Research Article

An Alternating Direction Method for Convex Quadratic Second-Order Cone Programming with Bounded Constraints

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An alternating direction method is proposed for convex quadratic second-order cone programming problems with bounded constraints. In the algorithm, the primal problem is equivalent to a separate structure convex quadratic programming over second-order cones and a bounded set. At each iteration, we only need to compute the metric projection onto the second-order cones and the projection onto the bound set. The result of convergence is given. Numerical results demonstrate that our method is efficient for the convex quadratic second-order cone programming problems with bounded constraints.

1. Introduction

In this paper, we consider a convex quadratic second-order cone programming (CQSOCP) problem with bounded constraints which is defined by minimizing a convex quadratic function over the intersection of an affine set, a bounded set, and the product of second-order cones. The primal convex quadratic second-order cone programming problem is defined as

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Qx + c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in K, \quad x \in \Omega, \end{aligned} \quad (1)$$

where $\Omega = \{x \mid l \leq x \leq u\}$ is a bounded set, Q is an $n \times n$ symmetric positive semidefinite matrix, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $x = [x_1, \dots, x_N] \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}$ is viewed as a column vector in $\mathbb{R}^{n_1 + \dots + n_N}$ with $\sum_{i=1}^N n_i = n$. In addition, $K = K^{n_1} \times K^{n_2} \times \dots \times K^{n_N}$ and $x_i \in K^{n_i}$, where K^{n_i} is the n_i -dimensional second-order cone given by

$$K^{n_i} = \left\{ x_i = \begin{bmatrix} x_{i_1} \\ x_{i_0} \end{bmatrix} \in \mathbb{R}^{n_i-1} \times \mathbb{R} : \|x_{i_1}\|_2 \leq x_{i_0} \right\}, \quad (2)$$

where $\|\cdot\|_2$ is the standard Euclidean norm.

Convex quadratic second-order cone programming problem with bounded constraints is a nonlinear programming problem, which can be seen as a trust region subproblem in the trust region method for the nonlinear second-order cone programming [1, 2]. Since Q is symmetric positive semidefinite, we can compute its positive semidefinite square root $Q^{1/2}$ by the Cholesky method. Then problem (1) can be equivalently transformed as the following mix linear and second-order cone programming (MLSOCP) [3]:

$$\begin{aligned} \min \quad & t + c^T x \\ \text{s.t.} \quad & Ax = b \\ & \sqrt{(t-1)^2 + 2\|Q^{1/2}x\|_2^2} \leq t + 1 \\ & l \leq x \leq u \\ & x \in K. \end{aligned} \quad (3)$$

In paper [1, 2], the authors use those well developed and publicly available softwares, based on interior-point methods, such as SeDuMi [4] and SDPT3 [5] to solve the equivalent MLSOCP (3).

Interior-point methods have been well developed for linear symmetric cone programming [6–8]. However, at each

iteration these solvers require to formulate and solve a dense Schur complement matrix, which for the CQSOCP problem with bounded constraints amounts to a linear system of dimension $(m + 3n + 2) \times (m + 3n + 2)$. In addition, the transformed method needs to compute the square root of semidefinite matrix Q . When n is large, because of the very large size and ill-conditioning of the linear system of equations, interior-point methods are difficult to solve the transformed MLSOCP problem efficiently [3].

The alternating direction method (ADM) has been an effective first-order approach for solving large optimization problems, such as linear programming [9], linear semidefinite programming (LSDP) [10, 11], nonlinear convex optimization [12], and nonsmooth l_1 minimization arising from compressive sensing [13, 14]. A modified alternating direction method is proposed for convex quadratically constrained quadratic semidefinite programs in paper [15]. In the thesis [3], a semismooth Newton-CG augmented Lagrangian method is proposed for large scale convex quadratic symmetric cone programming. In paper [16], an alternating direction dual augmented Lagrangian method for solving linear semidefinite programming problems in standard form is presented and extended to the SDP with inequality constraints and positivity constraints.

In the paper, an alternating direction method for the CQSOCP problem with bounded constraints is proposed. Firstly, the primal problem is equivalent to a separate structure convex quadratic programming over second-order cones and a bounded set. Then the alternating direction method is proposed to solve the separate structure convex quadratic programming. In the alternating direction method, we only need to compute the metric projection onto the second-order cones and projection onto the bounded set at each iteration. We also give the convergence results and the numerical results.

2. The Projection on the Second-Order Cone and the Bounded Set

In this section, we will give the projection results on the second-order cones and the bounded set.

Let $x_i = \begin{bmatrix} x_{i1} \\ x_{i0} \end{bmatrix} \in \mathfrak{R}^{n_i-1} \times \mathfrak{R}$ for $i = 1, 2, \dots, N$; then the spectral decomposition of x_i associate with second-order cone K^{n_i} can be described as [17–19]

$$x_i = \lambda_1(x_i) c_1(x_i) + \lambda_2(x_i) c_2(x_i), \quad i = 1, 2, \dots, N, \quad (4)$$

where

$$\begin{aligned} \lambda_1(x_i) &= x_{i0} - \|x_{i1}\|_2, \\ \lambda_2(x_i) &= x_{i0} + \|x_{i1}\|_2, \\ c_1(x_i) &= \frac{1}{2} \begin{bmatrix} -w \\ 1 \end{bmatrix}, \\ c_2(x_i) &= \frac{1}{2} \begin{bmatrix} w \\ 1 \end{bmatrix} \end{aligned} \quad (5)$$

with $w = -x_{i1}/\|x_{i1}\|_2$ if $x_{i1} \neq 0$ and any vector in $\mathfrak{R}^{n_i-1}$ satisfying $\|w\|_2 = 1$ if $x_{i1} = 0$.

