Research Article

H_2/H_∞ Control Design of Detectable Periodic Markov Jump Systems

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An infinite horizon H_2/H_∞ control problem is addressed for discrete-time periodic Markov jump systems with (x, u, v) -dependent noise. Above all, by use of the spectral criterion of detectability, an extended Lyapunov stability theorem is developed for the concerned dynamics. Further, based on a game theoretic approach, a state-feedback H_2/H_∞ control design is proposed. It is shown that under the condition of detectability H_2/H_∞ feedback gain can be constructed through the solution of a group of coupled periodic difference equations.

1. Introduction

H_∞ control has been one of the most active areas of modern control theory since the 1970s. Owing to the introduction of state-space approach [1], many researchers have been inspired to extend the deterministic H_∞ control theory to various stochastic systems; see [2–5]. In the development of stochastic H_∞ theory, [2] can be regarded as a pioneering work, which firstly established a stochastic version of bounded real lemma for linear Itô-type differential systems. Besides, initiated from [6], considerable progress has been made in the study of stochastic H_2/H_∞ control. By combining H_∞ index with an quadratic cost performance, the resulting multiobjective control strategy is more attractive than the sole H_∞ control in engineering applications.

The main objective of this paper is to settle an infinite horizon H_2/H_∞ control problem for periodic Markov jump systems with multiplicative noises. By now, Markov jump systems have been extensively investigated [7–9]. For example, stochastic and robust stability have been elaborately discussed in [10, 11] for networked dynamics with Markovian jump. As concerns H_∞ theory, an H_∞ estimation problem was tackled in [12] for a class of discrete homogeneous

Markov jump systems. On the other hand, an infinite horizon H_∞ control problem was handled in [13] for nonlinear Itô systems with homogeneous Markov process. However, few results have been reported for H_2/H_∞ control of periodic Markov jump systems. To some extent, this study will generalize the work of [14] to the case of periodically time-varying coefficients and transition probabilities, as in [15–17].

The remainder of this paper is organized as follows. Section 2 gives basic preliminaries and problem formulations. In Section 3, the intrinsic relationship between asymptotic mean square stability and detectability is addressed. As a result, a Barbashin-Krasovskii-type theorem is established for periodic Markov jump systems with state-dependent noises. Section 4 contains an internally stabilizing control design, which can not only fulfill the prescribed disturbance attenuation level, but also minimize the output energy. To verify the effectiveness of the proposed approach, a numerical example is supplied in Section 5. Finally, Section 6 concludes this paper with a concluding remark.

Notations. R^n ($\mathbb{C}^n$) is n -dimensional real (complex) space with the usual Euclidean norm $\|\cdot\|$; $R^{n \times m}$ is the space of all $n \times m$ real matrices with the operator norm $\|\cdot\|_2$; S_n is

the set of all $n \times n$ symmetric matrices whose entries may be complex; $A > 0$ (≥ 0): $A \in S_n$ is a positive (semi)definite matrix; A' is the transpose of a matrix (vector) A ; I_n is the $n \times n$ identity matrix; $\mathbb{Z}_+ := \{0, 1, \dots\}$ and $\mathbb{Z}_{1+} := \{1, 2, \dots\}$; $\mathcal{D} = \{1, 2, \dots, N\}$; $\otimes$ is the operation of Kronecker product; $\text{Ker}(\cdot)$ is the kernel of a matrix; $\text{diag}\{\cdot\}$ is a (block) diagonal matrix.

2. Preliminaries

On a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we consider the following discrete-time periodic Markov jump system with (x, u, v) -dependent noise:

$$\begin{aligned} x(t+1) &= A_0(t, \eta_t) x(t) + B_0(t, \eta_t) v(t) + G_0(t, \eta_t) \\ &\quad \cdot u(t) + \sum_{s=1}^r [A_s(t, \eta_t) x(t) + B_s(t, \eta_t) v(t) \\ &\quad + G_s(t, \eta_t) u(t)] w_s(t), \\ z(t) &= \begin{bmatrix} C(t, \eta_t) x(t) \\ D(t, \eta_t) u(t) \end{bmatrix}, \\ D(t, \eta_t)' D(t, \eta_t) &= I_{n_u}, \\ t &\in \mathbb{Z}_+, \end{aligned} \quad (1)$$

