Research Article

Analysis and Synthesis of Global Nonlinear $\mathcal{H}_\infty$ Controller for Robot Manipulators

**Carlos Alberto Chavez Guzmán,¹ Luis Tupak Aguilar Bustos,¹
and Jován Oseas Mérida Rubio²**

¹*Instituto Politécnico Nacional, Centro de Investigación y Desarrollo de Tecnología Digital,
Avenida Instituto Politécnico Nacional 1310, Mesa de Otay, 22510 Tijuana, BC, Mexico*

²*Departamento de Ingeniería Eléctrica y Electrónica, Instituto Tecnológico de Tijuana, Calzada Tecnológico S/N,
Fraccionamiento Tomas Aquino, 22414 Tijuana, BC, Mexico*

Correspondence should be addressed to Luis Tupak Aguilar Bustos; laguilarb@ipn.mx

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The $\mathcal{H}_\infty$ regulation problem for robot manipulators using gravitational force compensation or precompensation has been solved locally while global asymptotical stability (or global stability) has been demonstrated using other methodologies. A solution to the global nonlinear $\mathcal{H}_\infty$ regulation problem for l -degrees-of-freedom (l -DOF) robot manipulators, affected by external disturbances, is presented. We showed that the Hamilton-Jacobi-Isaacs (HJI) inequality, inherited in the solution of the $\mathcal{H}_\infty$ control problem, is satisfied by defining a strict Lyapunov function. The performance issues of the nonlinear $\mathcal{H}_\infty$ regulator are illustrated in experimental and simulation studies made for a 3-DOF rigid links robot manipulator.

1. Introduction

Industrial systems currently in use, including robotic systems, can be subject to uncertain environments and be mainly affected by friction, external disturbances, backlash, input dead zone, and noise. Moreover, imprecision in the knowledge of the parameters of the model of the robot and actuators used to compensate nonlinearities as part of model-based controllers also affects the performance of mechanical systems. As a result of such uncertainties and disturbances, poor motion repeatability, imprecisions, and weak exactness are reflected in the tasks for which robots were programmed. For example, chained robots are the typical systems affected by not only the abovementioned nonidealities but also positioning error, due to the gap between the actuator and the link that often exists, which occurs when the measured angular position, used for feedback, is provided from an encoder mounted at the actuator while the nonmeasurable angular position of the link is required to be driven to a desired point.

Syntheses of controllers ensuring global asymptotical stability (GAS) for nonperturbed robots have been widely proposed and reported in the literature (see, e.g., [1–3]) and the state-of-the-art of such systems is out of the scope of this paper. On the other hand, robust controllers for robots operating under uncertain conditions have been also investigated and reported concluding GAS as well. Several methodologies successfully solve the problems affecting robots but each nonlinearity was treated separately. For example, adaptive control was used to drive the trajectories of the closed-loop system to the desired point in spite of uncertain knowledge of the parameters of the robot. Variable structure control [4, 5] has been used to deal with parametric uncertainties and matched external disturbances rejection while finite-time stability is also concluded. Passivity based control, LQR, and $\mathcal{H}_2$, among others, have been used for robust control and interesting material of additional robust controllers for robots can be found in [6].

$\mathcal{H}_\infty$ control has long been recognized not only for attenuating the effects of a class of matched and unmatched

disturbances, including noise, but also for ensuring robust performance possessing the capability to reduce the effects due to uncertainties (see, e.g., [7–12]). Despite the fact that global solution of nonlinear $\mathcal{H}_\infty$ control has been already addressed (see, e.g., van der Schaft [11], Isidori and Astolfi [12], and Orlov and Aguilar [10]), the main drawback of the practical implementation of global $\mathcal{H}_\infty$ controllers for certain nonlinear systems has been the difficulty to solve the HJI inequality since the design procedure results in an infinite-dimensional problem [13, 14]. Indeed, while GAS for the position regulation problem for robot manipulators using gravitational force compensation or precompensation coupled with other control approaches has been demonstrated (see, e.g., [2, 6]), for the $\mathcal{H}_\infty$ controller such problem has been solved locally [15].

For nonlinear systems, the difficulty of dealing with partial differential equations can be circumvented by solving algebraic or differential Riccati equations considering the linearized model of the plant around an equilibrium point turning out to be a local solution of the nonlinear $\mathcal{H}_\infty$ control problem (see, e.g., [15–18]). While dealing with the latter, the HJI equation may be numerically solved where relevant contributions to the matter can be found in the literature. For example, Aliyu [19] addressed the solution of the HJI equation for a general class of affine nonlinear systems, where a computational procedure to find a symmetric solution to the HJI equation has been developed. A numerical method to solve the HJI equation proposed by Ferreira et al. [20] consists in reducing the HJI equation to an infinite sequence of linear partial differential equations where the solutions can be obtained by the Galerkin approximation method. Linear and nonlinear matrix inequalities, given in Boyd et al. [21], and their numerical techniques have been also applied to solve the nonlinear $\mathcal{H}_\infty$ control [22, 23]. For additional material and brief survey of numerical solutions for the nonlinear $\mathcal{H}_\infty$ control problem, the reader is referred to [24, 25].

