Research Article

Painlevé Integrability and New Exact Solutions of the $(4 + 1)$ -Dimensional Fokas Equation

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Received 15 October 2015; Accepted 14 December 2015

Academic Editor: Valentin Lychagin

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The Painlevé integrability of the $(4 + 1)$ -dimensional Fokas equation is verified by the WTC method of Painlevé analysis combined with a new and more general transformation. By virtue of the truncated Painlevé expansion, two new exact solutions with arbitrary differentiable functions are obtained. Thanks to the arbitrariness of the included functions, the obtained exact solutions not only possess rich spatial structures but also help to bring about two-wave solutions and three-wave solutions. It is shown that the transformation adopted in this work plays a key role in testing the Painlevé integrability and constructing the exact solutions of the Fokas equation.

1. Introduction

Integrable partial differential equations (PDEs), especially the ones possessing soliton solutions [1], have had a significant impact on both theory and phenomenology [2]. One of the most important features of integrable equations is that they have Painlevé integrability [3]. With the development of soliton theory, testing Painlevé integrabilities and finding soliton solutions of nonlinear PDEs have gradually developed into significant directions in nonlinear science. Since the soliton phenomena were first observed in 1834 and the Korteweg-de Vries (KdV) equation was solved by the inverse scattering method [4], many methods have been proposed for solving nonlinear PDEs, such as Bäcklund transformation [5], Darboux transformation [6], Hirota's bilinear method [7], homogeneous balance method [8], simplest equation method [9], and function expansion methods [10–20]. In 1983, Weiss et al. [21] proposed the Painlevé analysis approach of PDEs, called the WTC method of Painlevé analysis, which is an effective method for not only testing Painlevé integrabilities but also constructing exact solutions of nonlinear PDEs. Later, Kruskal et al. [22] simplified the WTC method. Kruskal et al.'s simplified method is often used to test the Painlevé integrabilities of some complicated nonlinear PDEs without losing the effectiveness of WTC method.

The present paper is motivated by the desire to extend the WTC method to a $(4 + 1)$ -dimensional nonlinear PDE in the form

$$u_{tx_1} - \frac{1}{4}u_{x_1x_1x_1x_2} + \frac{1}{4}u_{x_2x_2x_2x_1} + \frac{3}{2}(u^2)_{x_1x_2} - \frac{3}{2}u_{y_1y_2} = 0, \quad (1)$$

which is derived by Fokas [2] in the process of extending the integrable Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations to some new higher-dimensional nonlinear wave equations. In nonlinear wave theory, the KP equation and the DS equation are two important models, which have been proposed to describe the surface waves and internal waves in straits or channels of varying depth and width [3] and the evolution of a three-dimensional wave-packet on water of finite depth [23], respectively. The importance of the Fokas equation (1) not only embodies the fact that (1) gives an extension of the KP and DS equations to $(4 + 1)$ -dimensional space but also suggests that the idea of complexifying time should be investigated in the context of modern field theories through the existence of integrable nonlinear equations in four spatial dimensions involving complex time [2]. In [24–29], Jacobi elliptic double periodic

solutions, hyperbolic function solutions, trigonometric function solutions, and rational solutions of (1) were obtained. To our knowledge, the Painlevé integrability of (1) has not been studied. Does the Fokas equation (1) pass the Painlevé test and have soliton solutions with arbitrary functions? This paper will give a positive answer to the questions.

The rest of this paper is organized as follows. In Section 2, using the WTC method combined with a new and more general transformation, we verify the Painlevé integrability of (1). In Section 3, we truncate the Painlevé expansion to construct exact solutions with arbitrary differentiable functions of (1). Inspired by the one-wave solution reduced from one of the exact solutions, we also give two-wave solution and three-wave solution of (1). Section 4 contains conclusions and some further discussions.

