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Research Article

Approximate Solution of Two-Dimensional Nonlinear Wave Equation by Optimal Homotopy Asymptotic Method

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The two-dimensional nonlinear wave equations are considered. Solution to the problem is approximated by using optimal homotopy asymptotic method (OHAM). The residual and convergence of the proposed method to nonlinear wave equation are presented through graphs. The resultant analytic series solution of the two-dimensional nonlinear wave equation shows the effectiveness of the proposed method. The comparison of results has been made with the existing results available in the literature.

1. Introduction

The wave equations play a vital role in diverse areas of engineering, physics, and scientific applications. An enormous amount of research work is already available in the study of wave equations [1, 2]. This paper deals with the two-dimensional nonlinear wave equation of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} = 1 - \frac{x^2+t^2}{2}, \quad 0 \le x,\ t \le 1. \tag{1}$$

The differential equations (DEs) can be solved analytically by a number of perturbation techniques [3, 4]. These techniques are fairly simple in calculating the solutions, but their limitations are based on the assumption of small parameters. Therefore, the researchers are on the go for some new techniques to overcome these limitations.

The idea of homotopy was pooled with perturbation. Liao [5] proposed homotopy analysis method (HAM) in his Ph.D. dissertation and applied it to various nonlinear engineering problems [6–8]. The homotopy perturbation method (HPM) was initially introduced by He [9–13]. HPM has been extensively used by several researchers successfully for physical models [14–16]. Some useful comparisons between HAM and HPM were done by Domairry and Liang [17, 18].

Recently Marinca and Herişanu [19–21] introduced OHAM for the solution of nonlinear problems which made the perturbation methods independent of the assumption of small parameters, and Ullah et al. [22–26] have extended and applied OHAM successfully for numerous nonlinear phenomena.

The motive of this paper is to apply OHAM for the solution of two-dimensional nonlinear wave equations. In [19–21] OHAM has been proved to be useful for obtaining an approximate solution of nonlinear differential equations. Here, we have proved that OHAM is more useful and reliable for the solution of two-dimensional nonlinear wave equations, hence, showing its validity and greater potential for the solution of transient physical phenomenon in science and engineering.

Section 2 has the basic idea of OHAM formulated for the solution of partial differential equations. In Section 3, the effectiveness of OHAM for two-dimensional nonlinear wave equation has been studied.

2. Basic Formulation of OHAM

Consider the partial differential equation of the following form:

$$\begin{aligned} &\mathscr{A}(u(x,t)) + f(x,t) = 0, \quad x \in \Omega \\ &\mathscr{B}\left(u, \frac{\partial u}{\partial x}\right) = 0, \quad x \in \Gamma, \end{aligned} \tag{2}$$

where $\mathscr{A}$ is a differential operator, $u(x,t)$ is an unknown function, x and t denote spatial and temporal independent variables, respectively, Γ is the boundary of Ω, and $f(x,t)$ is a known analytic function. $\mathscr{A}$ can be divided into two parts: $\mathscr{L}$ and $\mathscr{N}$ such that

$$\mathscr{A} = \mathscr{L} + \mathscr{N}. \tag{3}$$

where $\mathscr{L}$ is the simpler part of the partial differential equation which is easier to solve and $\mathscr{N}$ contains the remaining part of $\mathscr{A}$.

According to OHAM, one can construct an optimal homotopy $\phi(x,t;p) : \Omega \times [0,1] \to \Re$ which satisfies

$$\begin{aligned} H(\phi(x,t;p),p) &= (1-p)\{\mathscr{L}(\phi(x,t;p)) + f(x,t)\} \\ &\quad - H(p)\{\mathscr{A}(\phi(x,t;p)) + f(x,t)\} = 0. \end{aligned} \tag{4}$$