where $x(t) \in R^n$, $u(t) \in R^{n_u}$, $v(t) \in R^{n_v}$, and $z(t) \in R^{n_z}$ denote the system state, control input, exogenous disturbance, and measurement output, respectively. Assume that $\{w(t) \mid w(t) = (w_1(t), \dots, w_r(t))', t \in \mathbb{Z}_+\}$ is a sequence of independent random vectors such that $E[w(t)] = 0$ and $E[w(t)w(s)'] = I_r \delta(t-s)$ ($\delta(\cdot)$ is a Kronecker function). The Markov chain $\{\eta_t\}_{t \in \mathbb{Z}_+}$ takes values in $\mathcal{D}$ with a nondegenerate transition probability matrix $\mathcal{P}_t = [p_t(i, j)]_{N \times N}$ ($p_t(i, j) := \mathcal{P}(\eta_{t+1} = j \mid \eta_t = i)$) and the initial distribution $\pi_0(i) = \mathcal{P}(\eta_0 = i) > 0$ for all $i \in \mathcal{D}$. As usual, we set that $\{\eta_t\}_{t \in \mathbb{Z}_+}$ is independent of the stochastic process $\{w(t)\}_{t \in \mathbb{Z}_+}$ and its mode is measurable in real time. Moreover, the coefficients of (1) are θ -periodic (e.g., $A_s(t, i) = A_s(t + \theta, i)$) and the transition probability of η_t satisfies $p_t(i, j) = p_{t+\theta}(i, j)$ ($i, j \in \mathcal{D}$), where $\theta \in \mathbb{Z}_{1+}$. Let $\mathcal{F}_k$ be σ -algebra generated by $\{\eta_t, w(s) \mid 0 \leq t \leq k, 0 \leq s \leq k-1\}$. In the case of $k = 0$, $\mathcal{F}_0 = \sigma\{\eta_0\}$. Denote by $l^2(0, \infty; R^m)$ the space of R^m -valued, nonanticipative square summable stochastic processes $\{y(t, \omega) : \mathbb{Z}_+ \times \Omega \rightarrow R^m\}$ which are $\mathcal{F}_k$ -measurable for all $k \in \mathbb{Z}_+$ and $\sum_{t=0}^{\infty} E\|y(t)\|^2 < +\infty$. It is clear that $l^2(0, \infty; R^m)$ is a real Hilbert space with the norm induced by the usual inner product: $\|y(\cdot)\|_{l^2(0, \infty; R^m)} = (\sum_{t=0}^{\infty} E\|y(t)\|^2)^{1/2} < +\infty$.

Definition 1 (see [18]). The zero-state equilibrium of discrete-time periodic Markov jump systems

$$x(t+1) = A(t, \eta_t) x(t) + \sum_{s=1}^r A_s(t, \eta_t) x(t) w_s(t), \quad (2)$$

or $(\mathbb{A}; \mathbb{P})$ is called asymptotically mean square stable (AMSS) if $\lim_{t \rightarrow \infty} E\|x(t; \xi, i, t_0)\|^2 = 0$ for all $(\xi, i) \in R^n \times \mathcal{D}$ and

$t_0 \in \mathbb{Z}_+$. Here, $x(t; \xi, i, t_0)$ is the state of (2) corresponding to the initial state $x(t_0) = \xi \in R^n$ and $\eta_{t_0} = i \in \mathcal{D}$. Moreover, if there exists θ -periodic sequence $\{K(t, i)\}_{t \in \mathbb{Z}_+} \in R^{n \times n_u}$ ($i \in \mathcal{D}$) such that the zero-state equilibrium of the closed-loop system

$$\begin{aligned} x(t+1) &= [A_0(t, \eta_t) + G_0(t, \eta_t) K(t, \eta_t)] x(t) \\ &\quad + \sum_{s=1}^r [A_s(t, \eta_t) + G_s(t, \eta_t) K(t, \eta_t)] x(t) w_s(t) \end{aligned} \quad (3)$$

is AMSS for any $(x_0, \eta_0) \in R^n \times \mathcal{D}$, then $(\mathbb{A}, \mathbb{G}; \mathbb{P})$ is called stochastically stabilizable and $u(t) = K(t, \eta_t)x(t)$ is called a stabilizing feedback.

By Theorem 3.10 [18], we know that system (2) is asymptotically mean square stable if and only if it is exponentially mean square stable.

