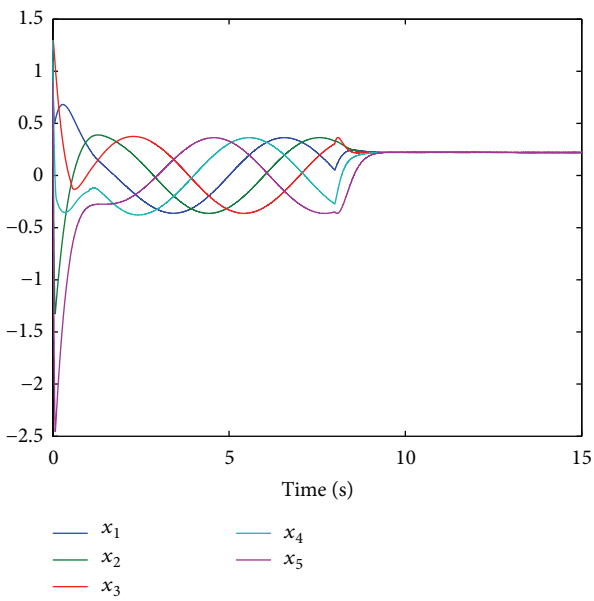

Figure 3: Agents positions x.

$y(0) = (1, 1.1, 1.2, 1.3, 0.9)$, and $\theta(0) = (5, 6, 7, 4, 3)$. The parameters are selected as

$$\begin{aligned} k_1 &= 1, \\ k_2 &= 40, \\ k_3 &= 2.7, \\ k_4 &= 1, \\ k_5 &= 1, \\ k_6 &= 2.7, \\ k_7 &= 0, \\ \eta &= 1, \\ p &= \frac{9}{7}, \\ T_1 &= 10\,\text{s}. \end{aligned} \tag{66}$$

Figures 3, 4, and 5 show the consensus of the positions and orientations, respectively.

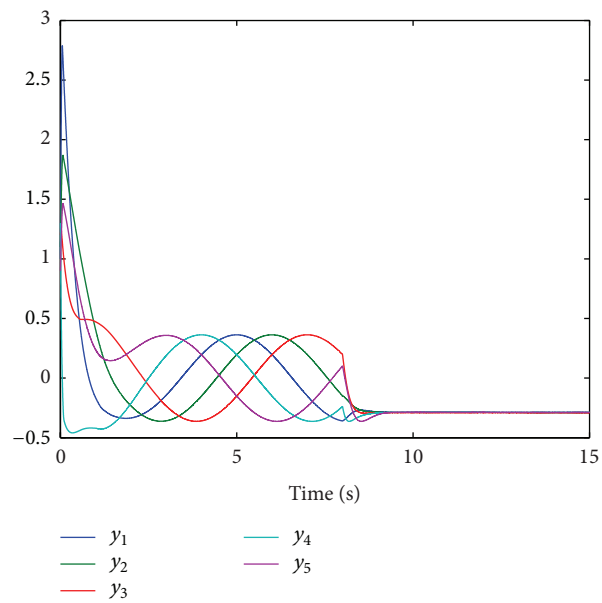

Figure 4: Agents positions y.

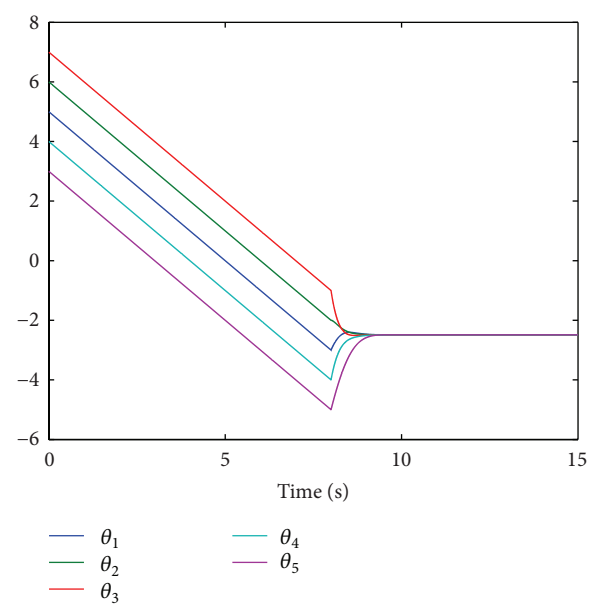

Figure 5: Agents orientation θ.