Next we introduce the projection lemma over the second-order cone [17–19].

Lemma 1 (see [17–19]). *For any $x_i = \begin{bmatrix} x_{i1} \\ x_{i0} \end{bmatrix} \in \mathfrak{R}^{n_i-1} \times \mathfrak{R}$, let $P_{K^{n_i}}(x_i)$ be the projection of x_i onto the second-order cone K^{n_i} ; then we have*

$$P_{K^{n_i}}(x_i) = (\lambda_1(x_i))_+ c_1(x_i) + (\lambda_2(x_i))_+ c_2(x_i), \quad (6)$$

$$i = 1, 2, \dots, N,$$

where $s_+ := \max(0, s)$ for $s \in \mathfrak{R}$.

Let $x = [x_1, \dots, x_N] \in \mathfrak{R}^{n_1} \times \dots \times \mathfrak{R}^{n_N}$; then the projection $P_K(x)$ of x over the cone K is described as

$$P_K(x) = [P_{K^{n_1}}(x_1), \dots, P_{K^{n_N}}(x_N)] \in \mathfrak{R}^{n_1} \times \dots \times \mathfrak{R}^{n_N}. \quad (7)$$

Let $y \in \mathfrak{R}^n$; then the projection on the bounded set Ω is easy to carry out, namely, through an element by element method:

$$P_\Omega(y) = \max\{l, \min\{y, u\}\}. \quad (8)$$

3. An Alternating Direction Method for CQSOCP Problems with Bounded Constraints

In this section, we give an alternating direction method for convex quadratic second-order cone programming problems with bounded constraints.

Firstly, we give an equivalent separate structure convex quadratic programming over second-order cone and bounded set as follows:

$$\begin{aligned} \min \quad & \frac{1}{2} x^T Q x + c^T y \\ \text{s.t.} \quad & A x = b, \quad x \in K \\ & x = y, \quad y \in \Omega. \end{aligned} \quad (9)$$

The Lagrangian function for the separate structure convex quadratic programming problem is written as

$$\begin{aligned} \max_{\lambda, \mu} \min_{x \in K, y \in \Omega} L(x, y, \lambda, \mu) \\ = \frac{1}{2} x^T Q x + c^T y - \lambda^T (A x - b) - \mu^T (x - y), \end{aligned} \quad (10)$$

where $\lambda \in \mathfrak{R}^m, \mu \in \mathfrak{R}^n$.

Under mild constraint qualifications (e.g., Slater condition), strong duality holds for problem (9), and hence, x^* is an optimal solution of (9) if and only if there exists

$(x^*, y^*, \lambda^*, \mu^*) \in K \times \Omega \times \mathfrak{R}^m \times \mathfrak{R}^n$ satisfying the following KKT system in variational inequality form:

$$\begin{aligned} \langle x - x^*, Qx^* - A^T \lambda^* - \mu^* \rangle &\geq 0, \quad \forall x \in K, \\ \langle y - y^*, c + \mu^* \rangle &\geq 0, \quad \forall y \in \Omega, \\ Ax^* &= b, \\ x^* &= y^*. \end{aligned} \quad (11)$$

The augmented Lagrangian function for the the separate structure convex quadratic programming problem is defined as

$$\begin{aligned} L(x, y, \lambda, \mu) &= \frac{1}{2} x^T Q x + c^T y - \lambda^T (Ax - b) \\ &\quad - \mu^T (x - y) + \frac{1}{2\beta_1} \|Ax - b\|_2^2 \\ &\quad + \frac{1}{2\beta_2} \|x - y\|_2^2, \end{aligned} \quad (12)$$

where $\beta_1, \beta_2 > 0$.

The variational inequality form of alternating direction method for (12) is as follows.

3.1. The Original Alternating Direction Method. Given $x^0 \in K, y^0 \in \Omega, \lambda^0 \in \mathfrak{R}^m, \mu^0 \in \mathfrak{R}^n$, and $\beta_1, \beta_2 > 0$. For $k = 0, 1, 2, \dots$, then consider the following.

Step 1. Consider $(x^k, y^k, \lambda^k, \mu^k) \rightarrow (x^{k+1}, y^k, \lambda^k, \mu^k)$; we compute x^{k+1} , which satisfies

$$\begin{aligned} \left\langle x - x^{k+1}, Qx^{k+1} - A^T \lambda^k - \mu^k + \frac{1}{\beta_1} A^T (Ax^{k+1} - b) \right. \\ \left. + \frac{1}{\beta_2} (x^{k+1} - y^k) \right\rangle \geq 0, \quad \forall x \in K. \end{aligned} \quad (13)$$

Step 2. Consider $(x^{k+1}, y^k, \lambda^k, \mu^k) \rightarrow (x^{k+1}, y^{k+1}, \lambda^k, \mu^k)$; we compute y^{k+1} , which satisfies

$$\left\langle y - y^{k+1}, c + \mu^k - \frac{1}{\beta_2} (x^{k+1} - y^{k+1}) \right\rangle \geq 0, \quad \forall y \in \Omega. \quad (14)$$

Step 3. Consider $(x^{k+1}, y^{k+1}, \lambda^k, \mu^k) \rightarrow (x^{k+1}, y^{k+1}, \lambda^{k+1}, \mu^k)$; update the Lagrange multiplier by

$$\lambda^{k+1} = \lambda^k - \frac{1}{\beta_1} (Ax^{k+1} - b). \quad (15)$$

Step 4. Consider $(x^{k+1}, y^{k+1}, \lambda^{k+1}, \mu^k) \rightarrow (x^{k+1}, y^{k+1}, \lambda^{k+1}, \mu^{k+1})$; update the Lagrange multiplier by

$$\mu^{k+1} = \mu^k - \frac{1}{\beta_2} (x^{k+1} - y^{k+1}). \quad (16)$$

In Steps 1 and 2, we should solve variational inequalities. In the following analysis, we will convert them to simple projection operations.