2. Painlevé Integrability

To begin with, we suppose that (1) has the following Laurent expansion:

$$u = \varphi^{-\alpha} \sum_{j=0}^{\infty} u_j \varphi^j, \quad (2)$$

where φ and u_j are functions of $\{x_1, x_2, y_1, y_2, t\}$ and $u_0 \neq 0$. Using the leading order analysis, we take $u \sim u_0 \varphi^{-\rho}$ and obtain

$$\begin{aligned} & u_{x_1 x_1 x_1 x_2} \\ & \sim -\rho(-\rho-1)(-\rho-2)(-\rho-3) u_0 \varphi^{-\rho-4} \varphi_{x_1}^3 \varphi_{x_2}, \\ & u_{x_2 x_2 x_2 x_1} \\ & \sim -\rho(-\rho-1)(-\rho-2)(-\rho-3) u_0 \varphi^{-\rho-4} \varphi_{x_1} \varphi_{x_2}^3, \\ & (u^2)_{x_1 x_2} \sim -2\rho(-2\rho-1) u_0^2 \varphi^{-2\rho-2} \varphi_{x_1} \varphi_{x_2}. \end{aligned} \quad (3)$$

Substituting (3) into (1) and balancing the highest order derivative term $-(1/4)u_{x_1 x_1 x_1 x_2} + (1/4)u_{x_2 x_2 x_2 x_1}$ with the highest order nonlinear term $(3/2)(u^2)_{x_1 x_2}$ yield

$$\begin{aligned} & -\frac{1}{4} [-\rho(-\rho-1)(-\rho-2)(-\rho-3) u_0 \varphi_{x_1}^3 \varphi_{x_2} \varphi^{-\rho-4}] \\ & + \frac{1}{4} [-\rho(-\rho-1)(-\rho-2)(-\rho-3) u_0 \varphi_{x_1} \varphi_{x_2}^3 \varphi^{-\rho-4}] \\ & + \frac{3}{2} [-2\rho(-2\rho-1) u_0^2 \varphi_{x_1} \varphi_{x_2} \varphi^{-2\rho-2}] = 0, \end{aligned} \quad (4)$$

which gives

$$\begin{aligned} & \rho = 2, \\ & u_0 = \varphi_{x_1}^2 - \varphi_{x_2}^2. \end{aligned} \quad (5)$$

Substituting (2) along with $\rho = 2$ into (1) and then comparing the coefficient of φ^{j-6} , we have

