Research Article

Forced Oscillation Criteria for a Class of Fractional Partial Differential Equations with Damping Term

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Sufficient conditions are established for the forced oscillation of fractional partial differential equations with damping term of the form $(\partial/\partial t)(D_{+,t}^\alpha u(x,t)) + p(t)D_{+,t}^\alpha u(x,t) = a(t)\Delta u(x,t) - q(x,t)u(x,t) + f(x,t)$, $(x,t) \in \Omega \times R_+ \equiv G$, with one of the two following boundary conditions: $\partial u(x,t)/\partial N = \psi(x,t)$, $(x,t) \in \partial\Omega \times R_+$ or $u(x,t) = 0$, $(x,t) \in \partial\Omega \times R_+$, where Ω is a bounded domain in R^n with a piecewise smooth boundary, $\partial\Omega, R_+ = [0, \infty)$, $\alpha \in (0, 1)$ is a constant, $D_{+,t}^\alpha u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t , Δ is the Laplacian in R^n , N is the unit exterior normal vector to $\partial\Omega$, and $\psi(x,t)$ is a continuous function on $\partial\Omega \times R_+$. The main results are illustrated by some examples.

1. Introduction

Fractional differential equations are generalizations of classical differential equations to an arbitrary (noninteger) order and have gained increasing attention because of their varied applications in various fields of applied sciences and engineering. In the past few years, the theory of fractional differential equations and their applications have been investigated extensively; for example, see the monographs [1–4].

Recently, the oscillatory behavior of solutions for fractional differential equations was discussed in [5–13] and so forth. In [13], the authors studied the oscillation of solutions to nonlinear fractional differential equations of this form

$$D_{0+}^{1+\alpha} y(t) + p(t) D_{0+}^\alpha y(t) + q(t) f(y(t)) = g(t), \quad (E)$$

$$t > 0,$$

with initial condition $I_{0+}^{1-\alpha} y(0+) = b$, where b is a real number, $\alpha \in (0, 1)$ is a constant, and $D_{0+}^\alpha y$ is the Riemann-Liouville fractional derivative of order α of y .

However, to the best of author's knowledge very little is known regarding the oscillatory behavior of fractional partial differential equations up to now; we refer to [14–16]. In particular, nothing is known regarding the oscillation properties of the problems (1)-(2) and (1)-(3) up to now.

In this paper we investigate the forced oscillation of fractional partial differential equations with damping term of the form

$$\begin{aligned} \frac{\partial}{\partial t} (D_{+,t}^\alpha u(x,t)) + p(t) D_{+,t}^\alpha u(x,t) \\ = a(t) \Delta u(x,t) - q(x,t) u(x,t) + f(x,t), \end{aligned} \quad (1)$$

$$(x,t) \in \Omega \times R_+ \equiv G,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega, R_+ = [0, \infty)$, $\alpha \in (0, 1)$ is a constant, $D_{+,t}^\alpha u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t , and Δ is the Laplacian in R^n .

We assume throughout this paper that

- (A1) $a \in C(R_+; (0, \infty))$, $p \in C(R_+; R)$;
- (A2) $q \in C(\bar{G}; R_+)$, and $q(t) = \min_{x \in \bar{\Omega}} q(x,t)$;
- (A3) $f \in C(\bar{G}; R)$.

Consider one of the two following boundary conditions:

$$\frac{\partial u(x,t)}{\partial N} = \psi(x,t), \quad (x,t) \in \partial\Omega \times R_+, \quad (2)$$

or

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+, \quad (3)$$

where N is the unit exterior normal vector to $\partial\Omega$ and $\psi(x, t)$ is a continuous function on $\partial\Omega \times R_+$.

By a solution of the problems (1)-(2) (or (1)-(3)), we mean a function $u(x, t)$ which satisfies (1) on $\bar{G}$ and the boundary condition (2) (or (3)).

A solution $u(x, t)$ of the problems (1)-(2) (or (1)-(3)) is said to be oscillatory in G if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

Definition 1. The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x, t)$ is given by

$$D_{+,t}^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\xi)^{-\alpha} u(x, \xi) d\xi, \quad (4)$$

provided the right hand side is pointwise defined on R_+ , where Γ is the gamma function.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : R_+ \rightarrow R$ on the half-axis R_+ is given by

$$I_+^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} y(\xi) d\xi \quad \text{for } t > 0, \quad (5)$$

provided the right hand side is pointwise defined on R_+ .