Lemma 2 (see [20]). *Let Θ be a closed convex set in a Hilbert space and let $P_\Theta(x)$ be the projection of x onto Θ . Then*

$$\langle z - y, y - x \rangle \geq 0, \quad \forall z \in \Theta \iff y = P_\Theta(x). \quad (17)$$

Taking $x = x^{k+1} - \alpha_1(Qx^{k+1} - A^T \lambda^k - \mu^k + (1/\beta_1)A^T(Ax^{k+1} - b) + (1/\beta_2)(x^{k+1} - y^k))$ and $y = x^{k+1}$ in (17), we see that (13) is equivalent to the following nonlinear equation:

$$\begin{aligned} x^{k+1} &= P_K \left[x^{k+1} - \alpha_1 \left(Qx^{k+1} - A^T \lambda^k - \mu^k \right. \right. \\ &\quad \left. \left. + \frac{1}{\beta_1} A^T (Ax^{k+1} - b) + \frac{1}{\beta_2} (x^{k+1} - y^k) \right) \right], \end{aligned} \quad (18)$$

where α_1 can be any positive number.

Taking $x = y^{k+1} - \alpha_2(c + \mu^k - (1/\beta_2)(x^{k+1} - y^{k+1}))$ and $y = y^{k+1}$ in (17), we see that (14) is equivalent to the following nonlinear equation:

$$y^{k+1} = P_\Omega \left[y^{k+1} - \alpha_2 \left(c + \mu^k - \frac{1}{\beta_2} (x^{k+1} - y^{k+1}) \right) \right], \quad (19)$$

where α_2 can be any positive number.

Due to the existence of the terms Qx^{k+1} and $A^T Ax^{k+1}$ in (18), we can not compute x^{k+1} directly. We therefore use the following approximate approach which is similar to the one in paper [15]. For certain constants γ_1 and γ_2 , let

$$\begin{aligned} R_1(x^k, x^{k+1}) &= Qx^{k+1} - Qx^k - \gamma_1(x^{k+1} - x^k), \\ R_2(x^k, x^{k+1}) &= A^T Ax^{k+1} - A^T Ax^k - \gamma_2(x^{k+1} - x^k) \end{aligned} \quad (20)$$

be the residual between $Qx^{k+1}, A^T Ax^{k+1}$ and their linearization at x^k , respectively.

Instead of computing (18), we compute

$$\begin{aligned} x^{k+1} &= P_K \left[x^{k+1} - \alpha_1 \left(Qx^{k+1} - A^T \lambda^k - \mu^k \right. \right. \\ &\quad \left. \left. + \frac{1}{\beta_1} A^T (Ax^{k+1} - b) + \frac{1}{\beta_2} (x^{k+1} - y^k) \right. \right. \\ &\quad \left. \left. - R_1(x^k, x^{k+1}) - \frac{1}{\beta_1} R_2(x^k, x^{k+1}) \right) \right] = P_K \left[x^{k+1} \right. \\ &\quad \left. - \alpha_1 \left(\left(\frac{1}{\beta_2} + \gamma_1 + \frac{\gamma_2}{\beta_1} \right) x^{k+1} - A^T \lambda^k - \mu^k - \frac{1}{\beta_1} A^T b \right. \right. \\ &\quad \left. \left. - \frac{1}{\beta_2} y^k + Qx^k - \gamma_1 x^k + \frac{1}{\beta_1} A^T Ax^k - \frac{\gamma_2}{\beta_1} x^k \right) \right]. \end{aligned} \quad (21)$$

We choose γ_1, γ_2 so that $\gamma_1 > \lambda_{\max}(Q), \gamma_2 > \lambda_{\max}(A^T A)$, where $\lambda_{\max}(Q)$ and $\lambda_{\max}(A^T A)$ are the largest eigenvalues of Q and $A^T A$, respectively.

Setting

$$\alpha_1 = \left(\frac{1}{\beta_2} + \gamma_1 + \frac{\gamma_2}{\beta_1} \right)^{-1} \quad (22)$$

in (21), we have

$$\begin{aligned} x^{k+1} = P_K \left[\alpha_1 \left(A^T \lambda^k + \mu^k + \frac{1}{\beta_1} A^T b + \frac{1}{\beta_2} y^k - Qx^k \right. \right. \\ \left. \left. + \gamma_1 x^k - \frac{1}{\beta_1} A^T A x^k + \frac{\gamma_2}{\beta_1} x^k \right) \right], \end{aligned} \quad (23)$$

which will be used as an approximation to the solution of variational inequality (13).

Let $\alpha_2 = \beta_2$ in (19); we have

$$y^{k+1} = P_\Omega \left[-\alpha_2 \left(c + \mu^k - \frac{1}{\beta_2} x^{k+1} \right) \right]. \quad (24)$$

In summary, the modified alternating direction method is given as follows.