Here the auxiliary function $H(p)$ is nonzero for $p \neq 0$ and $H(0) = 0$. Equation (4) is called optimal homotopy equation. Clearly, we have

$$\begin{aligned} p = 0 &\Longrightarrow H(\phi(x,t;0),0) = \mathscr{L}(\phi(x,t;0)) + f(x,t) = 0, \\ p = 1 &\Longrightarrow H(\phi(x,t;1),1) \\ &\quad = H(1)\{\mathscr{A}(\phi(x,t;1)) + f(x,t)\} = 0. \end{aligned} \tag{5}$$

Obviously, when $p = 0$ and $p = 1$ we obtain $\phi(x,t;0) = u_0(x,t)$ and $\phi(x,t;1) = u(x,t)$, respectively. Thus, as p varies from 0 to 1, the solution $\phi(x,t;p)$ approaches from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is obtained from (4) for $p = 0$:

$$\mathscr{L}(u_0(x,t)) + f(x,t) = 0, \qquad \mathscr{B}\left(u_0, \frac{\partial u_0}{\partial x}\right) = 0. \tag{6}$$

Next, we choose auxiliary function $H(p)$ in the form

$$H(p) = pC_1 + p^2C_2 + \cdots + p^mC_m. \tag{7}$$

To get an approximate solution, we expand $\phi(x,t;p,C_i)$ by Taylor's series about p in the following manner:

$$\phi(x,t;p,C_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t;C_i)\, p^k, \quad i = 1,2,\ldots. \tag{8}$$

Substituting (8) into (4) and equating the coefficient of like powers of p, we obtain zeroth-order problem, given by (6), the first- and second-order problems are given by (9) and (10), respectively, and the general governing equations for $u_k(x,t)$ are given by (11) as follows:

$$\mathscr{L}(u_1(x,t)) = C_1\mathscr{N}_0(u_0(x,t)), \qquad \mathscr{B}\left(u_1, \frac{\partial u_1}{\partial x}\right) = 0, \tag{9}$$

$$\begin{aligned} &\mathscr{L}(u_2(x,t)) - \mathscr{L}(u_1(x,t)) \\ &\quad = C_2\mathscr{N}_0(u_0(x,t)) \\ &\qquad + C_1\left[\mathscr{L}(u_1(x,t)) + \mathscr{N}_1(u_0(x,t),u_1(x,t))\right], \\ &\qquad \mathscr{B}\left(u_2, \frac{\partial u_2}{\partial x}\right) = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &\mathscr{L}(u_k(x,t)) - \mathscr{L}(u_{k-1}(x,t)) \\ &\quad = C_k\mathscr{N}_0(u_0(x,t)) \\ &\qquad + \sum_{i=1}^{k-1} C_i\left[\mathscr{L}(u_{k-i}(x,t))\right. \\ &\qquad\qquad \left. + \mathscr{N}_{k-i}(u_0(x,t),u_1(x,t),\ldots,u_{k-i}(x,t))\right], \\ &\qquad\qquad k = 2,3,\ldots, \\ &\qquad \mathscr{B}\left(u_k, \frac{\partial u_k}{\partial x}\right) = 0, \end{aligned} \tag{11}$$

where $\mathscr{N}_{k-i}(u_0(x,t),u_1(x,t),\ldots,u_{k-i}(x,t))$ are the coefficient of p^{k-i} in the expansion of $\mathscr{N}(\phi(x,t;p))$ about the embedding parameter p. One has

$$\begin{aligned} \mathscr{N}(\phi(x,t;p,C_i)) &= \mathscr{N}_0(u_0(x,t)) \\ &\quad + \sum_{k\geq 1} \mathscr{N}_k(u_0,u_1,u_2,\ldots,u_k)\, p^k. \end{aligned} \tag{12}$$

It should be underscored that the u_k for $k \geq 0$ is governed by the linear equations with linear boundary conditions that come from the original problem, which can be easily solved.