The main contribution of this paper is in addressing the global solution to the nonlinear $\mathcal{H}_\infty$ regulation problem for a class of fully actuated and revolute joint robot manipulators avoiding the numerical solution of partial differential equations. The main idea is to introduce a positive definite function and define necessary and sufficient conditions to guarantee that the HJI inequality is negative definite. The positive definite function was taken from [26], proven to be a strict Lyapunov function. The developed controller consists of a gravitational compensator, a proportional-derivative part, and a disturbance attenuator, tested experimentally in an industrial robot, showing considerable performance with respect to a controller without the disturbance attenuator.

The rest of the paper is organized as follows. In Section 2, the Euler-Lagrange model for robot manipulators with rotational joints and rigid links and some instrumental properties are described. In Section 3, a background of the solution to the global state feedback nonlinear $\mathcal{H}_\infty$ control problem for affine nonlinear systems is provided. In Section 4, the control objective and synthesis of the global nonlinear $\mathcal{H}_\infty$ regulator along with the verification of the HJI inequality via strict Lyapunov function are explained. Simulations and experiments results made for a 3-DOF industrial robot

are presented in Section 5. Finally, we give conclusions in Section 6.

2. Dynamic Model of Rigid Links Robot Manipulators and Useful Properties

The dynamic equation of l -DOF robots with revolute joints and rigid links can be written as Euler-Lagrange equations; that is, [3]

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) + \mathbf{F}_f(\dot{\mathbf{q}}) = \boldsymbol{\tau} + \mathbf{w}_\tau, \quad (1)$$

where $\mathbf{q}(t) \in \mathbb{R}^l$ is the vector of joint displacements, $\dot{\mathbf{q}}(t) \in \mathbb{R}^l$ is the vector of joint velocities, $\boldsymbol{\tau}(t) \in \mathbb{R}^l$ is the vector of applied torques, $t \in \mathbb{R}$ is the time, $M(\mathbf{q}) \in \mathbb{R}^{l \times l}$ is the symmetric positive definite inertia matrix, $C(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{l \times l}$ is the centrifugal and Coriolis forces matrix, $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^l$ is the vector of gravitational torques, $\mathbf{w}_\tau(t) \in \mathbb{R}^l$ is the disturbance vector, and $\mathbf{F}_f(\dot{\mathbf{q}}) \in \mathbb{R}^l$ represents the friction torques vector governed by

$$\begin{aligned} \mathbf{F}_f(\dot{\mathbf{q}}) &= F_c \mathbf{sign}(\dot{\mathbf{q}}) \\ \mathbf{sign}(\dot{\mathbf{q}}) &= [\mathbf{sign}(q_1), \dots, \mathbf{sign}(q_l)]^T. \end{aligned} \quad (2)$$

Here, the matrix $F_c \in \mathbb{R}^{l \times l}$ is assumed diagonal and positive definite where the main diagonal includes the constant coefficients of the Coulomb friction of each joint. Both the angular positions and velocities are available measurements for feedback. The following properties, taken from [2, 3], are valid for robots having only revolute joints.

Property 1. The inertia matrix $M(\mathbf{q})$ is symmetric and positive definite for all $\mathbf{q} \in \mathbb{R}^l$. The matrix $M^{-1}(\mathbf{q})$ exists and is positive definite as well.

Property 2. The centrifugal and Coriolis matrix $C(\mathbf{q}, \dot{\mathbf{q}})$ and the time derivative $\dot{M}(\mathbf{q})$ of the inertia matrix satisfy that skew-symmetry property as follows:

$$\dot{\mathbf{q}}^T \left[\frac{1}{2} \dot{M}(\mathbf{q}) - C(\mathbf{q}, \dot{\mathbf{q}}) \right] \dot{\mathbf{q}} = 0 \quad (3)$$

for all $\mathbf{q}, \dot{\mathbf{q}} \in \mathbb{R}^l$.

Property 3. There exists a positive constant k_c such that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^l$ we have

$$\|C(\mathbf{x}, \mathbf{y})\mathbf{z}\| \leq k_c \|\mathbf{y}\| \|\mathbf{z}\|. \quad (4)$$

3. Background on Global Nonlinear $\mathcal{H}_\infty$ Control for Autonomous Systems

The present study focuses on autonomous nonlinear systems of the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + G_1(\mathbf{x})\mathbf{w} + G_2(\mathbf{x})\mathbf{u} \\ \mathbf{z} &= \mathbf{h}_1(\mathbf{x}) + K_{12}(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \mathbf{h}_2(\mathbf{x}) + K_{21}(\mathbf{x})\mathbf{w}, \end{aligned} \quad (5)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^l$ is the control input, $\mathbf{w}(t) \in \mathbb{R}^{l+n}$ is the unknown disturbance vector, $\mathbf{z}(t) \in \mathbb{R}^{l+n}$ is the unknown output to be controlled, $\mathbf{y}(t) \in \mathbb{R}^n$ is the only available measurement on the system, $\mathbf{f}(\mathbf{x})$, $G_1(\mathbf{x})$, $G_2(\mathbf{x})$, $\mathbf{h}_1(\mathbf{x})$, $\mathbf{h}_2(\mathbf{x})$, $K_{12}(\mathbf{x})$, and $K_{21}(\mathbf{x})$ are vectors and matrices functions of appropriate dimensions.