Definition 2 (see [19]). The periodic Markov jump system with measurement equation

$$\begin{aligned} x(t+1) &= A_0(t, \eta_t) x(t) + \sum_{s=1}^r A_s(t, \eta_t) x(t) w_s(t), \\ z(t) &= C(t, \eta_t) x(t), \\ t &\in \mathbb{Z}_+, \end{aligned} \quad (4)$$

or $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is called (uniformly) detectable if for any $t_0 \in \mathbb{Z}_+$, $\xi \in R^n$, and $\eta_{t_0} \in \mathcal{D}$, there holds

$$\begin{aligned} z(t) &\equiv 0 \quad (\text{a.s.}), \quad t \in [t_0, T], \quad \forall T > t_0 \\ &\implies \lim_{t \rightarrow \infty} E\|x(t; \xi, i, t_0)\|^2 = 0. \end{aligned} \quad (5)$$

In this paper, we will deal with the infinite horizon optimal H_2/H_∞ control problem about (1). More specifically, for a prescribed disturbance attenuation level $\gamma > 0$, we aim to find a linear, memoryless, periodic state-feedback controller $u^*(t) \in l^2(0, \infty; R^{n_u})$ such that [20]

- (i) when $v(t) \equiv 0$, the closed-loop state of (1) corresponding to $u = u^*(t)$ is AMSS;
- (ii) the l^2 -induced norm of $L_\infty^{u^*}$ satisfies $\|L_\infty^{u^*}\| < \gamma$, where $L_\infty^{u^*}$ is the perturbation operator defined by $L_\infty^{u^*}(v) = z(t; 0, v, u^*)$; it is notable that $z(t; 0, v, u^*)$ is the output of (1) corresponding to $x(0) = 0$ and $u = u^*(t)$, while $v(t) \in l^2(0, \infty; R^{n_v})$ is arbitrary random disturbance;
- (iii) when the worst-case disturbance $v^*(t)$, if existing, is imposed on (1), $u^*(t)$ minimizes the corresponding output energy $J_2(u, v^*) := \sum_{t=0}^{\infty} E\|z(t)\|^2$.

3. Stability and Detectability

In this section, we will focus on the detectability of periodic Markov jump system (1). This structural property will play

an essential role in the treatment of H_2/H_∞ control problem. Firstly, we present several instrumental operators.

Let S_n^N (resp., S_n^{N+}) indicate the set of all N sequences $V = (V(1), \dots, V(N))$ with $V(i) \in S_n$ (resp., $V(i) \geq 0$). Thus, S_n^N is a Hilbert space with the inner product:

$$\langle U, V \rangle = \sum_{i=1}^N \text{Tr}(U(i) V(i)), \quad \forall U, V \in S_n^N. \quad (6)$$

Given $U \in S_n^N$, let $\mathcal{L}_t : S_n^N \rightarrow S_n^N$ be a Lyapunov operator defined as $\mathcal{L}_t(U) = (\mathcal{L}_t(U, 1), \dots, \mathcal{L}_t(U, N))$, where

$$\mathcal{L}_t(U, i) = \sum_{s=0}^r \sum_{j=1}^N p_t(j, i) A_s(t, j) U(j) A_s(t, j)'. \quad (7)$$

$$\psi(X) = [\psi(X(1))' \cdots \psi(X(N))']',$$

$$\varphi(X) = [\varphi(X(1))' \cdots \varphi(X(N))']',$$

$$\psi(X(i)) = (x_{11}(i) \cdots x_{1n}(i) \cdots x_{n1}(i) \cdots x_{nn}(i))',$$

$$\varphi(X(i)) = (x_{11}(i) \cdots x_{1n}(i) \ x_{22}(i) \cdots x_{2n}(i) \cdots x_{n-1,n-1}(i) \ x_{n-1,n}(i) \ x_{nn}(i))',$$

where $X \in S_n^N$ and $x_{jk}(i)$ is the entry of $X(i)$. It is easy to verify that $\psi : S_n^N \rightarrow \mathbb{C}^{n^2N}$ and $\varphi : S_n^N \rightarrow \mathbb{C}^{(n(n+1)/2)N}$ are both invertible and satisfy

$$\begin{aligned} \psi(\mathcal{L}_t(X)) &= M_t \psi(X), \\ \varphi(\mathcal{L}_t(X)) &= L_t \varphi(X), \\ \psi(X) &= H \varphi(X), \end{aligned} \quad (10)$$

$$\forall X \in S_n^N,$$

where $H \in R^{n^2N \times (n(n+1)/2)N}$ is a constant matrix of full column rank and

$$M_t = (\mathcal{P}_t' \otimes I_{n^2}) \Delta \left\{ \sum_{s=1}^r A_s(t, i) \otimes A_s(t, i) \right\}, \quad (11)$$

$$L_t = (H' H)^{-1} H M_t H.$$

In (11), L_t is called the induced matrix of $\mathcal{L}_t$ and $\Delta\{A(i)\} := \text{diag}\{A(1), \dots, A(N)\}$ (if $A(i) \equiv A$ for $i \in \mathcal{D}$, then $\Delta(A) := \text{diag}\{\underbrace{A, \dots, A}_N\}$). Repeating the above steps, the induced matrix of $\mathcal{T}_{t,s}$ is realized to be $\mathcal{A}_{t,s} = L_{t-1} \cdots L_s$. Particularly, the induced matrix of $\mathcal{T}_t^\theta$ is denoted by $\mathcal{A}_t^\theta := \mathcal{A}_{t+\theta,t}$.