It has been observed that the convergence of the series equation (8) depends on the auxiliary constants $C_1, C_2, \ldots$. If it is convergent at $p = 1$, one has

$$\tilde{u}(x,t;C_i) = u_0(x,t) + \sum_{k\geq 1} u_k(x,t;C_i). \tag{13}$$

Substituting (13) into (1), it results in the following expression for residual:

$$R(x,t;C_i) = \mathscr{L}(\tilde{u}(x,t;C_i)) + f(x,t) + \mathscr{N}(\tilde{u}(x,t;C_i)). \tag{14}$$

In actual computation $k = 1,2,3,\ldots,m$. If $R(x,t;C_i) = 0$ then $\tilde{u}(x,t;C_i)$ is the Exact solution of the problem. Generally it does not happen, especially in nonlinear problems.

For determining auxiliary constants, C_i, $i = 1,2,\ldots,m$, there are a number of methods like Galerkin's method,

Ritz method, least squares method, and collocation method. The method of least squares can be applied as follows:

$$J(C_i) = \int_0^t \int_\Omega R^2(x,t;C_i)\,dx\,dt \tag{15}$$

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \cdots = \frac{\partial J}{\partial C_m} = 0. \tag{16}$$

The mth-order approximate solution can be obtained by these optimal constants. The more general auxiliary function $H(p)$ is useful for convergence, which depends on constants $C_1, C_2, \ldots, C_m$, can be optimally identified by (16), and is useful in error minimization.

3. Application of OHAM to Two-Dimensional Nonlinear Wave Equations

To demonstrate the effectiveness of the formulation of OHAM, we consider two-dimensional nonlinear wave equations of the form (1) with initial conditions

$$u(0,t) = \frac{t^2}{2}, \qquad \frac{\partial}{\partial x}u(0,t) = 0. \tag{17}$$

Applying the method formulated in Section 2 leads to

$$\mathscr{L} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad \mathscr{N} = -u(x,t)\frac{\partial^2 u(x,t)}{\partial t^2}, \qquad f(x,t) = 1 - \frac{x^2+t^2}{2}. \tag{18}$$

Zeroth-Order Problem. Consider

$$\frac{\partial u_0(x,t)}{\partial t} = 1 - \frac{x^2+t^2}{2}, \qquad u_0(0,t) = \frac{t^2}{2}, \qquad \frac{\partial}{\partial x}u_0(0,t) = 0. \tag{19}$$

Its solution is

$$u_0(x,t) = \frac{x^2}{2} + \frac{t^2}{2} - \frac{x^2t^2}{4} - \frac{x^4}{24}. \tag{20}$$

First-Order Problem. Consider

$$\frac{\partial^2 u_1(x,t)}{\partial x^2} = (1+C_1)\frac{\partial^2 u_0(x,t)}{\partial x^2} - C_1 u_0(x,t)\frac{\partial^2 u_0(x,t)}{\partial t^2} + (1+C_1)\left(\frac{x^2}{2} + \frac{t^2}{2} - 1\right), \qquad u_1(0,t) = 0, \qquad \frac{\partial}{\partial x}u_1(0,t) = 0. \tag{21}$$

Its solution is

$$u_1(x,t) = \left(-\frac{1}{4}t^2x^2 - \frac{x^2}{24} + \frac{t^2x^4}{24} + \frac{7x^6}{720} - \frac{x^8}{2688}\right)C_1. \tag{22}$$

Second-Order Problem. Consider

$$\begin{aligned}\frac{\partial^2 u_2(x,t)}{\partial x^2} &= \Bigg[(1+C_1)\frac{\partial^2 u_1(x,t)}{\partial x^2} + C_2\frac{\partial^2 u_0(x,t)}{\partial x^2} \\ &\quad - C_1u_0(x,t)\frac{\partial^2 u_1(x,t)}{\partial t^2} - C_1u_1(x,t)\frac{\partial^2 u_0(x,t)}{\partial t^2} \\ &\quad - C_2u_0(x,t)\frac{\partial^2 u_0(x,t)}{\partial t^2} + C_2\left(\frac{x^2}{2} + \frac{t^2}{2} - 1\right)\Bigg], \\ u_2(0,t) &= 0, \qquad \frac{\partial}{\partial x}u_2(0,t) = 0.\end{aligned} \tag{23}$$