For the underlying system, the following assumptions are made throughout.

Assumption 1. The functions $\mathbf{f}(\mathbf{x})$, $G_1(\mathbf{x})$, $G_2(\mathbf{x})$, $\mathbf{h}_1(\mathbf{x})$, $\mathbf{h}_2(\mathbf{x})$, $K_{12}(\mathbf{x})$, $K_{21}(\mathbf{x})$ are smooth.

Assumption 2. We have

$$\mathbf{f}(\mathbf{0}) = \mathbf{0}, \mathbf{h}_1(\mathbf{0}) = \mathbf{0}, \text{ and } \mathbf{h}_2(\mathbf{0}) = \mathbf{0}.$$

Assumption 3. We have

$$\begin{aligned} \mathbf{h}_1^T(\mathbf{x}) K_{12}(\mathbf{x}) &= \mathbf{0}, & K_{12}^T(\mathbf{x}) K_{12}(\mathbf{x}) &= I, \\ K_{21}(\mathbf{x}) G_1^T(\mathbf{x}) &= \mathbf{0}, & K_{21}(\mathbf{x}) K_{21}^T(\mathbf{x}) &= I. \end{aligned} \quad (6)$$

Assumption 1 guarantees the well-posedness of the above dynamic system, under integrable exogenous inputs. Assumption 2 ensures that the origin is an equilibrium point of the nondriven ($\mathbf{u}(t) = \mathbf{0}$) and disturbance-free ($\mathbf{w}(t) = \mathbf{0}$) dynamic systems (5). Assumption 3 is a simplifying assumption inherited from the standard $\mathcal{H}_\infty$ control problem (see, e.g., [12, 27, 28]).

A state feedback controller

$$\mathbf{u} = \mathcal{K}(\mathbf{x}) \quad (7)$$

is said to be a globally admissible controller under the available measurement $\mathbf{y}(t)$ if the closed-loop system (5), (7) is globally asymptotically stable when $\mathbf{w}(t) = \mathbf{0}$. System (5) has $\mathcal{L}_2$ -gain less than γ if the response $\mathbf{z}(t)$ resulting from $\mathbf{w}(t)$ for initial state $\mathbf{x}(\mathbf{0}) = \mathbf{0}$ satisfies

$$\int_{t_0}^{t_1} \|\mathbf{z}(t)\|^2 dt < \gamma^2 \int_{t_0}^{t_1} \|\mathbf{w}(t)\|^2 dt \quad (8)$$

for all $t_1 > t_0$ and all piecewise continuous functions $\mathbf{w}(t)$.

3.1. Global State-Space Solution. The result, given below, globally solves the $\mathcal{H}_\infty$ control problem for nonlinear autonomous systems (5).

Hypothesis 1 (see [10]). There exist a positive definite function $F(\mathbf{x})$ and a smooth positive definite function $V(\mathbf{x})$, locally defined in the neighborhood of the origin $\mathbf{x} = \mathbf{0}$, which solves the Hamilton-Jacobi-Isaacs inequality

$$\begin{aligned} \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \gamma^2 \alpha_1^T(\mathbf{x}) \alpha_1(\mathbf{x}) - \alpha_2^T(\mathbf{x}) \alpha_2(\mathbf{x}) \\ + \mathbf{h}_1^T(\mathbf{x}) \mathbf{h}_1(\mathbf{x}) + F(\mathbf{x}) \leq 0 \end{aligned} \quad (9)$$

with

$$\alpha_1(\mathbf{x}) = \frac{1}{2\gamma^2} G_1^T(\mathbf{x}) \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \quad (10)$$

$$\alpha_2(\mathbf{x}) = -\frac{1}{2} G_2^T(\mathbf{x}) \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T. \quad (11)$$

Under Hypothesis 1, a solution of the $\mathcal{H}_\infty$ control problem stated above is as follows.

Theorem 4 (see [10]). *Let Hypothesis 1 be satisfied. Then the state feedback*

$$\mathbf{u} = \alpha_2(\mathbf{x}) \quad (12)$$

globally asymptotically stabilizes the disturbance-free system (5) and makes the $\mathcal{L}_2$ -gain of the closed-loop system less than γ .

4. Global Nonlinear $\mathcal{H}_\infty$ Regulator for Robot Manipulators

The nonlinear $\mathcal{H}_\infty$ regulation problem can be established as follows: given a desired constant ending vector position $\mathbf{q}_d \in \mathbb{R}^l$, the control objective for the regulator under study consists in making the system globally asymptotically stable and ensuring that

$$\lim_{t \rightarrow \infty} \|\mathbf{q}(t) - \mathbf{q}_d\| = 0 \quad (13)$$

for arbitrary initial condition $\mathbf{q}(\mathbf{0}) \in \mathbb{R}^l$ in spite of the presence of external disturbances.

Proposition 5. *The stabilizing controller of the form*

$$\boldsymbol{\tau} = \mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d) + \mathbf{F}_f(\mathbf{x}_2) - K_p \mathbf{x}_1 - K_v \mathbf{x}_2 + \mathbf{u}, \quad (14)$$

where $\mathbf{x}_1 = \mathbf{q} - \mathbf{q}_d$ is the position error vector, $\mathbf{x}_2 = \dot{\mathbf{q}}$, and K_p and K_v are $l \times l$ symmetric positive definite matrices, imposes the manipulator motion around $\mathbf{q}_d$, when $\mathbf{w} = \mathbf{0}$.