Next, we will give two useful lemmas, which have been shown in [19].

Lemma 3. $(\mathbb{A}; \mathbb{P})$ is AMSS if and only if $\Lambda(\mathcal{T}_t^\theta) = \Lambda(\mathcal{A}_t^\theta) \subset \odot$, where $\Lambda(\cdot)$ denotes the spectral set of an operator (or a matrix) and $\odot := \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

Then, associated with inner product (6), the adjoint operator of $\mathcal{L}_t$ is given by $\mathcal{L}_t^*(U) = (\mathcal{L}_t^*(U, 1), \dots, \mathcal{L}_t^*(U, N))$:

$$\mathcal{L}_t^*(U, i) = \sum_{s=0}^r A_s(t, i)' \sum_{j=1}^N p_t(i, j) U(j) A_s(t, i). \quad (8)$$

In terms of $\mathcal{L}_t$, we can construct a causal evolution $\mathcal{T}_{t,s} = \mathcal{L}_{t-1} \cdots \mathcal{L}_s$ ($t > s \geq 0$); when $t = s$, $\mathcal{T}_{t,t} = \mathcal{I}$ (i.e., the identity operator).

To proceed, let us introduce the following two linear operators (cf. [14, 21]):

Lemma 4. $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is detectable if and only if for some $t \in \mathbb{Z}_+$, there does not exist any nonzero $X \in S_n^N$ such that

$$\mathcal{T}_t^\theta(X, i) = \lambda X(i), \quad \lambda \in \mathbb{C}, \quad |\lambda| \geq 1, \quad i \in \mathcal{D}, \quad (12)$$

$$C(j, i) \mathcal{T}_{j,t}(X, i) = 0, \quad j = t, \dots, t + \theta - 1.$$

We are prepared to establish the following Barbashin-Krasovskii stability criterion for (4).

Theorem 5. If $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is detectable, then $(\mathbb{A}; \mathbb{P})$ is AMSS if and only if the PLE

$$X_t(i) = \mathcal{L}_t^*(X_{t+1}, i) + C(t, i)' C(t, i), \quad t \in \mathbb{Z}_+ \quad (13)$$

has a unique θ -periodic solution $X_t \in S_n^{N+}$.

Proof. By Theorem 2.5 [18], if $(\mathbb{A}; \mathbb{P})$ is AMSS, then the PLE (13) admits a unique θ -periodic solution $X_t \in S_n^{N+}$. Next, we will show the converse assertion. If (13) has θ -periodic solution $X_t \in S_n^{N+}$ but $(\mathbb{A}; \mathbb{P})$ is not AMSS, by Lemma 3, there must exist $\lambda \in \Lambda(\mathcal{T}_t^\theta)$ with $|\lambda| \geq 1$. Denote by $\rho(\mathcal{T}_t^\theta)$ the spectral radius of $\mathcal{T}_t^\theta$; then $\rho(\mathcal{T}_t^\theta) \geq 1$. According to the Krein-Rutman theorem, there is a positive definite $\bar{X} \in S_n^N$ such that $\mathcal{T}_t^\theta(\bar{X}) = \rho(\mathcal{T}_t^\theta) \bar{X}$. Since $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is detectable, by Lemma 4, for some $t \in \mathbb{Z}_+$, there exists at least one $k \in \{t, \dots, t + \theta - 1\}$ such that

$$C(k) \mathcal{T}_{k,t}(\bar{X}) \neq 0. \quad (14)$$

Thus, for inner product (6), it can be computed from (13) that

$$\begin{aligned} 0 &\leq \sum_{k=t}^{t+\theta-1} \sum_{i=1}^N \text{Tr} \left(C(k, i) \mathcal{T}_{k,t}(\bar{X}, i) C(k, i)' \right) \\ &= \sum_{k=t}^{t+\theta-1} \langle C(k) C(k)', \mathcal{T}_{k,t}(\bar{X}) \rangle \\ &= \langle X_t, \bar{X} \rangle - \langle X_{t+\theta}, \mathcal{T}_t^\theta(\bar{X}) \rangle. \end{aligned} \quad (15)$$