Its solution is

$$\begin{aligned}u_2(x,t) &= \frac{1}{2661120}\Bigg[-\frac{11}{5}\left(-173+56t^2\right)x^{10}C_1^2 \\ &\quad - \frac{43}{4}x^{12}C_1^2 + 66x^8 - 15C_1 \\ &\quad + \left(-15C_1 + 36t^2\right)C_1^2 - 15C_2 \\ &\quad - 665280t^2x^2\left(C_1 + C_2 + C_1^2\right) + 110880x^4 \\ &\quad \cdot\left(\left(-1+t^2\right)C_1 + \left(-1+2t^2\right)C_1^2 + \left(-1+t^2\right)C_2\right) \\ &\quad - 3696x^6\left(\left(-7+3t^2\right)C_1 + \left(-14+11t^2\right)C_1^2 + \left(-7+3t^2\right)C_2\right)\Bigg].\end{aligned} \tag{24}$$

Third-Order Problem. Consider

$$\begin{aligned}\frac{\partial^2 u_3(x,t)}{\partial x^2} &= \Bigg[(1+C_1)\frac{\partial^2 u_2(x,t)}{\partial x^2} + C_2\frac{\partial^2 u_1(x,t)}{\partial x^2} + C_3\frac{\partial^2 u_0(x,t)}{\partial x^2} \\ &\quad - C_1u_0(x,t)\frac{\partial^2 u_2(x,t)}{\partial t^2} - C_1u_2(x,t)\frac{\partial^2 u_0(x,t)}{\partial t^2} \\ &\quad - C_1u_1(x,t)\frac{\partial^2 u_1(x,t)}{\partial t^2} - C_2u_0(x,t)\frac{\partial^2 u_1(x,t)}{\partial t^2} \\ &\quad - C_2u_1(x,t)\frac{\partial^2 u_0(x,t)}{\partial t^2} - C_3u_0(x,t)\frac{\partial^2 u_0(x,t)}{\partial t^2} \\ &\quad + C_3\left(\frac{x^2}{2} + \frac{t^2}{2} - 1\right)\Bigg], \\ u_3(0,t) &= 0, \qquad \frac{\partial}{\partial x}u_3(0,t) = 0.\end{aligned} \tag{25}$$

Its solution is

$$
\begin{aligned}
&u_3(x,t)\\
&= \frac{1}{319334400}\\
&\cdot\Big[-12x^2\Big(352t^2\Big(37800-12600x^2\\
&\qquad+2310x^4-135x^6+7x^8\Big)\\
&\qquad+x^2\Big(2217600-1034880x^2+108240x^4\\
&\qquad-7612x^6+215x^8\Big)\Big)C_1^2\\
&+\Big(-13305600x^4+9313920x^6\\
&\quad-1591920x^8+195800x^{10}-12586x^{12}\\
&\quad+\frac{49221}{91}x^{14}-\frac{2867}{240}x^{16}-\frac{8}{13}x^2t^2\Big)\\
&\cdot\Big(129729600-64864800x^4-1904760x^6\\
&\quad+166452x^8-6604x^{10}+231x^{12}\Big)C_1^3\\
&-12x^2C_1\Big(660\Big(168t^2\Big(60-10x^2+x^4\Big)\\
&\qquad+x^2\Big(1680-392x^2+15x^4\Big)\Big)\\
&\qquad+\Big(352t^2\Big(37800-12600x^2\\
&\qquad+2310x^4-135x^6+7x^8\Big)\\
&\qquad+x^2\Big(2217600-1034880x^2\\
&\qquad+108240x^4-7612x^6\\
&\qquad+215x^8\Big)\Big)C_2\Big)\\
&-7920x^2\Big(168t^2\Big(60-10x^2+x^4\Big)\\
&\qquad+x^2\Big(1680-392x^2+15x^4\Big)\Big)\\
&\cdot(C_2+C_3)\Big].
\end{aligned}
\tag{26}
$$