3.2. The Modified Alternating Direction Method. Given $x^0 \in K$, $y^0 \in \Omega$, $\lambda^0 \in \mathfrak{R}^m$, $\mu^0 \in \mathfrak{R}^n$, and $\beta_1, \beta_2 > 0$. For $k = 0, 1, 2, \dots$, then consider the following.

Step 1. Consider $(x^k, y^k, \lambda^k, \mu^k) \rightarrow (x^{k+1}, y^k, \lambda^k, \mu^k)$; we compute x^{k+1} , which satisfies

$$\begin{aligned} x^{k+1} = P_K \left[\left(\frac{1}{\beta_2} + \gamma_1 + \frac{\gamma_2}{\beta_1} \right)^{-1} \left(A^T \lambda^k + \mu^k + \frac{1}{\beta_1} A^T b \right. \right. \\ \left. \left. + \frac{1}{\beta_2} y^k - Qx^k + \gamma_1 x^k - \frac{1}{\beta_1} A^T A x^k + \frac{\gamma_2}{\beta_1} x^k \right) \right]. \end{aligned} \quad (25)$$

Step 2. Consider $(x^{k+1}, y^k, \lambda^k, \mu^k) \rightarrow (x^{k+1}, y^{k+1}, \lambda^k, \mu^k)$; we compute y^{k+1} , which satisfies

$$y^{k+1} = P_\Omega \left[x^{k+1} - \beta_2 (c + \mu^k) \right]. \quad (26)$$

Step 3. Consider $(x^{k+1}, y^{k+1}, \lambda^k, \mu^k) \rightarrow (x^{k+1}, y^{k+1}, \lambda^{k+1}, \mu^k)$; update the Lagrange multiplier by

$$\lambda^{k+1} = \lambda^k - \frac{1}{\beta_1} (A x^{k+1} - b). \quad (27)$$

Step 4. Consider $(x^{k+1}, y^{k+1}, \lambda^{k+1}, \mu^k) \rightarrow (x^{k+1}, y^{k+1}, \lambda^{k+1}, \mu^{k+1})$; update the Lagrange multiplier by

$$\mu^{k+1} = \mu^k - \frac{1}{\beta_2} (x^{k+1} - y^{k+1}). \quad (28)$$

From Steps 1 and 2, the modified alternating direction method only needs to compute the metric projection of vectors onto K and Ω . From Steps 3 and 4, we could interpret $1/\beta_1$ and $1/\beta_2$ as the dual stepsizes. Therefore, the iteration of our method is simple and fast.

4. The Convergence Result

In this section, we extended and modified the convergence results of the alternating direction methods for convex quadratically constrained quadratic semidefinite programs in paper [15] and then give the convergence analysis of the alternating direction method for CQSOCP problems with bounded constraints.

Lemma 3. *The sequence $\{x^k, y^k, \lambda^k, \mu^k\}$ generated by the modified alternating direction method satisfies*

$$\begin{aligned} & \left\langle x^{k+1} - x^*, R_1(x^k, x^{k+1}) + \frac{1}{\beta_1} R_2(x^k, x^{k+1}) \right\rangle \\ & + \frac{1}{\beta_2} \langle y^{k+1} - y^*, y^k - y^{k+1} \rangle \\ & + \beta_1 \langle \lambda^{k+1} - \lambda^*, \lambda^k - \lambda^{k+1} \rangle \\ & + \beta_2 \langle \mu^{k+1} - \mu^*, \mu^k - \mu^{k+1} \rangle \geq 0, \end{aligned} \quad (29)$$

where $\{x^*, y^*, \lambda^*, \mu^*\}$ is a KKT point of system (11).

Proof. Let $y = y^{k+1}$ in the second inequality in system (11); we have

$$\langle y^{k+1} - y^*, c + \mu^* \rangle \geq 0. \quad (30)$$

Let $y = y^*$ in (14), and coupled with (16), we have

$$\langle y^* - y^{k+1}, c + \mu^{k+1} \rangle \geq 0. \quad (31)$$

Adding (30) and (31) together, we have

$$\langle y^{k+1} - y^*, \mu^* - \mu^{k+1} \rangle \geq 0. \quad (32)$$

In addition, from (14) and (16), we have

$$\begin{aligned} & \langle y^k - y^{k+1}, c + \mu^{k+1} \rangle \geq 0, \\ & \langle y^{k+1} - y^k, c + \mu^k \rangle \geq 0. \end{aligned} \quad (33)$$

Adding the two inequality above, we have

$$\langle y^{k+1} - y^k, \mu^k - \mu^{k+1} \rangle \geq 0. \quad (34)$$

Note that (21) can be written equivalently as

$$\begin{aligned} & \left\langle x - x^{k+1}, Qx^{k+1} - A^T \lambda^{k+1} - \mu^{k+1} + \frac{1}{\beta_2} (y^{k+1} - y^k) \right. \\ & \left. - R_1(x^k, x^{k+1}) - \frac{1}{\beta_1} R_2(x^k, x^{k+1}) \right\rangle \geq 0, \end{aligned} \quad (35)$$

$\forall x \in K$.