$$\begin{aligned} & (j+1)(j-4)(j-5)(j-6) u_0 u_j \varphi_{x_1} \varphi_{x_2} \\ & = 4u_{j-4,tx_1} + [(j-5)u_{j-3}\varphi_t]_{x_1} + [(j-5)u_{j-3}\varphi_{x_1}]_t \\ & \quad - (j-5)u_{j-3}\varphi_{tx_1} + 4(j-4)(j-5)u_{j-2}\varphi_t\varphi_{x_1} \\ & \quad - u_{j-4,x_1x_1x_1x_2} - (j-5)u_{j-3,x_1x_1x_1}\varphi_{x_2} \\ & \quad - 3[(j-5)u_{j-3,x_1x_1}\varphi_{x_1}]_{x_2} \\ & \quad - 3[(j-4)(j-5)u_{j-2,x_1}\varphi_{x_1}\varphi_{x_2}]_{x_1} \\ & \quad + 3(j-4)(j-5)u_{j-2,x_1}\varphi_{x_1}\varphi_{x_1x_2} \\ & \quad - 3[(j-4)(j-5)u_{j-2,x_1}\varphi_{x_1}^2]_{x_2} \\ & \quad - 3[(j-3)(j-4)(j-5)u_{j-1}\varphi_{x_1}^2\varphi_{x_2}]_{x_1} \\ & \quad + 3(j-3)(j-4)(j-5)u_{j-1}\varphi_{x_1}\varphi_{x_2}\varphi_{x_1x_1} \\ & \quad - 3[(j-5)u_{j-3,x_1}\varphi_{x_1x_1}]_{x_2} \\ & \quad - (j-3)(j-4)(j-5)u_{j-1,x_2}\varphi_{x_1}^3 \\ & \quad - 3[(j-4)(j-5)u_{j-2}\varphi_{x_1}\varphi_{x_1x_1}]_{x_2} \\ & \quad - [(j-5)u_{j-3}\varphi_{x_1x_1x_1}]_{x_2} \\ & \quad - (j-4)(j-5)u_{j-2}\varphi_{x_2}\varphi_{x_1x_1x_1} + u_{j-4,x_1x_2x_2x_2} \\ & \quad + (j-5)u_{j-3,x_2x_2x_2}\varphi_{x_1} \\ & \quad + 3[(j-5)u_{j-3,x_2x_2}\varphi_{x_2}]_{x_1} \\ & \quad + 3[(j-4)(j-5)u_{j-2,x_2}\varphi_{x_1}\varphi_{x_2}]_{x_2} \\ & \quad + 3(j-4)(j-5)u_{j-2,x_2}\varphi_{x_2}\varphi_{x_1x_2} \\ & \quad + 3[(j-5)u_{j-3,x_2}\varphi_{x_2x_2}]_{x_1} \\ & \quad + 3[(j-4)(j-5)u_{j-2,x_1}\varphi_{x_2}^2]_{x_2} \\ & \quad - 3(j-4)(j-5)u_{j-2,x_1}\varphi_{x_2}\varphi_{x_2x_2} \\ & \quad + 3[(j-3)(j-4)(j-5)u_{j-1}\varphi_{x_1}\varphi_{x_2}^2]_{x_2} \\ & \quad - 3(j-3)(j-4)(j-5)u_{j-1}\varphi_{x_1}\varphi_{x_2}\varphi_{x_2x_2} \\ & \quad + (j-3)(j-4)(j-5)u_{j-1,x_1}\varphi_{x_2}^3 \\ & \quad + 3[(j-4)(j-5)u_{j-2}\varphi_{x_2}\varphi_{x_2x_2}]_{x_1} \\ & \quad + 3(j-4)(j-5)u_{j-2,x_1}\varphi_{x_2}\varphi_{x_2x_2} \\ & \quad + (j-5)u_{j-3,x_1}\varphi_{x_2x_2x_2} \end{aligned}$$

$$\begin{aligned}
& + (j-4)(j-5)u_{j-2}\varphi_{x_1}\varphi_{x_2x_2x_2} \\
& + 12\sum_{k=0}^{j-2}u_ku_{j-2-k,x_1x_2} \\
& + 12\sum_{k=0}^{j-1}(j-3-k)u_ku_{j-1-k,x_2}\varphi_{x_1} \\
& + 12\sum_{k=0}^{j-1}(j-3-k)u_ku_{j-1-k,x_1}\varphi_{x_2} \\
& + 12\sum_{k=1}^{j-1}(j-2-k)(j-3-k)u_ku_{j-k}\varphi_{x_1}\varphi_{x_2} \\
& + 12\sum_{k=0}^{j-1}(j-3-k)u_ku_{j-1-k}\varphi_{x_1x_2} \\
& + 12\sum_{k=0}^{j-2}u_{k,x_1}u_{j-2-k,x_2} \\
& + 12\sum_{k=0}^{j-1}(j-3-k)u_{k,x_2}u_{j-1-k}\varphi_{x_1} \\
& + 12\sum_{k=0}^{j-1}(j-3-k)u_{k,x_1}u_{j-1-k}\varphi_{x_2} \\
& + 12\sum_{k=1}^{j-1}(j-2-k)u_ku_{j-k}\varphi_{x_1}\varphi_{x_2} - 6u_{j-4,y_1y_2} \\
& - 6(j-5)u_{j-3,y_1}\varphi_{y_2} - 6(j-5)u_{j-3,y_2}\varphi_{y_1} \\
& - 6(j-4)(j-5)u_{j-2}\varphi_{y_1}\varphi_{y_2} \\
& - 6(j-5)u_{j-3}\varphi_{y_1y_2},
\end{aligned} \tag{6}$$