The controller to be constructed consists of the gravitational torque ($\mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d)$) and friction torque ($\mathbf{F}_f(\mathbf{x}_2)$) compensators, a proportional-differential part that imposes the desired stability properties on the disturbance-free system motion, and a disturbance attenuator $\mathbf{u}(t)$, internally stabilizing the closed-loop system around the desired position.

The state-space representation of the closed-loop system (1), (14), in terms of the errors $\mathbf{x}(t) = [\mathbf{x}_1(t) \ \mathbf{x}_2(t)]^T$, is given by

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= M^{-1}(\mathbf{x}_1 + \mathbf{q}_d) \\ &\quad \cdot [-C(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2) \mathbf{x}_2 - K_p \mathbf{x}_1 - K_v \mathbf{x}_2 + \mathbf{w}_\tau + \mathbf{u}]. \end{aligned} \quad (15)$$

We confine our design objective to a position regulation, where the output to be controlled is given by

$$\mathbf{z} = \begin{bmatrix} \mathbf{u} \\ \rho \tanh(\mathbf{x}) \end{bmatrix} \quad (16)$$

with a positive weight coefficient ρ , where $\mathbf{tanh}(\mathbf{x}) = [\mathbf{tanh}(\mathbf{x}_1), \mathbf{tanh}(\mathbf{x}_2)]^T$ is the $n \times 1$ hyperbolic tangent vector. Here, $\mathbf{tanh}(\mathbf{x}_i) = [\tanh(x_{i,1}), \dots, \tanh(x_{i,l})]^T$ ($i = 1, 2$). The position and velocities are available for feedback; that is,

$$\mathbf{y} = \mathbf{x} + \mathbf{w}_y, \quad (17)$$

and these measurements are corrupted by the vector $\mathbf{w}_y(t) \in \mathbb{R}^n$.

Setting $\mathbf{w} = [\mathbf{w}_\tau \ \mathbf{w}_y]^T$, (15)–(17) can be rewritten in the generalized form (5) with

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_2 \\ M^{-1}(\mathbf{x}_1 + \mathbf{q}_d) [-C(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2) \mathbf{x}_2 - K_p \mathbf{x}_1 - K_v \mathbf{x}_2] \end{bmatrix}, \quad (18)$$

$$G_1(\mathbf{x}) = \begin{bmatrix} 0_{(l \times l)} & 0_{(l \times n)} \\ M^{-1}(\mathbf{x}_1 + \mathbf{q}_d) & 0_{(l \times n)} \end{bmatrix}, \quad (19)$$

$$G_2(\mathbf{x}) = \begin{bmatrix} 0_{(l \times l)} \\ M^{-1}(\mathbf{x}_1 + \mathbf{q}_d) \end{bmatrix},$$

$$\mathbf{h}_1(\mathbf{x}) = \begin{bmatrix} 0_{(l \times 1)} \\ \rho \mathbf{tanh}(\mathbf{x}) \end{bmatrix}, \quad \mathbf{h}_2(\mathbf{x}) = \mathbf{x}, \quad (20)$$

$$K_{12}(\mathbf{x}) = \begin{bmatrix} I_l \\ 0_{(n \times l)} \end{bmatrix}, \quad K_{21}(\mathbf{x}) = [0_{(n \times l)} \ I_n].$$

Hereinafter, I_n and $0_{n \times m}$ stand for the $n \times n$ identity matrix and the $n \times m$ matrix of zeros, respectively.

Theorem 6. Let Hypothesis 1 be satisfied with the following positive definite function [26]:

$$V(\mathbf{x}) = \frac{1}{2} \mathbf{x}_1^T K_p \mathbf{x}_1 + \frac{1}{2} \mathbf{x}_2^T M(\mathbf{x}_1 + \mathbf{q}_d) \mathbf{x}_2 + \beta \mathbf{tanh}(\mathbf{x}_1)^T M(\mathbf{x}_1 + \mathbf{q}_d) \mathbf{x}_2, \quad (21)$$

which is positive definite for all $\mathbf{x} \in \mathbb{R}^n$ and radially unbounded if

$$\lambda_{\min}\{K_p\} > \frac{\beta^2 \lambda_{\max}^2\{M\}}{\lambda_{\min}\{M\}}, \quad (22)$$

and the time derivative of $V(\mathbf{x})$ along the trajectories of system (18) is negative definite if

$$\lambda_{\min}\{K_v\} > \beta(\lambda_{\max}\{M\} + \sqrt{n}k_{c_1}) + \rho^2 + \varepsilon,$$

$$\begin{aligned} & \lambda_{\min}\{K_p\} \\ & > \left(\frac{\beta}{4} \lambda_{\max}^2\{K_v\} - \frac{1}{\beta} (\rho^2 + \varepsilon)^2 + (\rho^2 + \varepsilon) \right. \\ & \quad \cdot \left(\frac{1}{\beta} \lambda_{\min}\{K_v\} - \lambda_{\max}\{M\} - \sqrt{n}k_{c_1} \right) \\ & \quad \cdot \left(\lambda_{\min}\{K_v\} - \beta(\lambda_{\max}\{M\} + \sqrt{n}k_{c_1}) - \rho^2 - \varepsilon \right)^{-1} \end{aligned} \quad (23)$$

holds ($\lambda_{\max}$ and $\lambda_{\min}$ are defined as the largest and smaller eigenvalues, resp.), where $\varepsilon > 0$ is a sufficiently small constant. Then, the state feedback

$$\mathbf{u} = -\frac{1}{2} G_2^T(\mathbf{x}) \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T = -\frac{1}{2} [\beta \mathbf{tanh}(\mathbf{x}_1) + \mathbf{x}_2] \quad (24)$$

globally asymptotically stabilizes the disturbance-free system (18), (19), (24) and makes the $\mathcal{L}_2$ -gain of the closed-loop system less than γ .