Due to the periodicity of X_t , (15) leads to the fact that

$$\begin{aligned} 0 &\leq \sum_{k=t}^{t+\theta-1} \sum_{i=1}^N \text{Tr} \left(C(k, i) \mathcal{T}_{k,t}(\bar{X}, i) C(k, i)' \right) \\ &= \langle X_t, \bar{X} - \mathcal{T}_t^\theta(\bar{X}) \rangle = \langle X_t, [1 - \rho(\mathcal{T}_t^\theta)] \bar{X} \rangle \leq 0, \end{aligned} \quad (16)$$

which implies $C(k, i) \mathcal{T}_{k,t}(\bar{X}, i) C(k, i)' = 0$ for $t \leq k \leq t + \theta - 1$ and $i \in \mathcal{D}$. That is, $C(k) \mathcal{T}_{k,t}(\bar{X}) = 0$ for $t \leq k \leq t + \theta - 1$, which contradicts (14). Hence, $(\mathbb{A}; \mathbb{P})$ is AMSS. $\square$

Remark 6. In [18], a similar result has been proven under the condition of stochastic detectability. According to [19], (uniform) detectability is a weaker prerequisite than stochastic detectability. Therefore, Theorem 5 has improved the result of Theorem 4.1 [18] within the concerned framework.

4. H_2/H_∞ Control

In this section, a game theoretic approach will be employed to deal with the infinite horizon H_2/H_∞ control problem of (1). Under the assumption of detectability, a necessary and sufficient condition can be provided for the existence of H_2/H_∞ controller.

Theorem 7. For system (1), if the following coupled periodic difference equations (CPDEs) admit θ -periodic quaternion solution $(X_1(t), F_1(t); X_2(t), F_2(t)) \in S_n^{N+} \times R_{n \times n_v}^N \times S_n^{N+} \times R_{n \times n_u}^N$ on $\mathbb{Z}_+ \times \mathcal{D}$,

$$\begin{aligned} X_1(t, i) &= \sum_{s=0}^r \bar{A}_s^{-2}(t, i)' \mathcal{E}_t(X_1(t+1), i) \bar{A}_s^{-2}(t, i) \\ &\quad + C(t, i)' C(t, i) + F_2(t, i)' F_2(t, i) \\ &\quad + F_3(t, i) H_1(t, i)^{-1} F_3(t, i)', \end{aligned} \quad (17)$$

$$\begin{aligned} H_1(t, i) &\geq \varepsilon_0 I_{n_v}, \quad \varepsilon_0 \in \left(0, \gamma^2 - \|L_\infty^{u^*}\|^2\right), \\ F_1(t, i) &= H_1(t, i)^{-1} F_3(t, i)', \end{aligned} \quad (18)$$

$$\begin{aligned} X_2(t, i) &= \sum_{s=0}^r \bar{A}_s^{-1}(t, i)' \mathcal{E}_t(X_2(t+1), i) \bar{A}_s^{-1}(t, i) \\ &\quad + C(t, i)' C(t, i) \\ &\quad - F_4(t, i) H_2(t, i)^{-1} F_4(t, i)', \end{aligned} \quad (19)$$

$$F_2(t, i) = -H_2(t, i)^{-1} F_4(t, i)', \quad (20)$$

where

$$\begin{aligned} \bar{A}_s^{-1}(t, i) &= A_s(t, i) + B_s(t, i) F_1(t, i), \\ \bar{A}_s^{-2}(t, i) &= A_s(t, i) + G_s(t, i) F_2(t, i), \\ H_1(t, i) &= \gamma^2 I_{n_v} \\ &\quad - \sum_{s=0}^r B_s(t, i)' \mathcal{E}_t(X_1(t+1), i) B_s(t, i), \end{aligned} \quad (21)$$

$$H_2(t, i) = I_{n_u} + \sum_{s=0}^r G_s(t, i)' \mathcal{E}_t(X_2(t+1), i) G_s(t, i),$$

$$F_3(t, i) = \sum_{s=0}^r \bar{A}_s^{-2}(t, i)' \mathcal{E}_t(X_1(t+1), i) B_s(t, i),$$

$$F_4(t, i) = \sum_{s=0}^r \bar{A}_s^{-1}(t, i)' \mathcal{E}_t(X_2(t+1), i) G_s(t, i),$$

and $(\mathbb{A}, \mathbb{C}; \mathbb{P})$, $(\mathbb{A} + \mathbb{B}F_1, \mathbb{C}; \mathbb{P})$ are detectable, then the state-feedback H_2/H_∞ control is given by $(u^*(t) = F_2(t, \eta_t)x(t), v^*(t) = F_1(t, \eta_t)x(t))$.