which can be written as

$$\begin{aligned}
& (j+1)(j-4)(j-5)(j-6)u_j \\
& = F(u_0, u_1, u_2, \dots, u_{j-1}, \varphi_{x_1}, \varphi_{x_2}\varphi_{y_1}, \varphi_{y_2}, \varphi_t, \dots),
\end{aligned} \tag{7}$$

where the right hand side of (7) is a function of $u_0, u_1, u_2, \dots, u_{j-1}, \varphi$ and their partial derivatives. We can see from (7) that $j = -1, 4, 5, 6$ are the resonance points. The resonance at $j = -1$ corresponds to the arbitrariness of φ , which describes the singular hypersurface. In order to test whether the resonance points $j = 4, 5, 6$ satisfy the compatibility condition (7), following Kruskal et al.'s simplified idea [22] of the WTC method, we take the following transformation:

$$\begin{aligned}
\varphi &= \varphi(\xi), \\
\xi &= k_1x_1 + k_2x_2 + \eta, \\
\eta &= \eta(y_1, y_2, t).
\end{aligned} \tag{8}$$

Substituting (8) into (7), we obtain the resonance conditions that $j = 4, 5, 6$ pass through the Painlevé test as follows:

$$\begin{aligned}
u_0 &= (k_1^2 - k_2^2)\varphi_\xi^2, \\
u_1 &= -(k_1^2 - k_2^2)\varphi_{\xi\xi}, \\
u_2 &= \frac{k_1k_2(k_1^2 - k_2^2)(4\varphi_\xi\varphi_{\xi\xi\xi} - 3\varphi_{\xi\xi}^2) - 2\varphi_\xi^2(2k_1\eta_t - 3\eta_{y_1}\eta_{y_2})}{12k_1k_2\varphi_\xi^2}, \\
u_3 &= \frac{k_1k_2(k_1^2 - k_2^2)(4\varphi_\xi\varphi_{\xi\xi}\varphi_{\xi\xi\xi} - 3\varphi_{\xi\xi}^3 - \varphi_\xi^2\varphi_{\xi\xi\xi\xi}) - 6\varphi_\xi^3\eta_{y_1y_2}}{12k_1k_2\varphi_\xi^4}, \\
2k_1k_2\varphi_{\xi\xi}\varphi_{\xi\xi}(k_1^2 + k_2^2)(k_1u_{4y_2} - k_2u_{4y_1}) \\
&+ 2k_1k_2\varphi_\xi^3(k_1u_{5y_2} - k_2u_{5y_1}) \\
&+ 2k_1k_2\varphi_\xi^2[k_1k_2u_{4y_2y_2} + (k_1^2 - k_2^2)u_{4y_1y_2} - k_1k_2u_{4y_1y_1}] \\
&+ \eta_{y_1y_1}^2 - \eta_{y_1y_1}\eta_{y_2y_2} = 0.
\end{aligned} \tag{9}$$

This shows that the Fokas equation (1) possesses Painlevé integrability in the sense of WTC method.

3. Exact Solutions

It is easy to see from (7) that u_j ($j = 7, 8, \dots$) can be determined except for u_4, u_5, u_6 . In order to truncate Painlevé expansion (2) to construct exact solutions of the Fokas equation (1), we further set u_j ($j > 3$) = 0 and take

$$\varphi(\xi) = 1 + e^{k_1x_1 + k_2x_2 + \eta(y_1, y_2, t)}. \tag{10}$$

In this case, (2) is truncated as

$$u = u_0\varphi^{-2}(\xi) + u_1\varphi^{-1}(\xi) + u_2 + u_3\varphi(\xi), \tag{11}$$

where u_0, u_1, u_2, u_3 are determined by (9) and (10) and the following system of PDEs:

$$\begin{aligned}
\eta_{y_1y_2} &= 0, \\
\eta_{y_1y_1}\eta_{y_2y_2} &= 0.
\end{aligned} \tag{12}$$

Obviously, the system of PDEs (12) has two solutions

$$\eta(y_1, y_2, t) = f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t), \tag{13}$$

$$\eta(y_1, y_2, t) = f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t), \tag{14}$$

where $f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)$, and $f_6(t)$ are differentiable functions with respect to t , while $g_1(y_2)$ and $g_2(y_1)$ are differentiable functions of y_2 and y_1 , respectively.