Adding (20), (22), (24), and (26), we obtain

$$
\begin{aligned}
u(x,t,C_1,C_2,C_3) &= u_0(x,t)+u_1(x,t,C_1)\\
&\quad+u_2(x,t,C_1,C_2).
\end{aligned}
\tag{27}
$$

The residual can be calculated by using (14). For calculations of the constants C_1, C_2, and C_3, using (27) in (17) and applying the procedure mentioned in (13) and (14), we get

$$
\begin{aligned}
C_1 &= -1.0639118119872306,\\
C_2 &= -9.762978332188523\times10^{-4},\\
C_3 &= 1.287603572987478\times10^{-4},\\
u(x,t)&\\
&=\Big[x^2\Big(0.5+3.12815\times10^{-7}x^2\\
&\quad-1.0409\times10^{-4}x^4+2.812\times10^{-4}x^6\\
&\quad-2.52425\times10^{-5}x^8+3.37375\\
&\quad\times10^{-5}x^{10}-2.03977\times10^{-6}x^{12}\\
&\quad+4.50493\times10^{-8}x^{14}\Big)\\
&\times t^2\Big(0.5+1.87689\times10^{-3}x^2\\
&\quad-4.4610\times10^{-4}x^4+1.53732\\
&\quad\times10^{-3}x^6-1.38661\times10^{-3}x^8\\
&\quad+2.28978\times10^{-4}x^{10}-1.53259\\
&\quad\times10^{-5}x^{12}+5.3608\times10^{-7}x^{14}\Big)\Big].
\end{aligned}
\tag{28}
$$

The Exact solution is [2]

$$
u(x,t)=\frac{x^2+t^2}{2},
\tag{29}
$$

and HPM solution is [2]

$$
\begin{aligned}
&u(x,t)\_\text{HPM}\\
&=\left[\frac{t^2}{2}+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^8}{2688}+\frac{x^6t^2}{240}-\frac{327x^8}{241920}\right.\\
&\quad\left.+\frac{x^8t^2}{1120}-\frac{x^{10}t^2}{21600}+\frac{1038x^{10}}{3628800}-\frac{x^{12}}{107520}\right].
\end{aligned}
\tag{30}
$$

4. Results and Discussions

The formulation presented in Section 2 provides accurate solutions for the problems demonstrated in Section 3. We have used Mathematica 7 for most of our computational work. In Table 1 and Figures 1, 2, 3, we have compared the OHAM results with the results obtained by HPM and Exact for various values of x and t at spatial domain $[0,1]$ for Table 1 and at different values of x and fixed value of $t = 1$ for Figures 1–3. In Table 2, we have presented absolute errors at different values of x and t. Figure 4 presents the residual at a spatial domain $[0,1]$ at $t = 1$. The convergence of OHAM is presented in Figure 5 at $t = 1$.

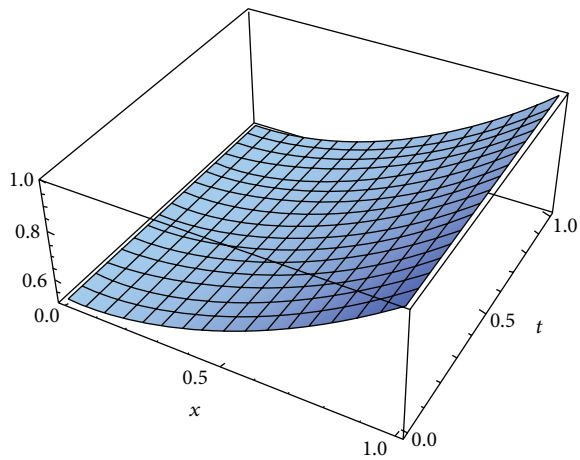

Figure 1: 3D plot for the OHAM solution at $t = 1$.