Setting $x = x^*$, we have

$$\begin{aligned} & \left\langle x^* - x^{k+1}, Qx^{k+1} - A^T \lambda^{k+1} - \mu^{k+1} + \frac{1}{\beta_2} (y^{k+1} - y^k) \right. \\ & \left. - R_1(x^k, x^{k+1}) - \frac{1}{\beta_1} R_2(x^k, x^{k+1}) \right\rangle \geq 0. \end{aligned} \quad (36)$$

Let $x = x^{k+1}$ in the first inequality in system (11); we have

$$\langle x^{k+1} - x^*, Qx^* - A^T \lambda^* - \mu^* \rangle \geq 0. \quad (37)$$

Adding (36) and (37) together, we have

$$\begin{aligned} & \langle x^{k+1} - x^*, A^T (\lambda^{k+1} - \lambda^*) \rangle \\ & + \langle x^{k+1} - x^*, \mu^{k+1} - \mu^* \rangle \\ & + \left\langle x^{k+1} - x^*, \frac{1}{\beta_2} (y^k - y^{k+1}) \right\rangle \\ & + \left\langle x^{k+1} - x^*, R_1 (x^k, x^{k+1}) + \frac{1}{\beta_1} R_2 (x^k, x^{k+1}) \right\rangle \\ & \geq \langle x^{k+1} - x^*, Q(x^{k+1} - x^*) \rangle \geq 0. \end{aligned} \quad (38)$$

From first part at the left side of (38) and the third equation in system (11), we have

$$\begin{aligned} & \langle x^{k+1} - x^*, A^T (\lambda^{k+1} - \lambda^*) \rangle \\ & = \langle Ax^{k+1} - Ax^*, \lambda^{k+1} - \lambda^* \rangle \\ & = \langle Ax^{k+1} - b, \lambda^{k+1} - \lambda^* \rangle \\ & = \beta_1 \langle \lambda^{k+1} - \lambda^*, \lambda^k - \lambda^{k+1} \rangle. \end{aligned} \quad (39)$$

From (16), (36), the last equation in system (11), and the second part at the left side of (38), we have

$$\begin{aligned} & \langle x^{k+1} - x^*, \mu^{k+1} - \mu^* \rangle + \langle y^{k+1} - y^*, \mu^* - \mu^{k+1} \rangle \\ & = \langle \mu^{k+1} - \mu^*, x^{k+1} - x^* - y^{k+1} + y^* \rangle \\ & = \langle \mu^{k+1} - \mu^*, x^{k+1} - y^{k+1} \rangle \\ & = \beta_2 \langle \mu^{k+1} - \mu^*, \mu^k - \mu^{k+1} \rangle. \end{aligned} \quad (40)$$

In addition, from the third part at the left side of (38), we have

$$\begin{aligned} & \frac{1}{\beta_2} \langle x^{k+1} - x^*, y^k - y^{k+1} \rangle \\ & = \frac{1}{\beta_2} \langle y^{k+1} - y^*, y^k - y^{k+1} \rangle \\ & + \frac{1}{\beta_2} \langle x^{k+1} - y^{k+1}, y^k - y^{k+1} \rangle \\ & = \frac{1}{\beta_2} \langle y^{k+1} - y^*, y^k - y^{k+1} \rangle \\ & - \langle y^{k+1} - y^k, \mu^k - \mu^{k+1} \rangle. \end{aligned} \quad (41)$$

It follows from (32)-(34) and (38)-(41) that

$$\begin{aligned} & \left\langle x^{k+1} - x^*, R_1 (x^k, x^{k+1}) + \frac{1}{\beta_1} R_2 (x^k, x^{k+1}) \right\rangle \\ & + \frac{1}{\beta_2} \langle y^{k+1} - y^*, y^k - y^{k+1} \rangle \\ & + \beta_1 \langle \lambda^{k+1} - \lambda^*, \lambda^k - \lambda^{k+1} \rangle \\ & + \beta_2 \langle \mu^{k+1} - \mu^*, \mu^k - \mu^{k+1} \rangle \geq 0. \end{aligned} \quad (42)$$

□

Now, we give the convergent conclusion.

Theorem 4. *The sequence $\{x^k\}$ generated by the modified alternating direction method converges to a solution point x^* of problem (9).*

Proof. We denote

$$w = \begin{bmatrix} x \\ y \\ \lambda \\ \mu \end{bmatrix},$$

$$G \quad (43)$$

$$= \begin{bmatrix} (\gamma_1 I_n - Q) + \frac{1}{\beta_1} (\gamma_2 I_n - A^T A) & 0 & 0 & 0 \\ 0 & \frac{1}{\beta_2} I_n & 0 & 0 \\ 0 & 0 & \beta_1 I_m & 0 \\ 0 & 0 & 0 & \beta_2 I_n \end{bmatrix},$$

where I_n denotes the n -dimensional unit matrix and G is positive definite. Here, we define the G -inner product of w and $\bar{w}$ as

$$\begin{aligned} \langle w, \bar{w} \rangle_G & = \left\langle x, (\gamma_1 I_n - Q) \bar{x} + \frac{1}{\beta_1} (\gamma_2 I_n - A^T A) \bar{x} \right\rangle \\ & + \frac{1}{\beta_2} \langle y, \bar{y} \rangle + \beta_1 \langle \lambda, \bar{\lambda} \rangle + \beta_2 \langle \mu, \bar{\mu} \rangle \end{aligned} \quad (44)$$

and the associated G -norm as

$$\begin{aligned} \|w\|_G & = \left\{ \|x\|_{(\gamma_1 I_n - Q) + (1/\beta_1)(\gamma_2 I_n - A^T A)}^2 + \frac{1}{\beta_2} \|y\|_2^2 \right. \\ & \left. + \beta_1 \|\lambda\|_2^2 + \beta_2 \|\mu\|_2^2 \right\}^{0.5}, \end{aligned} \quad (45)$$

where $\|x\|_{(\gamma_1 I_n - Q) + (1/\beta_1)(\gamma_2 I_n - A^T A)}^2 = x^T \{(\gamma_1 I_n - Q) + (1/\beta_1)(\gamma_2 I_n - A^T A)\} x$.