Substituting (13) into (9), we have

$$\begin{aligned}
 u_0 &= (k_1^2 - k_2^2) e^{2k_1 x_1 + 2k_2 x_2 + 2f_1(t)y_1 + 2f_2(t)g_1(y_2) + 2f_3(t)}, \\
 u_1 &= -(k_1^2 - k_2^2) e^{k_1 x_1 + k_2 x_2 + f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t)} \\
 u_2 &= \frac{6f_1(t)f_2(t)g_1'(y_2) + k_1^3 k_2 - k_1 k_2^3 - 4k_1 y_1 f_1'(t) - 4k_1 g_1(y_2) f_2'(t) - 4k_1 f_3'(t)}{12k_1 k_2}, \\
 u_3 &= 0,
 \end{aligned} \tag{15}$$

and hence obtain an exact solution of the Fokas equation (1):

$$\begin{aligned}
 u &= \frac{(k_1^2 - k_2^2) e^{2k_1 x_1 + 2k_2 x_2 + 2f_1(t)y_1 + 2f_2(t)g_1(y_2) + 2f_3(t)}}{[1 + e^{k_1 x_1 + k_2 x_2 + f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t)}]^2} - \frac{(k_1^2 - k_2^2) e^{k_1 x_1 + k_2 x_2 + f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t)}}{1 + e^{k_1 x_1 + k_2 x_2 + f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t)}} \\
 &+ \frac{6f_1(t)f_2(t)g_1'(y_2) + k_1^3 k_2 - k_1 k_2^3 - 4k_1 y_1 f_1'(t) - 4k_1 g_1(y_2) f_2'(t) - 4k_1 f_3'(t)}{12k_1 k_2} \\
 &= -\frac{(k_1^2 - k_2^2)}{k_1 k_2} \frac{\partial^2}{\partial x_1 \partial x_2} \ln [1 + e^{k_1 x_1 + k_2 x_2 + f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t)}] \\
 &+ \frac{6f_1(t)f_2(t)g_1'(y_2) + k_1^3 k_2 - k_1 k_2^3 - 4k_1 y_1 f_1'(t) - 4k_1 g_1(y_2) f_2'(t) - 4k_1 f_3'(t)}{12k_1 k_2}.
 \end{aligned} \tag{16}$$

Similarly, substituting (14) into (9) yields

$$\begin{aligned}
 u_0 &= (k_1^2 - k_2^2) e^{2k_1 x_1 + 2k_2 x_2 + 2f_4(t)g_2(y_1) + 2f_5(t)y_2 + 2f_6(t)}, \\
 u_1 &= -(k_1^2 - k_2^2) e^{k_1 x_1 + k_2 x_2 + f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t)}, \\
 u_2 &= \frac{6f_4(t)f_5(t)g_2'(y_1) + k_1^3 k_2 - k_1 k_2^3 - 4k_1 g_2(y_1) f_4'(t) - 4k_1 y_2 f_5'(t) - 4k_1 f_6'(t)}{12k_1 k_2}, \\
 u_3 &= 0
 \end{aligned} \tag{17}$$

and hence obtains another exact solution of the Fokas equation (1):