Proof. To better explain the analysis, we split the HJI inequality (9) in two parts; that is,

$$H(\mathbf{x}) = H_1(\mathbf{x}) + H_2(\mathbf{x}), \quad (25)$$

where

$$H_1(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \mathbf{h}_1^T(\mathbf{x}) \mathbf{h}_1(\mathbf{x}) + F(\mathbf{x}) \quad (26)$$

$$H_2(\mathbf{x}) = \gamma^2 \alpha_1^T(\mathbf{x}) \alpha_1(\mathbf{x}) - \alpha_2^T(\mathbf{x}) \alpha_2(\mathbf{x}).$$

For $H_1(\mathbf{x})$ we have that

$$\begin{aligned} & H_1 \\ & = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \\ & \quad + \underbrace{\rho^2 (\mathbf{tanh}(\mathbf{x}_1)^T \mathbf{tanh}(\mathbf{x}_1) + \mathbf{tanh}(\mathbf{x}_2)^T \mathbf{tanh}(\mathbf{x}_2))}_{=h_1^T h_1} \\ & \quad + F(\mathbf{x}). \end{aligned} \quad (27)$$

From Hypothesis 1, $F(\mathbf{x})$ is a positive definite function chosen, a priori, as

$$F(\mathbf{x}) = \varepsilon \mathbf{tanh}(\mathbf{x}_1)^T \mathbf{tanh}(\mathbf{x}_1) + \varepsilon \mathbf{tanh}(\mathbf{x}_2)^T \mathbf{tanh}(\mathbf{x}_2). \quad (28)$$

Developing $H_1(\mathbf{x})$, we have

$$\begin{aligned} H_1 & = -\beta \mathbf{tanh}(\mathbf{x}_1)^T K_p \mathbf{x}_1 + \beta \mathbf{x}_2^T M(\mathbf{x}_1 + \mathbf{q}_d) \text{Sech}^2(\mathbf{x}_1) \mathbf{x}_2 \\ & \quad - \mathbf{x}_2^T K_v \mathbf{x}_2 + \beta \mathbf{tanh}(\mathbf{x}_1)^T C(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)^T \mathbf{x}_2 \\ & \quad - \beta \mathbf{tanh}(\mathbf{x}_1)^T K_v \mathbf{x}_2 \\ & \quad + (\rho^2 + \varepsilon) \mathbf{tanh}(\mathbf{x}_1)^T \mathbf{tanh}(\mathbf{x}_1) \\ & \quad + (\rho^2 + \varepsilon) \mathbf{x}_2^T \mathbf{x}_2. \end{aligned} \quad (29)$$

Here, $\partial \tanh(\mathbf{x}_1)/\partial \mathbf{x}_1 = \text{Sech}^2(\mathbf{x}_1) = \text{diag}\{\text{sech}^2(x_{1,1}), \dots, \text{sech}^2(x_{1,l})\}$. Consider the following well-known inequalities for a vectorial tangent and secant hyperbolic functions:

- (i) $\|\tanh(\mathbf{x})\| \leq \alpha_1 \|\mathbf{x}\|$,
- (ii) $\|\tanh(\mathbf{x})\| \leq \alpha_2$,
- (iii) $\|\tanh(\mathbf{x})\|^2 \leq \alpha_3 \tanh(\mathbf{x})^T \mathbf{x}$,
- (iv) $\|\text{Sech}^2(\mathbf{x})\dot{\mathbf{x}}\| \leq \alpha_4 \|\dot{\mathbf{x}}\|$,

for any $\mathbf{x}, \dot{\mathbf{x}} \in \mathbb{R}^n$ and $\alpha_1, \dots, \alpha_4 > 0$. With these inequalities and using Properties 1–3, provided in Section 2, we can obtain

$$H_1 \leq -\lambda_{\min}\{K_p\} \|\tanh(\mathbf{x}_1)\|^2 + \beta \lambda_{\max}\{M(\mathbf{x}_1 + \mathbf{q}_d)\} \|\mathbf{x}_2\|^2$$

$$\begin{aligned} & -\lambda_{\min}\{K_v\} \|\mathbf{x}_2\|^2 + \beta \sqrt{n} k_{c_1} \|\mathbf{x}_2\|^2 \\ & + \beta \lambda_{\max}\{K_v\} \|\tanh(\mathbf{x}_1)\| \|\mathbf{x}_2\| \\ & + (\rho^2 + \varepsilon) \|\mathbf{x}_2\|^2 + (\rho^2 + \varepsilon) \|\tanh(\mathbf{x}_1)\|^2. \end{aligned} \quad (30)$$