Definition 3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : R_+ \rightarrow R$ on the half-axis R_+ is given by

$$\begin{aligned} D_+^\alpha y(t) &:= \frac{d^{[\alpha]}}{dt^{[\alpha]}} \left(I_+^{[\alpha]-\alpha} y \right) (t) \\ &= \frac{1}{\Gamma([\alpha]-\alpha)} \frac{d^{[\alpha]}}{dt^{[\alpha]}} \int_0^t (t-\xi)^{[\alpha]-\alpha-1} y(\xi) d\xi \quad (6) \\ &\quad \text{for } t > 0, \end{aligned}$$

provided the right hand side is pointwise defined on R_+ , where $[\alpha]$ is the ceiling function of α .

2. Main Results

In this section, we establish the oscillation of the problems (1)-(2) and (1)-(3). We firstly introduce the following lemmas which are very useful in the proof of our main results.

Lemma 4 (see [17]). *The smallest eigenvalue β_0 of the Dirichlet problem,*

$$\begin{aligned} \Delta \omega(x) + \beta \omega(x) &= 0, \quad \text{in } \Omega, \\ \omega(x) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (7)$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω .

Lemma 5 (see [2, page 74]). *Let $\alpha \geq 0$, $m \in N$, and $D = d/dx$. If the fractional derivatives $D_+^\alpha y(t)$ and $D_+^{\alpha+m} y(t)$ exist, then*

$$D^m (D_+^\alpha y(t)) = D_+^{\alpha+m} y(t). \quad (8)$$

Lemma 6 (see [2, page 75]). *Let $\alpha \in (0, 1)$ and $I_+^{1-\alpha} y(t)$ be the fractional integral (5) of order $1 - \alpha$, then*

$$I_+^\alpha (D_+^\alpha y(t)) = y(t) - \frac{I_+^{1-\alpha} y(0)}{\Gamma(\alpha)} t^{\alpha-1}. \quad (9)$$

For convenience, one uses the following notations in this paper:

$$\begin{aligned} U(t) &= \int_\Omega u(x, t) dx, \\ \Psi(t) &= \int_{\partial\Omega} \psi(x, t) dS, \\ F(t) &= \int_\Omega f(x, t) dx, \\ H(t) &= F(t) + a(t) \Psi(t), \\ W(t) &= \int_\Omega u(x, t) \varphi(x) dx, \\ Y(t) &= \int_\Omega f(x, t) \varphi(x) dx, \\ &\quad t \geq 0, \end{aligned} \quad (10)$$

where dS is the surface element on $\partial\Omega$.

Theorem 7. *Assume that*

$$\lim_{t \rightarrow 0} I_+^{1-\alpha} U(t) = C_0, \quad (11)$$

where C_0 is a constant. If

$$\begin{aligned} \liminf_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^\xi H(s) V(s) ds \right) d\xi &< 0, \\ \limsup_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^\xi H(s) V(s) ds \right) d\xi &> 0, \end{aligned} \quad (12)$$

then every solution $u(x, t)$ of the problems (1)-(2) is oscillatory in G , where $V(t) = \exp \int_{t_0}^t p(\xi) d\xi$ and C is a constant.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problems (1)-(2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 \geq 0$. Then $u(x, t) > 0$ or $u(x, t) < 0$ for $t \geq t_0$.

Case 1 ($u(x, t) > 0, t \geq t_0$). Integrating (1) with respect to x over the domain Ω , we have

$$\begin{aligned} \frac{d}{dt} \left(\int_\Omega D_{+,t}^\alpha u(x, t) dx \right) + p(t) \int_\Omega D_{+,t}^\alpha u(x, t) dx \\ = a(t) \int_\Omega \Delta u(x, t) dx - \int_\Omega q(x, t) u(x, t) dx \\ + \int_\Omega f(x, t) dx, \quad t \geq t_0. \end{aligned} \quad (13)$$

Green's formula and (2) yield

$$\begin{aligned} \int_{\Omega} \Delta u(x, t) dx &= \int_{\partial\Omega} \frac{\partial u(x, t)}{\partial N} dS = \int_{\partial\Omega} \psi(x, t) dS \\ &= \Psi(t), \quad t \geq t_0. \end{aligned} \quad (14)$$

From (A2), it is easy to see that

$$\int_{\Omega} q(x, t) u(x, t) dx \geq q(t) \int_{\Omega} u(x, t) dx, \quad t \geq t_0. \quad (15)$$

By Lemma 5, it follows from (13)–(15) that

$$D_+^{1+\alpha} U(t) + p(t) D_+^{\alpha} U(t) \leq -q(t) U(t) + H(t), \quad t \geq t_0. \quad (16)$$