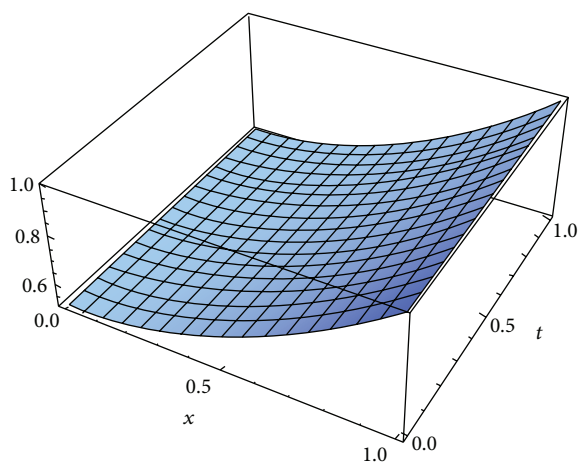

Figure 2: 3D plot for the Exact solution at $t = 1$.

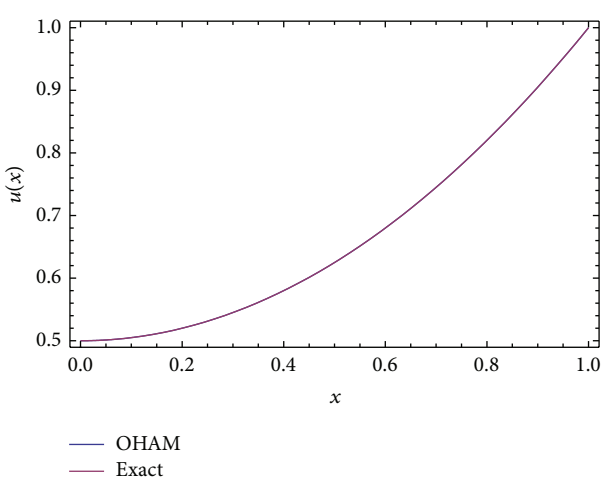

Figure 3: 2D plot for the OHAM and Exact solutions at $t = 1$.

Table 1: Comparison of Exact, HPM, and OHAM solutions for different values of x and t.

x	t	Exact solution	HPM solution	OHAM solution
0	0	0.00000	0.000000	0.00000
0.062	0.162	0.015044	0.015044	0.015044
0.156	0.156	0.024336	0.024336	0.024336
0.225	0.128	0.033504	0.033505	0.033504
0.187	0.275	0.055297	0.055297	0.055297
0.281	0.281	0.078961	0.078961	0.0789608
0.343	0.343	0.117649	0.117650	0.117648
0.350	0.263	0.095834	0.095835	0.095834
0.468	0.468	0.219024	0.219032	0.219022
0.369	0.384	0.141801	0.141810	0.141808
0.350	0.870	0.439700	0.439706	0.439697
0.539	0.403	0.226465	0.226476	0.226463
0.780	0.111	0.310360	0.310263	0.310358
0.593	0.593	0.351649	0.351704	0.351647
0.620	0.685	0.426813	0.426913	0.426808
0.656	0.656	0.430336	0.430462	0.430332
0.781	0.781	0.609961	0.610499	0.609963
0.843	0.843	0.710649	0.711668	0.710652
0.968	0.968	0.937024	0.940295	0.936958
0.975	0.692	0.714744	0.716206	0.714690
1.000	1.000	1.000000	1.004310	0.999877

Table 2: Comparison of absolute errors of HPM and OHAM solutions for different values of x and t.