Conversely, if $(\mathbb{A} + \mathbb{B}F_1, \mathbb{C}; \mathbb{P})$ is detectable and the H_2/H_∞ control problem about (1) is solved by $(u^*(t) = F_2(t, \eta_t)x(t), v^*(t) = F_1(t, \eta_t)x(t))$, then CPDEs (17)–(20) admit a unique θ -periodic quaternion solution $(X_1(t), F_1(t); X_2(t), F_2(t)) \in S_n^{N+} \times R_{n \times n_v}^N \times S_n^{N+} \times R_{n \times n_u}^N$ on $\mathbb{Z}_+ \times \mathcal{D}$.

Proof. “ $\Rightarrow$ ”: (a) Let us first show that u^* stabilizes system (1) internally ($v(t) \equiv 0$). To this end, we rewrite (19) as follows:

$$\begin{aligned} X_2(t, i) &= \sum_{s=0}^r \left[\bar{A}_s^{-1}(t, i) + G_s(t, i) F_2(t, i) \right]' \\ &\quad \cdot \mathcal{E}_t(X_1(t+1), i) \left[\bar{A}_s^{-1}(t, i) + G_s(t, i) F_2(t, i) \right] \\ &\quad + C_1(t, i)' C_1(t, i), \end{aligned} \quad (22)$$

where

$$C_1(t, i) = \begin{bmatrix} C(t, i) \\ H_1(t, i)^{-1/2} F_3(t, i) \\ F_2(t, i) \end{bmatrix}. \quad (23)$$

Since $(\mathbb{A} + \mathbb{B}F_1, \mathbb{C}; \mathbb{P})$ is detectable, by Lemma 4, we can prove that $(\mathbb{A} + \mathbb{B}F_1 + \mathbb{G}F_2, \mathbb{C}_1; \mathbb{P})$ is also detectable. From (22) and Theorem 5, it follows that $(\mathbb{A} + \mathbb{B}F_1 + \mathbb{G}F_2; \mathbb{P})$ is AMSS, which means $u^*(t) = F_2(r_t)x(t) \in l^2(0, \infty; R^{n_u})$ and $v^*(t) = F_1(r_t)x(t) \in l^2(0, \infty; R^{n_v})$. Similarly, we can prove that $(\mathbb{A} + \mathbb{G}F_2; \mathbb{P})$ is also AMSS. Hence, $u^*(t)$ can stabilize system (1) internally.

(b) Consider the following: $\|L_\infty^{u^*}\| < \gamma$. Implementing $u^*(t) = F_2(t, \eta_t)x(t)$ into system (1), we get

$$\begin{aligned} x(t+1) &= \bar{A}_0^2(t, \eta_t)x(t) + B_0(t, \eta_t)v(t) \\ &\quad + \sum_{s=1}^r \left\{ \bar{A}_s^1(t, \eta_t)x(t) + B_s(t, \eta_t)v(t) \right\} w_s(t), \\ z(t) &= \begin{bmatrix} C(t, \eta_t) \\ D(t, \eta_t)F_2(t, \eta_t) \end{bmatrix} x(t), \\ D(t, \eta_t)'D(t, \eta_t) &= I_{n_u}, \\ t &\in \mathbb{Z}_+. \end{aligned} \quad (24)$$

Noting that $(\mathbb{A} + \mathbb{B}F_1; \mathbb{P})$ is AMSS, $X_1 \in S_n^{N+}$ is a stabilizing solution of (17). By use of Theorem 1 [20], we deduce that system (24) satisfies $\|L_\infty^{u^*}\| < \gamma$.

(c) u^* minimizes the performance $J_2(u, v^*)$. By (17) and (24), we can complete square as follows:

$$\begin{aligned} J_\infty(u^*, v) &:= \sum_{t=0}^{\infty} \left[E\|z(t)\|^2 - \gamma^2\|v(t)\|^2 \right] = \sum_{i=1}^N \pi_0(i) \\ &\quad \cdot x_0' X_1(0, i) x_0 \\ &\quad - \sum_{t=0}^{\infty} E \left\{ \left[v(t) - H_1(t, \eta_t)^{-1} F_3(t, \eta_t)' x(t) \right]' \right. \\ &\quad \cdot H_1(t, \eta_t) \left[v(t) - H_1(t, \eta_t)^{-1} F_3(t, \eta_t)' x(t) \right] \Big\} \\ &\leq \sum_{i=1}^N \pi_0(i) x_0' X_1(0, i) x_0 = J_\infty(u^*, v^*), \end{aligned} \quad (25)$$