Observe that, by Lemma 2, solving the optimal condition (11) for problem (9) is equivalent to finding a zero point of the residual function:

$$e(w) = \begin{bmatrix} x - P_K(x - \alpha_1(Qx - A^T\lambda - \mu)) \\ y - P_\Omega(y - \alpha_2(c + \mu)) \\ Ax - b \\ x - y \end{bmatrix}_2. \quad (46)$$

From (15), (16), and the first equation in (21), we have that

$$\begin{aligned} x^{k+1} = & P_K \left[x^{k+1} - \alpha_1 \left(Qx^{k+1} - A^T\lambda^{k+1} - \mu^{k+1} \right. \right. \\ & + \frac{1}{\beta_2} (y^{k+1} - y^k) - R_1(x^k, x^{k+1}) \\ & \left. \left. - \frac{1}{\beta_1} R_2(x^k, x^{k+1}) \right) \right]. \end{aligned} \quad (47)$$

From (19) and (16), we have

$$\begin{aligned} y^{k+1} = & P_\Omega \left[y^{k+1} - \alpha_2 \left(c + \mu^k - \frac{1}{\beta_2} (x^{k+1} - y^{k+1}) \right) \right] \\ = & P_\Omega [y^{k+1} - \alpha_2(c + \mu^{k+1})]. \end{aligned} \quad (48)$$

Based on (47)-(48), (15)-(16), and the nonexpansion property of the projection operator, we have

$$\begin{aligned} & \|e(w^{k+1})\|_2 \\ \leq & \left\| \begin{array}{c} \frac{\alpha_1}{\beta_2} (y^k - y^{k+1}) + \alpha_1 R_1(x^k, x^{k+1}) + \frac{\alpha_1}{\beta_1} R_2(x^k, x^{k+1}) \\ 0 \\ \beta_1 (\lambda^k - \lambda^{k+1}) \\ \beta_2 (\mu^k - \mu^{k+1}) \end{array} \right\|_2 \\ \leq & \left\| \begin{array}{c} \alpha_1 R_1(x^k, x^{k+1}) + \frac{\alpha_1}{\beta_1} R_2(x^k, x^{k+1}) \\ 0 \\ \beta_1 (\lambda^k - \lambda^{k+1}) \\ \beta_2 (\mu^k - \mu^{k+1}) \end{array} \right\|_2 \\ & + \left\| \begin{array}{c} \frac{\alpha_1}{\beta_2} (y^k - y^{k+1}) \\ 0 \\ 0 \\ 0 \end{array} \right\|_2 \leq \delta \|w^k - w^{k+1}\|_G, \end{aligned} \quad (49)$$

where δ is a positive constant depending on parameters $\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2$, and the largest eigenvalue of Q and $A^T A$, for example, setting

$$\delta = \sqrt{\max \left\{ \beta_1, \beta_2, \frac{\alpha_1^2}{\beta_2}, \alpha_1^2 \lambda_{\max} \left(\gamma_1 I_n - Q + \frac{1}{\beta_1} (\gamma_2 I_n - A^T A) \right) \right\}}. \quad (50)$$

TABLE 1: The test problems with small scale.

Problems	m	n	SOC	Ratio
P01	40	100	1×100	1.00%
P02	40	100	$1 \times 40; 20 \times 3$	21.00%
P03	40	100	20×5	20.00%
P04	40	100	$1 \times 4; 32 \times 3$	33.00%
P05	120	200	1×200	0.50%
P06	120	200	$1 \times 100; 1 \times 4; 32 \times 3$	17.00%
P07	120	200	40×5	20.00%
P08	120	200	$1 \times 5; 65 \times 3$	33.00%
P09	200	400	1×400	0.25%
P10	200	400	$1 \times 200; 1 \times 5; 65 \times 3$	16.75%
P11	200	400	80×5	20.00%
P12	200	400	$1 \times 4; 132 \times 3$	33.25%
P13	300	600	1×600	0.16%
P14	300	600	$1 \times 400; 1 \times 5; 65 \times 3$	11.16%
P15	300	600	120×5	20.00%
P16	300	600	200×3	33.33%

From Lemma 3, we can write (29) as

$$\langle w^{k+1} - w^*, w^k - w^{k+1} \rangle_G \geq 0, \quad (51)$$

which implies that

$$\langle w^k - w^*, w^k - w^{k+1} \rangle_G \geq \|w^k - w^{k+1}\|_G^2. \quad (52)$$

Thus

$$\begin{aligned} \|w^{k+1} - w^*\|_G^2 &= \|(w^k - w^*) - (w^k - w^{k+1})\|_G^2 \\ &= \|w^k - w^*\|_G^2 \\ &\quad - 2 \langle w^k - w^*, w^k - w^{k+1} \rangle_G \\ &\quad + \|w^k - w^{k+1}\|_G^2 \\ &\leq \|w^k - w^*\|_G^2 - \|w^k - w^{k+1}\|_G^2 \\ &\leq \|w^k - w^*\|_G^2 - \frac{1}{\delta^2} \|e(w^{k+1})\|_2^2. \end{aligned} \quad (53)$$

From the above inequality, we have

$$\|w^{k+1} - w^*\|_G^2 \leq \|w^k - w^*\|_G^2, \quad k = 1, 2, \dots \quad (54)$$

That is, the sequence $\{w^k\}$ is bounded. Thus there exists at least one cluster point of $\{w_k\}$.