x	t	E^*	E^{**}
0	0	0.00000000	0.00000000
0.062	0.162	6.003×10^{-12}	1.74057×10^{-12}
0.156	0.156	1.128×10^{-9}	6.00903×10^{-9}
0.225	0.128	2.613×10^{-9}	2.49951×10^{-9}
0.187	0.275	1.213×10^{-8}	3.5136×10^{-8}
0.281	0.281	1.275×10^{-7}	1.91487×10^{-7}
0.343	0.343	6.370×10^{-7}	5.53487×10^{-7}
0.350	0.263	3.309×10^{-7}	4.03006×10^{-7}
0.468	0.468	7.922×10^{-7}	2.28841×10^{-7}
0.369	0.384	1.273×10^{-6}	8.50245×10^{-7}
0.350	0.870	5.736×10^{-6}	3.11566×10^{-6}
0.539	0.403	1.122×10^{-6}	2.45830×10^{-6}
0.780	0.111	9.873×10^{-5}	4.00013×10^{-6}
0.593	0.593	5.497×10^{-5}	4.1745×10^{-6}
0.620	0.685	1.010×10^{-4}	4.68453×10^{-6}
0.656	0.656	1.263×10^{-4}	3.64289×10^{-6}
0.781	0.781	5.378×10^{-4}	2.2877×10^{-6}
0.843	0.843	1.019×10^{-3}	2.6668×10^{-6}
0.968	0.968	3.271×10^{-3}	6.56401×10^{-5}
0.975	0.692	1.461×10^{-3}	5.41072×10^{-5}
1.000	1.000	4.310×10^{-3}	1.22591×10^{-4}

$E^* = |\text{HPM} - \text{Exact}|$, $E^{**} = |\text{OHAM} - \text{Exact}|$.

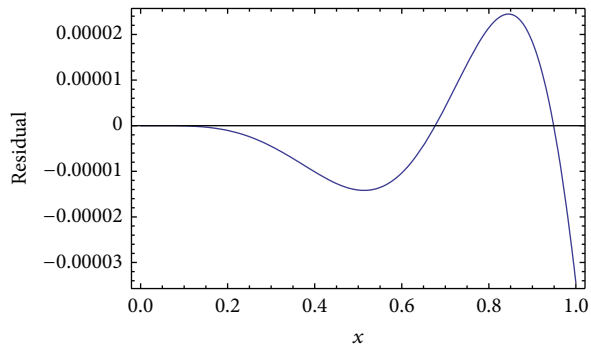

Figure 4: 2D plot for the residual of OHAM at $t = 1$.

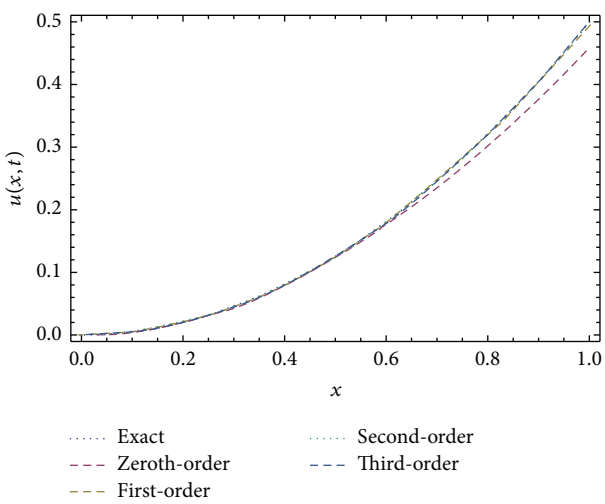

Figure 5: 2D plot for the convergence of orders of OHAM solution at $t = 1$.

5. Conclusion

In this paper, we studied the two-dimensional nonlinear wave equation. We have used the OHAM to approximate the solution of the titled problem. It is clear from the present work that OHAM can successfully be applied to nonlinear phenomena like the one we had. The obtained results are accurate which shows the effectiveness and validity of the proposed method. It is observed that OHAM is simpler in applicability and more convenient to control the convergence and involve less computational work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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