The latter bound can be rewritten in the following form:

$$H_1 < - \begin{bmatrix} \|\tanh(\mathbf{x}_1)\| \\ \|\mathbf{x}_2\| \end{bmatrix}^T Q_1 \begin{bmatrix} \|\tanh(\mathbf{x}_1)\| \\ \|\mathbf{x}_2\| \end{bmatrix}, \quad (31)$$

where

$$Q_1 = \begin{bmatrix} \beta \lambda_{\min}\{K_p\} - (\rho^2 + \varepsilon) & -\frac{\beta}{2} \lambda_{\max}\{K_v\} \\ -\frac{\beta}{2} \lambda_{\max}\{K_v\} & \lambda_{\min}\{K_v\} - \beta \sqrt{n} k_{c_1} - \beta \lambda_{\max}\{M\} - (\rho^2 + \varepsilon) \end{bmatrix}. \quad (32)$$

Applying the theorem of Silvester to Q_1 , we have that

$$\lambda_{\min}\{K_v\} > \beta (\lambda_{\max}\{M\} + \sqrt{n} k_{c_1}) + \rho^2 + \varepsilon, \quad (33)$$

$$\begin{aligned} & \lambda_{\min}\{K_p\} \\ & > \left(\frac{\beta}{4} \lambda_{\max}^2\{K_v\} - \frac{1}{\beta} (\rho^2 + \varepsilon)^2 + (\rho^2 + \varepsilon) \right. \\ & \quad \cdot \left(\frac{1}{\beta} \lambda_{\min}\{K_v\} - \lambda_{\max}\{M\} - \sqrt{n} k_{c_1} \right) \\ & \quad \cdot \left(\lambda_{\min}\{K_v\} - \beta (\lambda_{\max}\{M\} + \sqrt{n} k_{c_1}) - \rho^2 - \varepsilon \right)^{-1} \\ & \quad \left. > \frac{\rho^2 + \varepsilon}{\beta} \right). \end{aligned} \quad (34)$$

Choosing K_p and K_v satisfying the above bounds, then Q_1 will be a positive definite matrix, and consequently $H_1(\mathbf{x})$ will be a negative definite function.

Let us develop $H_2(\mathbf{x})$. For the proposed Lyapunov function we have that

$$\begin{aligned} \alpha_1(\mathbf{x}) &= \frac{1}{2\gamma^2} \begin{bmatrix} \beta \tanh(\mathbf{x}_1) + \mathbf{x}_2 \\ 0 \end{bmatrix}, \\ \alpha_2(\mathbf{x}) &= -\frac{1}{2} [\beta \tanh(\mathbf{x}_1) + \mathbf{x}_2]. \end{aligned} \quad (35)$$

Hence,

$$\begin{aligned} H_2 &= - \left(\frac{1}{4} - \frac{1}{4\gamma^2} \right) \mathbf{x}_2^T \mathbf{x}_2 - \left(\frac{\beta}{2} - \frac{\beta}{2\gamma^2} \right) \mathbf{x}_2^T \tanh(\mathbf{x}_1) \\ & \quad - \left(\frac{\beta^2}{4} - \frac{\beta^2}{4\gamma^2} \right) \tanh(\mathbf{x}_1)^T \tanh(\mathbf{x}_1) \\ & \leq - \left(1 - \frac{1}{\gamma^2} \right) \\ & \quad \cdot \left(\frac{1}{4} \|\mathbf{x}_2\|^2 - \frac{\beta}{2} \|\mathbf{x}_2\| \|\tanh(\mathbf{x}_1)\| + \frac{\beta^2}{4} \|\tanh(\mathbf{x}_1)\|^2 \right). \end{aligned} \quad (36)$$

The latter bound can be compactly expressed as

$$H_2 \leq -\frac{1}{4} \left(1 - \frac{1}{\gamma^2} \right) \begin{bmatrix} \|\tanh(\mathbf{x}_1)\| \\ \|\mathbf{x}_2\| \end{bmatrix}^T \underbrace{\begin{bmatrix} \beta^2 & \beta \\ \beta & 1 \end{bmatrix}}_{Q_2} \begin{bmatrix} \|\tanh(\mathbf{x}_1)\| \\ \|\mathbf{x}_2\| \end{bmatrix}. \quad (37)$$

Thus, choosing $\gamma > 1$ and $\beta > 0$, the matrix Q_2 will be positive semidefinite and consequently $H_2(\mathbf{x})$ will be a negative semidefinite function.

Finally, the Hamilton-Jacobi-Isaacs inequality $H(\mathbf{x})$ will be a negative definite function if $\lambda_{\min}\{Q_1\} + (1/4)(1 - 1/\gamma^2)\lambda_{\min}\{Q_2\} > 0$. The proof is completed. $\square$

The positive definite function (21), proven in [26] to be a strict Lyapunov function to conclude GAS for a disturbance-free robot, was crucial to prove that the Hamilton-Jacobi-Isaacs inequality is negative definite globally in contrast to the *standard* positive definite function $V = \mathbf{x}^T P \mathbf{x}$, $P = P^T > 0$, where only asymptotic stability in a neighborhood of the origin can be concluded [15].