It also follows from (53) that

$$\sum_{k=0}^{\infty} \frac{1}{\delta^2} \|e(w^{k+1})\|_2^2 < +\infty, \quad (55)$$

and thus

$$\lim_{k \rightarrow \infty} \|e(w^{k+1})\|_2 = 0. \quad (56)$$

TABLE 2: The results for the test problems with small scale.

Problems	MADM			SeDuMi		
	Iter.	Time	Value	Iter.	Time	Value
P01	216	0.14	6.8296723	9	0.34	6.8296742
P02	245	0.25	71.9788074	9	0.31	71.9788099
P03	264	0.30	76.2533942	11	0.39	76.2533927
P04	261	0.34	143.5300081	11	0.44	143.5300023
P05	269	0.33	47.1664568	10	0.81	47.1664516
P06	271	0.56	164.0914031	12	1.16	164.0914052
P07	293	0.66	467.8962158	13	1.28	467.8962123
P08	320	0.89	671.0620751	13	1.34	671.0620812
P09	322	1.53	71.2857981	12	5.19	71.2857926
P10	337	2.28	85.4731057	13	7.41	85.4731061
P11	304	2.14	1085.3976583	15	7.94	1085.3976475
P12	316	2.62	2256.4765633	16	10.11	2256.4765646
P13	351	4.17	89.3228874	11	15.03	89.3228832
P14	377	5.14	113.5586948	12	16.16	113.5586923
P15	327	5.13	2198.2136742	16	27.53	2198.2136739
P16	326	5.81	2727.7797204	16	32.67	2727.7797233

Let $\bar{w}$ be a cluster point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to $\bar{w}$. We have

$$\|e(\bar{w})\|_2 = \lim_{j \rightarrow \infty} \|e(w^{k_j})\|_2 = 0, \quad (57)$$

so $\bar{w}$ satisfies system (11). Setting $w^* = \bar{w}$, we have

$$\|w^{k+1} - \bar{w}\|_G \leq \|w^k - \bar{w}\|_G. \quad (58)$$

The sequence $\{w^k\}$ satisfies

$$\lim_{k \rightarrow \infty} w^k = \bar{w}. \quad (59)$$

□

5. Simulation Experiments

In this section we present computational results by comparing the modified alternating direction method with the interior-point method. The interior-point method is used to solve the transformed mix linear and second-order cone programming problems (3). All the algorithms are run in the MATLAB 7.0 environment on an Inter Core processor 1.80 GHz personal computer with 2.00 GB of Ram.

The test problems are formulated by random method as follows:

- (1) Given the values of $n, m, N, n_i, i = 1, 2, \dots, N$ with $\sum_{i=1}^N n_i = n$.
- (2) Generate a random matrix $\tilde{Q} \in R^{n \times n}$, and set $Q = \tilde{Q}^T \tilde{Q}$. At the same time, generate a random matrix $A \in R^{m \times n}$ with full row rank.
- (3) Set $l = -e, u = e$, where e is a vector whose components are all ones.

TABLE 3: The test problems with medium scale.

Problems	m	n	SOC	Ratio
P21	400	1000	100×10	10.00%
P22	400	1000	$1 \times 200; 160 \times 5$	16.10%
P23	400	1000	$1 \times 4; 332 \times 3$	33.30%
P24	600	2000	50×40	2.5%
P25	600	2000	$1 \times 400; 1 \times 4; 532 \times 3$	26.70%
P26	600	2000	$1 \times 5; 665 \times 3$	33.33%
P27	800	3000	100×30	3.33%
P28	800	3000	$1 \times 600; 800 \times 3$	26.70%
P29	800	3000	1000×3	33.33%
P30	1000	4000	100×40	2.50%
P31	1000	4000	$1 \times 200; 760 \times 5$	19.02%
P32	1000	4000	$1 \times 4; 1332 \times 3$	33.32%
P33	2000	5000	100×50	2.00%
P34	2000	5000	$1 \times 400; 920 \times 5$	18.42%
P45	2000	5000	$1 \times 5; 1665 \times 3$	33.32%

- (4) Given $x = [x_1, x_2, \dots, x_N] \in \Re^{n_1} \times \dots \times \Re^{n_N}$, then generate the random vector $x_i \in \{l_i, u_i\}$ and make it an interior point of second-order cone K^{n_i} for $i = 1, 2, \dots, N$.

- (5) We obtain b by computing $b = Ax$.

The first set of test problems includes 16 small scale CQSOCP problems with bounded constraints, which is shown in Table 1. In Tables 1 and 3, an entry of the form “ 20×5 ” in the “SOC” column means that there are 20 5-dimensional second-order cones, and the “ratio” denotes the ratio between the number of the second-order cones and the value of n .

TABLE 4: The results for the test problems with small scale.