From (16), we easily see that

$$\begin{aligned} ((D_+^{\alpha} U(t)) V(t))' &= (D_+^{1+\alpha} U(t)) V(t) \\ &\quad + (D_+^{\alpha} U(t)) p(t) V(t) \\ &\leq -q(t) U(t) V(t) + H(t) V(t) \\ &\leq H(t) V(t), \quad t \geq t_0. \end{aligned} \quad (17)$$

Integrating both sides of the above inequality from t_0 to t , we obtain

$$\begin{aligned} (D_+^{\alpha} U(t)) V(t) &\leq (D_+^{\alpha} U(t_0)) V(t_0) \\ &\quad + \int_{t_0}^t H(s) V(s) ds \\ &= C + \int_{t_0}^t H(s) V(s) ds, \end{aligned} \quad (18)$$

where $C = (D_+^{\alpha} U(t_0)) V(t_0)$. Using Lemma 6, it follows from (18) that

$$\begin{aligned} U(t) &\leq \frac{I_+^{1-\alpha} U(0)}{\Gamma(\alpha)} t^{\alpha-1} \\ &\quad + I_+^{\alpha} \left(\frac{C}{V(t)} + \frac{1}{V(t)} \int_{t_0}^t H(s) V(s) ds \right) \\ &= \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi. \end{aligned} \quad (19)$$

Taking $t \rightarrow \infty$ in (19), we have

$$\begin{aligned} \liminf_{t \rightarrow \infty} U(t) &\leq \limsup_{t \rightarrow \infty} \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1} + \liminf_{t \rightarrow \infty} \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi < 0, \end{aligned} \quad (20)$$

which contradicts the fact that $U(t) > 0$.

Case 2 ($u(x, t) < 0, t \geq t_0$). Using the procedure of the proof of Case 1, we conclude that (13) and (14) are satisfied. From (A2), we have

$$\int_{\Omega} q(x, t) u(x, t) dx \leq q(t) \int_{\Omega} u(x, t) dx, \quad t \geq t_0. \quad (21)$$

Combining (13), (14), and (21), we have

$$D_+^{1+\alpha} U(t) + p(t) D_+^{\alpha} U(t) \geq -q(t) U(t) + H(t), \quad t \geq t_0. \quad (22)$$

It follows from (22) that

$$\begin{aligned} ((D_+^{\alpha} U(t)) V(t))' &= (D_+^{1+\alpha} U(t)) V(t) \\ &\quad + (D_+^{\alpha} U(t)) p(t) V(t) \\ &\geq -q(t) U(t) V(t) + H(t) V(t) \\ &\geq H(t) V(t), \quad t \geq t_0. \end{aligned} \quad (23)$$

Integrating both sides of the above inequality from t_0 to t , we obtain

$$\begin{aligned} (D_+^{\alpha} U(t)) V(t) &\geq (D_+^{\alpha} U(t_0)) V(t_0) \\ &\quad + \int_{t_0}^t H(s) V(s) ds \\ &= C + \int_{t_0}^t H(s) V(s) ds. \end{aligned} \quad (24)$$

Using Lemma 6, from (24) we obtain

$$\begin{aligned} U(t) &\geq \frac{I_+^{1-\alpha} U(0)}{\Gamma(\alpha)} t^{\alpha-1} \\ &\quad + I_+^{\alpha} \left(\frac{C}{V(t)} + \frac{1}{V(t)} \int_{t_0}^t H(s) V(s) ds \right) \\ &= \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi. \end{aligned} \quad (25)$$

Taking $t \rightarrow \infty$ in (25), we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} U(t) &\geq \liminf_{t \rightarrow \infty} \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1} + \limsup_{t \rightarrow \infty} \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi > 0, \end{aligned} \quad (26)$$

which contradicts the fact that $U(t) < 0$. This completes the proof of Theorem 7. $\square$

Theorem 8. Assume that

$$\lim_{t \rightarrow 0} I_+^{1-\alpha} W(t) = C_1, \quad (27)$$

where C_1 is a constant. If

$$\begin{aligned} \liminf_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} Y(s) V(s) ds \right) d\xi &< 0, \\ \limsup_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} Y(s) V(s) ds \right) d\xi &> 0, \end{aligned} \quad (28)$$

then every solution $u(x, t)$ of the problems (1)–(3) is oscillatory in G , where $V(t) = \exp \int_{t_0}^t p(\xi) d\xi$ and C is a constant.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problems (1)–(3) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 \geq 0$. Then $u(x, t) > 0$ or $u(x, t) < 0$ for $t \geq t_0$.