FIGURE 1: The 3-DOF robot manipulator used for the experiments.

5. Application to a 3-DOF Industrial Robot

5.1. Experimental Setup. Experimental setup involves a robot manipulator (see Figure 1) manufactured by AMATROL. The base of the mechanical robot has a horizontal revolute joint (q_1), whereas two links have vertical revolute joints q_2 and q_3 . The remaining degrees-of-freedom correspond to the end effector orientation. Nominal parameters of mechanical manipulator are summarized in Table 1. Worm gear set, helicon gear set, and roller chain are used for torque transmission to joints q_1 , q_2 , and q_3 , respectively; there is a DC gear motor for each joint with a reduction ratio of 19.7:1 for q_1 and q_2 , and 127.8:1 for q_3 . These gears are the main source of friction. The PCI multifunction I/O board model 626 from Sensoray Co., Inc. [29], is employed for the real time control system and it consists of four analog outputs (13-bit resolution), 20 digital I/O channels with edge detection, and interrupt capability. The controllers are implemented using MATLAB/SIMULINK running on a Pentium PC. Position measurements of each joint of the robot are obtained using the channels of quadrature encoders available on all DC gear motors which are connected to the I/O card, programmed to provide the encoder signal processing each millisecond. The resolutions of encoders are 52×10^{-3} rad, 62×10^{-3} rad, and 34×10^{-3} rad for q_1 , q_2 , and q_3 , respectively. Linear power amplifiers are installed in each servomotor which applies a variable torque to each joint. These amplifiers accept control inputs from D/A converter in the range of ± 10 V. Evidence of presence of Coulomb friction and backlash has been reported in [15].

5.2. Experimental Results. The robot manipulator was required to move in the space from the origin $q_1(0) = q_2(0) = q_3(0) = 0^\circ$ (the references for each joint are shown in Figure 1) to the desired positions $q_{d1} = 180^\circ$, $q_{d2} = 145^\circ$, and $q_{d3} = 100^\circ$, chosen deliberately to be far from the origin. The initial velocity $\dot{q}(0) \in \mathbb{R}^3$ was set to zero for the

TABLE 1: Parameter of the mechanical manipulator.

Description	Notation	Value	Units
Length of link 1	l_1	0.297	m
Length of link 2	l_2	0.297	m
Mass of link 1	m_1	0.38	Kg
Mass of link 2	m_2	0.34	Kg
Inertial 1	I_1	0.243×10^{-3}	Kgm ²
Inertial 2	I_2	0.068×10^{-3}	Kgm ²
Inertial 3	I_3	0.015×10^{-3}	Kgm ²
Friction coeff. 1	F_{c1}	1.6	Nm
Friction coeff. 2	F_{c2}	1.08	Nm
Friction coeff. 3	F_{c3}	0.81	Nm
Gravity acceleration	g	9.8	m/s ²

experiment. We achieved the control goal by implementing the global $\mathcal{H}_\infty$ regulators (11) and (14) with the weight parameters $\rho = 0.1$, $\beta = 0.02$, and $\varepsilon = 0.01$ on the 3-DOF manipulator. The gains of the linear part in (14) were set to

$$K_p = \text{diag}\{4, 3.1, 2.81\}, \quad (38)$$

$$K_v = \text{diag}\{0.4947, 0.3947, 0.2947\}.$$

As predicted, Figure 2(a) shows convergence of the trajectories to the desired position and Figure 2(b) shows the control inputs for the disturbances-free system. In contrast to the nonlinear $\mathcal{H}_\infty$ controller for robot position regulation presented in [15], initial conditions have been prespecified relatively far from the desired positions noticing that the control objective is evidently achieved.

5.3. Simulation Results. Since experimental robot is not easy to be externally perturbed, simulation results, run in MATLAB/SIMULINK, are presented to provide evidence of robustness. The dynamic model of the experimental robot is given in the Appendix. We used the same initial conditions, desired position, and gains and parameters of the controller provided in the latter subsection.

Simulations were performed affecting the model with harmonic functions as external disturbances; that is,

$$w_i(t) = \frac{\pi}{10} \cos(2\pi it), \quad i = 1, \dots, 9. \quad (39)$$

Figure 3, which illustrates angular positions and control inputs, corroborates disturbance attenuation of the closed-loop system. For the sake of comparison, the disturbance attenuator $u(t) \in \mathbb{R}^3$ was set to zero. The angular positions in Figure 4 show underdamped position responses whose overshoot is larger than the position responses using the proposed global $\mathcal{H}_\infty$ regulator (Figure 3(a)). On the other hand, the applied torque, in Figure 4(b), made it evident that the closed-loop system with $\mathcal{H}_\infty$ regulator needs less energy to stabilize the trajectories to the desired position and attenuate external disturbances.

FIGURE 2: Experimental output responses (a) and input torques (b) of the unperturbed closed-loop system.

FIGURE 3: Simulation output responses (a) and input torques (b) of the perturbed closed-loop system.

6. Conclusions

We presented a global solution of the nonlinear $\mathcal{H}_\infty$ regulation problem, where the HJI inequality was satisfied verifying a strict Lyapunov function proposed in [2]. The key to guarantee that the HJI inequality was negative definite was proposing a bounded control objective ($\mathbf{z}$) and a sufficiently small parameter β . Experimental results, made for a 3-DOF robot manipulator with friction, verify the results.