$$\begin{aligned}
 u &= \frac{(k_1^2 - k_2^2) e^{2k_1 x_1 + 2k_2 x_2 + 2f_4(t)g_2(y_1) + 2f_5(t)y_2 + 2f_6(t)}}{[1 + e^{k_1 x_1 + k_2 x_2 + f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t)}]^2} - \frac{(k_1^2 - k_2^2) e^{k_1 x_1 + k_2 x_2 + f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t)}}{1 + e^{k_1 x_1 + k_2 x_2 + f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t)}} \\
 &+ \frac{6f_4(t)f_5(t)g_2'(y_1) + k_1^3 k_2 - k_1 k_2^3 - 4k_1 g_2(y_1) f_4'(t) - 4k_1 y_2 f_5'(t) - 4k_1 f_6'(t)}{12k_1 k_2} \\
 &= -\frac{(k_1^2 - k_2^2)}{k_1 k_2} \frac{\partial^2}{\partial x_1 \partial x_2} \ln [1 + e^{k_1 x_1 + k_2 x_2 + f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t)}] \\
 &+ \frac{6f_4(t)f_5(t)g_2'(y_1) + k_1^3 k_2 - k_1 k_2^3 - 4k_1 g_2(y_1) f_4'(t) - 4k_1 y_2 f_5'(t) - 4k_1 f_6'(t)}{12k_1 k_2}.
 \end{aligned} \tag{18}$$

FIGURE 1: Bell-shaped solitons determined by solution (16).

FIGURE 2: Space-time structures of solution (16).

Figure 1 shows two bell-shaped solitons determined by solution (16) at the time $t = 0$, where the parameters are selected as $f_1(t) = \text{cn}(t, 0.6)$, $f_2(t) = 1 + t^2$, $f_3(t) = \sin t$, $g_1(y_2) = y_2$, $k_1 = 0.8$, and $k_2 = 1.2$. Figure 2 describes two space-time structures of the solitons shown in Figure 1.

Since multiwave solutions are more crucial to the integrabilities of nonlinear PDEs, in what follows we would like to focus on two-wave solutions and three-wave solutions of the Fokas equation (1). For this purpose, we substitute $f_1(t) = p_1$, $f_2(t) = q_1$, $f_3(t) = c_1 t$, $g_2(y_1) = y_1$, and $g_1(y_2) = y_2$ into (16); then solution (16) can be rewritten as

$$\begin{aligned} u &= -(k_1^2 - k_2^2) \left[\ln(1 + e^{\xi_1}) \right]_{xx} \\ &= -(k_1^2 - k_2^2) \text{sech}^2 \frac{1}{2} \xi_1, \end{aligned} \quad (19)$$

where

$$\begin{aligned} \xi_1 &= x + p_1 y_1 + q_1 y_2 + c_1 t, \\ x &= k_1 x_1 + k_2 x_2, \\ c_1 &= \frac{1}{4} k_2 (k_1^2 - k_2^2) + \frac{3p_1 q_1}{2k_1}. \end{aligned} \quad (20)$$

Inspired by the one-wave solution (19), we obtain the two-wave solution of the Fokas equation (1):

$$u = -(k_1^2 - k_2^2) \left[\ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}) \right]_{xx}, \quad (21)$$

where

$$\begin{aligned} \xi_2 &= x + p_2 y_1 + q_2 y_2 + c_2 t, \\ x &= k_1 x_1 + k_2 x_2, \\ c_2 &= \frac{1}{4} k_2 (k_1^2 - k_2^2) + \frac{3p_2 q_2}{2k_1}, \end{aligned} \quad (22)$$

$$e^{A_{12}} = \frac{(p_2 - p_1)(q_2 - q_1)}{2k_1 k_2 (k_2^2 - k_1^2) + (p_2 - p_1)(q_2 - q_1)}.$$