Case 1 ($u(x, t) > 0, t \geq t_0$). Multiplying both sides of (1) by $\varphi(x)$ and integrating with respect to x over the domain Ω , we have

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\Omega} D_{+,t}^{\alpha} u(x, t) \varphi(x) dx \right) \\ &+ p(t) \int_{\Omega} D_{+,t}^{\alpha} u(x, t) \varphi(x) dx \\ &= a(t) \int_{\Omega} \Delta u(x, t) \varphi(x) dx \\ &- \int_{\Omega} q(x, t) u(x, t) \varphi(x) dx \\ &+ \int_{\Omega} f(x, t) \varphi(x) dx, \quad t \geq t_0. \end{aligned} \quad (29)$$

Green's formula and (3) yield

$$\begin{aligned} \int_{\Omega} \Delta u(x, t) \varphi(x) dx &= \int_{\Omega} u(x, t) \Delta \varphi(x) dx \\ &= -\beta_0 \int_{\Omega} u(x, t) \varphi(x) dx, \end{aligned} \quad (30)$$

$t \geq t_0$.

From (A2), it is easy to see that

$$\int_{\Omega} q(x, t) u(x, t) \varphi(x) dx \geq q(t) \int_{\Omega} u(x, t) \varphi(x) dx, \quad (31)$$

$t \geq t_0$.

Combining (29)–(31), we easily see that

$$\begin{aligned} &D_+^{1+\alpha} W(t) + p(t) D_+^{\alpha} W(t) \\ &\leq -(\beta_0 a(t) + q(t)) W(t) + Y(t), \quad t \geq t_0. \end{aligned} \quad (32)$$

The remainder of the proof is similar to that of Case 1 of Theorem 7 and we can obtain a contradiction to $W(t) > 0$.

Case 2 ($u(x, t) < 0, t \geq t_0$). Using the procedure of the proof of Case 1, we conclude that (29) and (30) are satisfied. From (A2), we have

$$\int_{\Omega} q(x, t) u(x, t) \varphi(x) dx \leq q(t) \int_{\Omega} u(x, t) \varphi(x) dx, \quad (33)$$

$t \geq t_0$.

Combining (29), (30), and (33), we obtain

$$\begin{aligned} &D_+^{1+\alpha} W(t) + p(t) D_+^{\alpha} W(t) \\ &\geq -(\beta_0 a(t) + q(t)) W(t) + Y(t), \quad t \geq t_0. \end{aligned} \quad (34)$$

Using a similar way in the proof of Case 2 of Theorem 7, we can obtain a contradiction to $W(t) < 0$. The proof of Theorem 8 is complete. $\square$

3. Examples

Example 1. Consider the following fractional partial differential equation:

$$\begin{aligned} &\frac{\partial}{\partial t} (D_{+,t}^{1/2} u(x, t)) - D_{+,t}^{1/2} u(x, t) \\ &= \frac{1}{\pi} \Delta u(x, t) - (x^2 + t^2) u(x, t) + e^t \sin t \sin x, \end{aligned} \quad (35)$$

$(x, t) \in (0, \pi) \times R_+$,

with the boundary condition

$$-\frac{\partial u(0, t)}{\partial x} = \frac{\partial u(\pi, t)}{\partial x} = -e^t \sin t, \quad t \geq 0. \quad (36)$$

Here $\alpha = 1/2$, $\Omega = (0, \pi)$, $n = 1$, $p(t) = -1$, $a(t) = 1/\pi$, $q(x, t) = x^2 + t^2$, and $f(x, t) = e^t \sin t \sin x$. It is obvious that $V(t) = e^{t_0-t}$, $q(t) = t^2$, $\Psi(t) = -\pi e^t \sin t$, and

$$\begin{aligned} F(t) &= \int_{\Omega} f(x, t) dx = \int_0^{\pi} e^t \sin t \sin x dx \\ &= 2e^t \sin t, \\ H(t) &= F(t) + a(t) \Psi(t) = e^t \sin t. \end{aligned} \quad (37)$$

Hence

$$\begin{aligned} &\int_{t_0}^{\xi} H(s) V(s) ds = \int_{t_0}^{\xi} e^s \sin s e^{t_0-s} ds \\ &= e^{t_0} (-\cos \xi + \cos t_0). \end{aligned} \quad (38)$$

Letting $t_0 = \pi/2$, we obtain

$$\begin{aligned} & \int_0^t \frac{(t-\xi)^{-1/2}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi \\ &= \int_0^t (t-\xi)^{-1/2} e^{\xi-\pi/2} \left(C + \int_{\pi/2}^{\xi} H(s) V(s) ds \right) d\xi \quad (39) \\ &= \int_0^t (t-\xi)^{-1/2} e^{\xi-\pi/2} \left(C - e^{\pi/2} \cos \xi \right) d\xi. \end{aligned}$$