Appendix

The equation of motion of the experimental manipulator governed by (1) was specified by applying the Euler-Lagrange formulation [3], where

$$M(\mathbf{q}) = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{bmatrix}, \quad (\text{A.1})$$

FIGURE 4: Simulation output responses (a) and input torques (b) of the perturbed closed-loop system under $\mathbf{u}(t) = \mathbf{0}$.

where

$$\begin{aligned}
 m_{11}(\mathbf{q}) &= m_2 l_1^2 \cos^2(q_2) + 2m_2 l_1 l_2 \cos(q_2) \cos(q_2 + q_3) \\
 &\quad + m_2 l_2^2 \cos^2(q_2 + q_3) + m_1 l_1^2 \cos^2(q_2) + I_1 \\
 m_{22}(\mathbf{q}) &= m_1 l_1^2 + m_2 l_1^2 + 2m_2 l_1 l_2 \cos(q_3) \\
 &\quad + m_2 l_2^2 + I_2 + I_3 \\
 m_{23}(\mathbf{q}) &= m_2 l_1 l_2 \cos(q_3) + m_2 l_2^2 + I_3 \\
 m_{33}(\mathbf{q}) &= m_2 l_2^2 + I_3; \\
 C(\mathbf{q}, \dot{\mathbf{q}}) &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix},
 \end{aligned} \tag{A.2}$$

where

$$\begin{aligned}
 c_{11}(\mathbf{q}, \dot{\mathbf{q}}) &= -m_2 l_1^2 S_2 C_2 \dot{q}_2 - m_2 l_1 l_2 S_2 C_{23} \dot{q}_2 \\
 &\quad - m_1 l_1^2 S_2 C_2 \dot{q}_2 - m_2 l_1 l_2 C_2 S_{23} (\dot{q}_2 + \dot{q}_3) \\
 &\quad - m_2 l_2^2 S_{23} C_{23} (\dot{q}_2 + \dot{q}_3) \\
 c_{12}(\mathbf{q}, \dot{\mathbf{q}}) &= -m_2 l_1^2 S_2 C_2 \dot{q}_1 - m_2 l_1 l_2 C_2 S_{23} \dot{q}_1 - m_2 l_1 l_2 S_2 C_{23} \dot{q}_1 \\
 &\quad - m_2 l_2^2 S_{23} C_{23} \dot{q}_1 - m_1 l_1^2 S_2 C_2 \dot{q}_1 \\
 c_{13}(\mathbf{q}, \dot{\mathbf{q}}) &= -m_2 l_1 l_2 C_2 S_{23} \dot{q}_1 - m_2 l_2^2 S_{23} C_{23} \dot{q}_1
 \end{aligned}$$

$$\begin{aligned}
 c_{21}(\mathbf{q}, \dot{\mathbf{q}}) &= m_2 l_1^2 S_2 C_2 \dot{q}_1 + m_2 l_1 l_2 C_2 S_{23} \dot{q}_1 \\
 &\quad + m_2 l_1 l_2 S_2 C_{23} \dot{q}_1 + m_2 l_2^2 S_{23} C_{23} \dot{q}_1 \\
 c_{22}(\mathbf{q}, \dot{\mathbf{q}}) &= -m_2 l_1 l_2 S_3 \dot{q}_3 \\
 c_{23}(\mathbf{q}, \dot{\mathbf{q}}) &= -m_2 l_1 l_2 S_3 (\dot{q}_2 + \dot{q}_3) \\
 c_{31}(\mathbf{q}, \dot{\mathbf{q}}) &= m_2 l_1 l_2 C_2 S_{23} \dot{q}_1 + m_2 l_2^2 S_{23} C_{23} \dot{q}_1 \\
 c_{32}(\mathbf{q}, \dot{\mathbf{q}}) &= m_2 l_1 l_2 S_3 \dot{q}_2
 \end{aligned} \tag{A.3}$$

with $S_i = \sin(q_i)$, $C_i = \cos(q_i)$, $S_{ij} = \sin(q_i + q_j)$, and $C_{ij} = \cos(q_i + q_j)$; and finally

$$\begin{aligned}
 \mathbf{g}(\mathbf{q}) &= g \begin{bmatrix} 0 \\ m_1 l_1 \cos(q_2) + m_2 l_1 \cos(q_2) + m_2 l_2 \cos(q_2 + q_3) \\ m_2 l_2 \cos(q_2 + q_3) \end{bmatrix}.
 \end{aligned} \tag{A.4}$$

The typical constant parameters, m_i , l_i , $i = 1, 2$, and I_j , $j = 1, 2, 3$, and their upper bounds are given in Tables 1 and 2, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

TABLE 2: Bounds of the matrices involved in the mechanical manipulator.

Bound	Definition	Value	Unit
K_{c_1}	$n^2(\max_{i,j,k,q} \ C_{kij}(q)\)$	2.2229	Kgm ²
$\lambda_{\max}\{M\}$	$n(\max_{i,j,q} \ M_{ij}(q)\)$	0.9224	Kgm ²
$\lambda_{\min}\{M\}$	I_1	0.243×10^{-3}	Kgm ²

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