which implies that $v^*(t) = F_1(t, \eta_t)x(t)$ is the worst-case disturbance associated with u^* . Applying v^* to system (1), we have

$$\begin{aligned} x(t+1) &= \bar{A}_0^1(t, \eta_t)x(t) + G_0(t, \eta_t)u(t) \\ &\quad + \sum_{s=1}^r \left\{ \bar{A}_s^1(t, \eta_t)x(t) + G_s(t, \eta_t)u(t) \right\} w_s(t), \\ z(t) &= \begin{bmatrix} C(t, \eta_t)x(t) \\ D(t, \eta_t)u(t) \end{bmatrix}, \\ D(t, \eta_t)'D(t, \eta_t) &= I_{n_u}, \\ t &\in \mathbb{Z}_+. \end{aligned} \quad (26)$$

It remains to show that u^* fulfills the following optimal index:

$$\begin{aligned} \min_{u(\cdot) \in l^2(0, \infty; R^{n_u})} \left\{ J_2(u, v^*) = \sum_{t=0}^{\infty} E \left[\|z(t)\|^2 \right] \right\}, \\ \text{subject to} \quad (26), \end{aligned} \quad (27)$$

which is an LQ optimal control problem. Since (19) is equivalent to (22) and $(\mathbb{A} + \mathbb{B}F_1 + \mathbb{G}F_2, \mathbb{C}_1; \mathbb{P})$ is detectable, by Theorem 5, $X_2 \in S_n^{N+}$ is a stabilizing solution of (19). Making use of Proposition 6.3 [18], we arrive at

$$\begin{aligned} J_2(u, v^*) &= \sum_{i=1}^N \pi_0(i) x_0' X_2(0, i) x_0 \\ &\quad + \sum_{t=0}^{\infty} E \left\{ \left[u(t) + H_2(t, \eta_t)^{-1} F_4(t, \eta_t)' x(t) \right]' \right. \\ &\quad \cdot H_2(t, \eta_t) \left[u(t) + H_2(t, \eta_t)^{-1} F_4(t, \eta_t)' x(t) \right] \Big\} \\ &\geq \sum_{i=1}^N \pi_0(i) x_0' X_2(0, i) x_0 = J_2(u^*, v^*), \end{aligned} \quad (28)$$

where $u^* = F_2(t, \eta_t)x(t)$. This justifies the sufficiency statement.

“ $\Leftarrow$ ”: Assume that $(u^*(t) = F_2(t, \eta_t)x(t)$ and $v^*(t) = F_1(t, \eta_t)x(t))$ solve the considered H_2/H_∞ control problem. Thus, u^* stabilizes system (1) internally and $\|L_\infty^{u^*}\| < \gamma$. By Theorem 1 [20], we conclude that (17) admits a stabilizing solution $X_1 \in S_n^{N+}$, which implies that $(\mathbb{A} + \mathbb{B}F_1 + \mathbb{G}F_2; \mathbb{P})$ is AMSS. Since system (24) is internally stable, by Corollary 3.9 [18], we deduce that $x(t) \in l^2(0, \infty; R^n)$ for any $v(t) \in l^2(0, \infty; R^{n_v})$. As shown in the sufficient part, by use of (17) and (24), we will come to (25), which indicates $F_1(t, \eta_t) = H_1(t, \eta_t)^{-1} F_3(t, \eta_t)'$. Further, imposing v^* on system (1) gives (26). Because u^* is the optimal H_2/H_∞ control, u^* solves LQ control problem (27). Moreover, from detectability of $(\mathbb{A} + \mathbb{B}F_1, \mathbb{C}; \mathbb{P})$, we have that $(\mathbb{A} + \mathbb{B}F_1 + \mathbb{G}F_2, \mathbb{C}_1; \mathbb{P})$ is detectable. Recalling that $(\mathbb{A} + \mathbb{B}F_1 + \mathbb{G}F_2; \mathbb{P})$ is AMSS, by Theorem 5, (22) (i.e., (19)) has a stabilizing solution $X_2 \in S_n^{N+}$. Finally, by completing square in terms of (19) and (26), we obtain (28), which justifies that $F_2(t, \eta_t) = -H_2(t, \eta_t)^{-1} F_4(t, \eta_t)'$. This ends the proof. $\square$

Remark 8. If the coefficients of (1) reduce to be time-invariant and the Markov chain $\{\eta_t\}_{t \in \mathbb{Z}_+}$ is homogeneous, then Theorem 7 is reduced to the conclusion of Theorem 3 [14]. Hence, the current study can be regarded as a periodic extension of [14]. At present, there still exists some difficulty in generalizing the above method to design H_2/H_∞ controller for time-varying Markov jump systems as in [22]. To this end, a time-varying version of PBH criterion has to be developed.