For the three-wave solution of the Fokas equation (1), we have

$$\begin{aligned} u &= -(k_1^2 - k_2^2) \left[\ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1 + \xi_2 + A_{12}} \right. \\ &\quad \left. + e^{\xi_1 + \xi_3 + A_{13}} + e^{\xi_2 + \xi_3 + A_{23}} \right. \\ &\quad \left. + e^{\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}}) \right]_{xx}, \end{aligned} \quad (23)$$

where

$$\begin{aligned}
 \xi_3 &= x + p_3 y_1 + q_3 y_2 + c_3 t, \\
 x &= k_1 x_1 + k_2 x_2, \\
 c_3 &= \frac{1}{4} k_2 (k_1^2 - k_2^2) + \frac{3 p_3 q_3}{2 k_1}, \\
 e^{A_{13}} &= \frac{(p_3 - p_1)(q_3 - q_1)}{2 k_1 k_2 (k_2^2 - k_1^2) + (p_3 - p_1)(q_3 - q_1)}, \\
 e^{A_{23}} &= \frac{(p_3 - p_2)(q_3 - q_2)}{2 k_1 k_2 (k_2^2 - k_1^2) + (p_3 - p_2)(q_3 - q_2)}.
 \end{aligned} \tag{24}$$

4. Conclusions and Discussions

In summary, we have employed the WCT method of Painlevé analysis to study the $(4 + 1)$ -dimensional Fokas equation (1). The obtained results show that the Fokas equation can pass through Painlevé test and hence possesses Painlevé integrability. With the help of the truncated Painlevé expansion (11), two exact solutions (16) and (18) with arbitrary differentiable functions of the Fokas equation (1) are obtained. Inspired by the one-wave solution (19) reduced from solution (16), two-wave solution (21) and three-wave solution (23) are obtained. To the best of our knowledge, the obtained solutions (16), (18), (21), and (23) are new. The arbitrariness of the differentiable functions $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$, $f_5(t)$, $f_6(t)$, $g_1(y_2)$, and $g_2(y_1)$ makes the obtained solutions (16) and (18) have rich spatial structures. If we select $f_1(t) = 2l_1$, $f_2(t) = 2l_2$, $f_3(t) = 2ct$, $g_2(y_1) = y_1$, $g_1(y_2) = y_2$, $k_1 = 2k_1$, and $k_2 = 2k_2$, then solution (16) can be simplified as the known solution [27]:

$$\begin{aligned}
 u &= (k_1^2 - k_2^2) \tanh^2(k_1 x_1 + k_2 x_2 + l_1 y_1 + l_2 y_2 + ct) \\
 &\quad - \frac{2ck_1 - 3l_1 l_2 + 4k_1^3 k_2 - 4k_1 k_2^3}{6k_1 k_2}.
 \end{aligned} \tag{25}$$

It should be noted that if $\varphi(\xi)$ in (10) is selected as some other forms, for example, $1 - e^{k_1 x_1 + k_2 x_2 + \eta(y_1, y_2, t)}$, we can obtain other exact solutions of the Fokas equation (1) by the similar processes. For the sake of simplification, we omit them here. With the help of symbolic computation system *Mathematica*, we have tested the validities of solutions (16) and (18), one-wave solution (19), two-wave solution (21), and three-wave solution (23) by substituting them back into the Fokas equation (1). In 2010, Ma et al. [30] proposed a multiple exp-function method to construct multiwave solutions, which is the most general approach for nonlinear PDEs in higher dimensions [31]. For the Fokas equation (1), solutions (19), (21), and (23) can be obtained by the multiple exp-function method. In general, we often associate multiwave solutions with bilinear differential equations derived from a given nonlinear PDE by the use of Hirota bilinear operator. Recently, new kinds of bilinear differential operators were introduced in [32], which substantially generalize Hirota bilinear operators. This kind of bilinear differential operator may provide a useful help in constructing multiwave solutions or testing integrability of nonlinear PDEs. Besides, there

are pretty interesting links between the truncated Painlevé series and other math research objects (see, e.g., [33]). Constructing multiwave solutions and testing integrability, as well as exploring the related problems of some other new nonlinear PDEs, are worthy of study. This is our task in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Natural Science Foundation of China (11547005), the PhD Start-up Funds of Liaoning Province of China (20141137) and Bohai University (bsqd2013025), the Liaoning BaiQianWan Talents Program (2013921055), and the Natural Science Foundation of Liaoning Province of China (L2012404).

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