Setting $t - \xi = s^2$, above integral (39) can be written as the following form:

$$\begin{aligned} & \int_{\sqrt{t}}^0 \frac{1}{s} \left(C e^{t-s^2-\pi/2} - e^{t-s^2} \cos(t-s^2) \right) (-2s) ds \\ &= 2C e^{t-\pi/2} \int_0^{\sqrt{t}} e^{-s^2} ds \\ & \quad - 2e^t \int_0^{\sqrt{t}} e^{-s^2} \cos(t-s^2) ds \\ &= 2C e^{t-\pi/2} \int_0^{\sqrt{t}} e^{-s^2} ds - 2e^t \cos t \int_0^{\sqrt{t}} e^{-s^2} \cos s^2 ds \quad (40) \\ & \quad - 2e^t \sin t \int_0^{\sqrt{t}} e^{-s^2} \sin s^2 ds \\ &= 2e^t \left[C e^{-\pi/2} \int_0^{\sqrt{t}} e^{-s^2} ds - \cos t \int_0^{\sqrt{t}} e^{-s^2} \cos s^2 ds \right. \\ & \quad \left. - \sin t \int_0^{\sqrt{t}} e^{-s^2} \sin s^2 ds \right]. \end{aligned}$$

Noting that

$$\begin{aligned} & \left| e^{-s^2} \cos s^2 \right| \leq e^{-s^2}, \\ & \left| e^{-s^2} \sin s^2 \right| \leq e^{-s^2}, \\ & \lim_{t \rightarrow \infty} \int_0^{\sqrt{t}} e^{-s^2} ds = \frac{\sqrt{\pi}}{2}, \end{aligned} \quad (41)$$

we obtain that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int_0^{\sqrt{t}} e^{-s^2} \cos s^2 ds, \\ & \lim_{t \rightarrow \infty} \int_0^{\sqrt{t}} e^{-s^2} \sin s^2 ds \end{aligned} \quad (42)$$

are convergent.

Therefore, combining (39)–(41) and noting the fact that (42) are convergent, by careful calculation, we can get

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi \\ & \leq -\infty < 0, \\ & \limsup_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} H(s) V(s) ds \right) d\xi \\ & \geq +\infty > 0, \end{aligned} \quad (43)$$

which shows that all the conditions of Theorem 7 are fulfilled. Then every solution of the problems (35)–(36) oscillates in $(0, \pi) \times R_+$.

Example 2. Consider the following fractional partial differential equation:

$$\begin{aligned} & \frac{\partial}{\partial t} \left(D_{+,t}^{1/2} u(x, t) \right) - D_{+,t}^{1/2} u(x, t) \\ &= e^{-t} \Delta u(x, t) - (2x^2 + t^2) u(x, t) \\ & \quad + e^{t-\pi} \sin t \sin x, \quad (x, t) \in (0, \pi) \times R_+, \end{aligned} \quad (44)$$

with the boundary condition

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0. \quad (45)$$

Here $\alpha = 1/2$, $\Omega = (0, \pi)$, $n = 1$, $p(t) = -1$, $a(t) = e^{-t}$, $q(x, t) = 2x^2 + t^2$, and $f(x, t) = e^{t-\pi} \sin t \sin x$.

It is obvious that $V(t) = e^{t_0-t}$, $q(t) = t^2$, $\beta_0 = 1$, and $\varphi(x) = \sin x$. Therefore,

$$\begin{aligned} Y(t) &= \int_{\Omega} f(x, t) \varphi(x) dx = \int_0^{\pi} e^{t-\pi} \sin t \sin^2 x dx \\ &= \frac{\pi e^{-\pi}}{2} e^t \sin t. \end{aligned} \quad (46)$$

Hence

$$\begin{aligned} & \int_{t_0}^{\xi} Y(s) V(s) ds = \frac{\pi e^{-\pi}}{2} \int_{t_0}^{\xi} e^s \sin s e^{t_0-s} ds \\ &= \frac{\pi e^{t_0-\pi}}{2} (-\cos \xi + \cos t_0). \end{aligned} \quad (47)$$

Using a similar way in Example 1, we can obtain

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} Y(s) V(s) ds \right) d\xi \\ & \leq -\infty < 0, \\ & \limsup_{t \rightarrow \infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} Y(s) V(s) ds \right) d\xi \\ & \geq +\infty > 0. \end{aligned} \quad (48)$$

Therefore, by Theorem 8, it is easy to see that every solution of the problems (44)–(45) is oscillatory in $(0, \pi) \times R_+$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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