Problems	MADM			SeDuMi		
	Iter.	Time	Value	Iter.	Time	Value
P21	326	10.88	1382.4709177	17	136.73	1382.4709168
P22	330	11.61	206.0023854	15	130.14	206.0023835
P23	378	16.78	7714.2171166	17	178.88	7714.2171243
P24	429	58.73	1678.7355144	19	1013.98	1678.7355313
P25	449	65.80	312.2865058	18	1596.53	312.2865057
P26	520	64.86	26103.1426757	19	1640.63	26103.1426826
P27	448	109.83	2882.0125678	/	/	/
P28	536	141.35	414.5181029	/	/	/
P29	566	150.15	35810.2685515	/	/	/
P30	425	190.92	6190.8005238	/	/	/
P31	506	239.69	801.0388729	/	/	/
P32	599	289.32	53966.4593065	/	/	/
P33	348	273.73	10471.3329318	/	/	/
P34	383	310.31	1145.2045565	/	/	/
P45	507	413.41	170350.7785126	/	/	/

As is known to all, the interior-point methods have proved to be one of the most efficient class of methods for SOCP. Here the Matlab program codes for the interior-point method are designed from the software package by SeDuMi [4]. In the SeDuMi software, we set the desired accuracy parameter $\text{pars.eps} = 10^{-6}$.

Let $\Delta f_k = |f(x^k) - f(x^{k-1})|$, where $f(x) = x^T Q x + c^T x$. In the alternating direction method, we stop our algorithm when

$$\max \{ \|x^k - x^{k-1}\|_2, \|y^k - y^{k-1}\|_2, \|\lambda^k - \lambda^{k-1}\|_2, \|\mu^k - \mu^{k-1}\|_2, \Delta f_k \} \leq \epsilon \quad (60)$$

for $\epsilon > 0$. Here we set $\beta_1 = 0.8, \beta_2 = 0.8, \gamma_1 = \lambda_{\max}(Q) + 0.0001, \gamma_2 = \lambda_{\max}(A^T A) + 0.0001$ and $\epsilon = 10^{-6}$. We choose the initial point $x^0 = e_n, y^0 = e_n, \lambda^0 = e_m$, and $\mu^0 = e_n$, where e_n is the n -dimensional vector of ones.

For the first set of test problems, the iteration number and average CPU time are used to evaluate the performances of the modified alternating direction method and the interior-point method by SeDuMi. The test results are shown in Table 2. In the Tables 2 and 4, "Time" represents the average CPU time (in seconds), and "Iter." denotes the average number of iteration. In addition, "MADM" represents the modified alternating direction method. In Table 4, "/" denotes that the method does not work in our personal computer because the method is "out of memory."

Table 2 shows that the modified alternating direction method costs less CPU time than the interior-point method by SeDuMi. But, the iteration number of the interior-point method is less than that of the modified alternating direction method.

In addition, Table 1 gives different kinds of test problem, including the problems with only one large second-order cone, such as P01, P05, P09, and P13, the problems with many small second-order cones, such as P04, P08, P12, and

P16, and the problems with one large second-order cone and some small second-order cones, such as P02, P06, P10, and P14. The test results in Table 2 show that the modified alternating direction method can solve different kinds of convex quadratic second-order cone programming problems within appropriate CPU time and accuracy.

The second set of test problems includes 15 medium scale problems, which is shown in Table 3. For the second set of test problems, the test results are shown in Table 4.

The results in Table 4 show the interior point method by SeDuMi does not work for the transformed problem (3) because of being "out of memory" in our personal computer when $n > 2000$, but the modified alternating direction method is still efficient because the modified alternating direction method needs less memory space than the interior-point method.

In addition, we add test results of P04 and P12 in smaller criteria and with random initial points. The smaller criteria of our method is 10^{-10} . In addition, we do one hundred experiments with the random initial point. The test results are shown in Table 5. In the SeDuMi software, we set the desired accuracy parameter $\text{pars.eps} = 10^{-8}$.

Table 5 shows that the performances of MADM with random initial points are a bit better than that of MADM with fixed initial points in two different stop criteria. In addition, the number of iteration of MADM with $\epsilon = 10^{-10}$ is more than that of MADM with $\epsilon = 10^{-6}$, and the CPU time of MADM with $\epsilon = 10^{-10}$ is longer than that of MADM with $\epsilon = 10^{-6}$.

6. Conclusion

In the paper, a modified alternating direction method is proposed for solving convex quadratic second-order cone programming problems with bounded constraints. The proposed method does not require solving subvariational inequality

TABLE 5: The results in smaller criteria and with random initial points.

Problems	MADM			SeDuMi		
	Iter.	Time	Value	Iter.	Time	Value
P04 (10^{-6} , random point)	275	0.37	169.24049123	21	0.88	169.24049003
P04 (10^{-6} , fixed point)	287	0.41	169.24049234	21	0.88	169.24049003
P04 (10^{-10} , random point)	485	0.62	169.24049007	21	0.88	169.24049003
P04 (10^{-10} , fixed point)	494	0.67	169.24049007	21	0.88	169.24049003
P12 (10^{-6} , random point)	355	2.55	1904.587401	27	20.52	1904.587416
P12 (10^{-6} , fixed point)	367	2.95	1904.587402	27	20.52	1904.587416
P12 (10^{-10} , random point)	628	4.37	1904.587409	27	20.52	1904.587416
P12 (10^{-10} , fixed point)	642	5.20	1904.587408	27	20.52	1904.587416

problems over the second cones and the bounded set. At each iteration, we only need to compute the metric projection onto the second-order cones and a projection onto the bounded set. The proposed modified method does not require second-order information and it is easy to implement. The random simulation results show that our method can efficiently solve some convex quadratic second-order cone programming problems of vector size up to 5000 within reasonable time and accuracy by using a desktop computer.

Disclosure

This work was conducted while Xuewen Mu has been visiting Ohio University, Department of Mathematics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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