5. Conclusion

This paper studies the problem of finite-time consensus problem of multiple nonholonomic systems in the presence of disturbances. At first, the triple chained form agent system is divided into two agent subsystems and these two agent subsystems are discussed, respectively. Then, two integral terminal sliding mode controllers are designed for the two subsystems. Finally, we demonstrate the effectiveness of the proposed consensus algorithms with application to multiple nonholonomic mobile robots.

Appendix

Proof of Proposition 8. Define $\beta = \max_{\forall i}\{\sum_{j\in N_i} a_{ij}\}$ and $\gamma = \max_{\forall i,j}\{a_{ij}\}$. With (13), the definition of e_{i2}, and the virtual controller $x_{i3}^* = -k_3 e_{i2}^{1/p}$, one has

$$\begin{aligned}\frac{dx_{i3}^{*p}}{dt} &= -k_3^p \dot{e}_{i2} = -k_3^p (k_1 + d_{i1}) \sum_{j\in N_i} a_{ij} (x_{i3} - x_{j3}) \\ &\le (k_1 + d_0) k_3^p \left(|x_{i3}| \sum_{j\in N_i} a_{ij} + \sum_{j\in N_i} a_{ij} |x_{j3}| \right) \\ &\le (k_1 + d_0) k_3^p \left(\beta |x_{i3}| + \gamma \sum_{m=1}^{n} |x_{m3}| \right) \\ &\le k_3^p c_1 \sum_{i=1}^{n} |x_{i3}|,\end{aligned} \tag{A.1}$$

where c_1 is a positive constant.

We can also verify that

$$\begin{aligned}&\left| \int_{x_{i3}^*}^{x_{i3}} \left(s^p - x_{i3}^{*p}\right)^{1-1/p} ds \right| \\ &\quad \le |x_{i3} - x_{i3}^*| \left| x_{i3}^p - x_{i3}^{*p} \right|^{1-1/p} \\ &\quad \le |x_{i3} - x_{i3}^*| |\xi_i|^{1-1/p}.\end{aligned} \tag{A.2}$$

□

Proof of Proposition 9. With Lemma 3 and the definition of ξ_i, x_{i3}^*, one has

$$\begin{aligned}&|x_{3m}| |x_{i3} - x_{i3}^*| |\xi_i|^{1-1/p} \\ &\quad \le 2^{1-1/p} |x_{3m}| |\xi_i|^{1/p} |\xi_i|^{1-1/p} \\ &\quad \le 2^{1-1/p} \left(|\xi_i| |x_{m3} - x_{m3}^*| + |\xi_i| |x_{m3}^*| \right) \\ &\quad \le 2^{1-1/p} \left(2^{1-1/p} |\xi_i| |\xi_m|^{1/p} + k_3 |\xi_i| |e_{m2}|^{1/p} \right) \\ &\quad \le |\xi_i|^d + c_2 |\xi_m|^d + k_3 \left(|\xi_i|^d + c_3 |e_{m2}|^d \right) \\ &\quad \le (k_3 + 1) |\xi_i|^d + c_2 |\xi_m|^d + k_3 c_3 |e_{m2}|^d.\end{aligned} \tag{A.3}$$

□

Proof of Proposition 10. With Lemma 3 and the definition of ξ_i, one gets

$$\begin{aligned}\left| \sum_{i=1}^{n} e_{i2} (x_{i3} - x_{i3}^*) \right| &\le \sum_{i=1}^{n} |e_{i2}| 2^{1-1/p} \left| x_{i3}^p - x_{i3}^{*p} \right|^{1/p} \\ &= 2^{1-1/p} \sum_{i=1}^{n} |e_{i2}| |\xi_i|^{1/p} \\ &\le \sum_{i=1}^{n} \left(\tilde{c}_4 |e_{i2}|^d + \tilde{c}_5 |\xi_i|^d \right) \\ &\le c_4 \sum_{i=1}^{n} \left(|e_{i2}|^d + |\xi_i|^d \right).\end{aligned} \tag{A.4}$$

□

Proof of Proposition 12. With Lemma 3, one obtains

$$\begin{aligned}&\sum_{i=1}^{n} V_i(t) \\ &\quad \le \frac{1}{(2 - 1/p) 2^{1-1/p} k_3^{1+p}} \sum_{i=1}^{n} |x_{i3} - x_{i3}^*| |\xi_i|^{2-1/p} \\ &\quad \le \frac{1}{(2 - 1/p) k_3^{1+p}} \sum_{i=1}^{n} |\xi_i|^{1/p} |\xi_i|^{2-1/p} \\ &\quad = \frac{1}{(2 - 1/p) k_3^{1+p}} \sum_{i=1}^{n} |\xi_i|^2.\end{aligned} \tag{A.5}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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