FIGURE 1: State responses and H_2 performance.

5. Numerical Example

Consider the following two-dimensional Markov jump system with the periodic coefficients listed as follows:

$$\begin{aligned}
 A_0(t, 1) &= \begin{bmatrix} 0.88 - 0.35 \cos \pi t & 0 \\ 0 & 0.6 \end{bmatrix}, \\
 A_0(t, 2) &= \begin{bmatrix} 0.66 & 0 \\ 0 & 0.73 \end{bmatrix}, \\
 B_0(t, 1) &= \begin{bmatrix} 0.52 \\ 0.31 \end{bmatrix}, \\
 B_0(t, 2) &= \begin{bmatrix} 0.76 \\ 0.53 \end{bmatrix}, \\
 G_0(t, 1) &= \begin{bmatrix} 1.4 + 0.2 \cos \pi t \\ 0.49 \end{bmatrix}, \\
 G_0(t, 2) &= \begin{bmatrix} 1.1 \\ 1.04 \end{bmatrix}, \\
 A_1(t, 1) &= \begin{bmatrix} 0.63 & 0 \\ 0 & 0.72 \end{bmatrix}, \\
 A_1(t, 2) &= \begin{bmatrix} 0.48 & -0.2 \cos \pi t \\ 0 & 0.6 \end{bmatrix}, \\
 B_1(t, 1) &= \begin{bmatrix} 0.4 - 0.1 \cos \pi t \\ 0.5 \end{bmatrix}, \\
 B_1(t, 2) &= \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, \\
 G_1(t, 1) &= \begin{bmatrix} 1.1 \\ 1.04 \end{bmatrix}, \\
 G_1(t, 2) &= \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}.
 \end{aligned} \tag{29}$$

Moreover, $C(t, i) \equiv [0.9 \ 0.56]$ and $D(t, i) \equiv 1$. It is clear that $\theta = 2$. The transition probability matrix $\mathcal{P}_t$ is determined by $p_{2k}(1, 1) = p_{2k}(2, 1) = 0.2$, $p_{2k}(1, 2) = p_{2k}(2, 2) = 0.8$, $p_{2k+1}(1, 1) = p_{2k+1}(2, 1) = 0.3$, and $p_{2k+1}(1, 2) = p_{2k+1}(2, 2) = 0.7$ ($k \in \mathbb{Z}_+$). For a prescribed disturbance attenuation level $\gamma = 2.3$, by use of the Runge-Kutta algorithm, we can solve CPDEs (17)–(20) and get the feedback gains of (u^*, v^*) :

$$\begin{aligned}
 F_1(2k, 1) &= [0.12 \ 0.024], \\
 F_1(2k, 2) &= [0.03 \ 0.05], \\
 F_2(2k, 1) &= [-0.49 \ -0.21], \\
 F_2(2k, 2) &= [-0.34 \ -0.27]; \\
 F_1(2k+1, 1) &= [-0.06 \ 0.05], \\
 F_1(2k+1, 2) &= [0.04 \ 0.016], \\
 F_2(2k+1, 1) &= [-0.38 \ -0.235], \\
 F_2(2k+1, 2) &= [-0.35 \ -0.25], \\
 k &\in \mathbb{Z}_+.
 \end{aligned} \tag{30}$$

By Lemma 4, it can be verified that $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ and $(\mathbb{A} + \mathbb{B}F_1, \mathbb{C}; \mathbb{P})$ are both detectable. Applying (u^*, v^*) to the periodic Markov jump system, we get the closed-loop state trajectory and corresponding H_2 performance. Figure 1(a) has displayed 50 sampled state trajectories originating from $(x_1(0), x_2(0)) = (10, 20)$, while Figure 1(b) demonstrates the cumulative energy of the system output.

6. Conclusion

In this paper, an infinite horizon H_2/H_∞ control problem has been settled for discrete-time periodic Markov jump systems with multiplicative noise. Under the condition of (uniform) detectability, a game theoretic H_2/H_∞ control is produced by

solving a group of CPDEs. Note that there remain some open topics on this issue. For example, it is interesting as well as challenging to investigate the H_2/H_∞ control problem with input or output saturation constraint [23], which no doubt deserves a further study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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