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Research Article

New Stability Analysis for Linear Systems with Time-Varying Delay Based on Combined Convex Technique

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A novel combined convex method is developed for the stability of linear systems with a time-varying delay. A new delay-dependent stability condition expressed in terms of linear matrix inequalities (LMIs) is derived by employing a dedicated constructed Lyapunov-Krasovskii functional (LKF), utilizing the Wirtinger inequality and the reciprocally convex approach to handle the integral term of quadratic quantities. Different from the previous convex techniques which only tackle the time-varying delay, our method adopts the idea of combined convex technique which can tackle not only the delay but also the delay variation. Four well-known examples are illustrated to show the effectiveness of the proposed results.

1. Introduction

In recent years, the stability of the time-delayed linear system is one of the hot issues in control theory, for time delay occurs in different physical, industrial, and engineering systems, such as aircraft, biological systems, population dynamics, and neural networks. It is well-known that time delay is often a source of the degradation of performance and/or the instability of the time-delayed linear system. Hence, the problem of the stability analysis of time-delayed systems has attracted considerable attention in recent years. For more details, see the literature [1–29].

Currently, many researchers have devoted time and effort to the stability analysis of linear-systems with time delay, and a great number of results on delay-dependent stability conditions for time-delayed systems have been reported in the briefs [6, 8, 11, 17, 20, 22, 28, 29] because it is well known that delay-dependent stability criteria which include the information on the size of time delay are generally less conservative than delay-independent ones when the size of time delay is small. The objective of the stability analysis is to find a less conservative condition to enlarge the feasibility region of stability criteria such that it guarantees asymptotic stability of time-delayed systems as large as possible. In order to reduce the conservatism of the stability criteria for linear time-delayed system, integral inequality lemma was used by Park and Ko [9]. He et al. presented some less conservative stability conditions using free-weighting matrix in [11, 23]. Descriptor model transform method was presented by Fridman and Shaked in [12]; Jensen's inequality and delay decomposition method were used in [17, 20, 21] and [30], respectively. Jensen's inequality introduces an undesirable conservatism in the stability conditions, so, some Wirtinger inequalities which allow consideration of more accurate integral inequalities are introduced by Seuret and Gouaisbaut to deal with the derivative of LKF recently in [18]. Notice that the reciprocal convex approach presented in [24] has been a popular method. Although this method can be more effective than earlier convex techniques in studying the time-varying delay systems, it still needs more improvements since it cannot tackle the delay variation or more complicated cases [15].

In the light of the discussion above, in this paper, the combined convex method which was presented in [31, 32] is further developed for the stability of the linear systems with time-varying delay. With the new method, both the time-varying delay and the variation of the delay can be tackled. We notice that some important terms are ignored during the construction of the LKF because of limitation of the previous method. First, we construct a new LKF and

use reciprocal convex approach and Wirtinger inequality to handle the integral term of quadratic quantities, and then we derive the stability condition in terms of the sum of two first-order convex functions with respect to the time-varying delay and its variation. Second, a novel delay-dependent stability criterion is presented in terms of LMIs which can be solved efficiently by convex optimization algorithm. Finally, four well-known examples are given to illustrate the effectiveness of the proposed method.

Throughout this paper, the following notations will be used: C^T represents the transposition of matrix C, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices. $P>0$ means that P is positively definite. Symbol $*$ represents the elements below the main diagonal of a symmetric block matrix. $\mathrm{Sym}(X)$ is defined as $\mathrm{Sym}(X) = X + X^T$.

2. Problem Statement

Consider the following linear systems with time-varying delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-h(t)), \quad t \geq 0, \\ x(t) &= \psi(t), \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A, B \in \mathbb{R}^{n\times n}$ are constant matrices with appropriate dimensions, $h(t)$ is the time-varying delay, and it is assumed to satisfy the following:

C1: $0 \leq h(t) \leq h$, $\mu_1 \leq \dot{h}(t) \leq \mu_2 < 1$, for all $t \geq 0$;

C2: $0 \leq h(t) \leq h$, for all $t \geq 0$.

The initial condition $\psi(t)$ is a continuously differentiable function on $[-h, 0]$.

For C1, let us define ∇_d in the following set:

$$\Psi_d := \left\{\nabla_d \mid \nabla_d \in \mathrm{conv}\left\{\nabla_d^1, \nabla_d^2\right\}\right\}, \tag{2}$$

where conv denotes the convex hull, $\nabla_d^1 = \mu_1$, and $\nabla_d^2 = \mu_2$. Then, there exists a parameter $\theta > 0$ such that $\dot{h}(t)$ can be expressed as convex combination of the vertices as follows:

$$\dot{h}(t) = \theta\nabla_d^1 + (1-\theta)\nabla_d^2. \tag{3}$$

If a matrix $M_{|\dot{h}(t)|}$ is affinely dependent on $\dot{h}(t)$, then $M_{|\dot{h}(t)|}$ can be expressed as convex combinations of the vertices

$$M_{[\dot{h}(t)]} = \theta M_{[\nabla_d^1]} + (1-\theta) M_{[\nabla_d^2]}. \tag{4}$$

From (4), if a stability condition is affinely dependent on $\dot{h}(t)$, then it needs only to check the vertex values of $\dot{h}(t)$ instead of checking all values of $\dot{h}(t)$ [33].

Before deriving the main results, the following lemmas are stated, which will be used in the proof of the main results.

Lemma 1 (see [18]). *For a given matrix $R > 0$, the following inequality holds for all continuously differentiable functions ω in $[a, b] \to \mathbb{R}^n$:*

$$\begin{aligned} \int_a^b \dot{\omega}^T(u) R\dot{\omega}(u)\,du \geq{}& \frac{1}{b-a}(\omega(b)-\omega(a))^T R(\omega(b)-\omega(a)) \\ &+ \frac{3}{b-a}\Psi^T R\Psi, \end{aligned} \tag{5}$$

where $\Psi = \omega(b) + \omega(a) - (2/(b-a))\int_a^b \omega(u)du$.

Lemma 2 (see [22]). *Suppose that $\Omega, \Xi_{1i}, \Xi_{2i}$ $(i = 1, 2)$ are the constant matrices of appropriate dimensions, $\alpha \in [0, 1]$, and $\beta \in [0, 1]$, then $\Omega + [\alpha\Xi_{11} + (1-\alpha)\Xi_{12}] + [\beta\Xi_{21} + (1-\beta)\Xi_{22}] < 0$ holds, if the following four inequalities hold simultaneously:*

$$\begin{aligned} &\Omega + \Xi_{11} + \Xi_{21} < 0, \qquad \Omega + \Xi_{11} + \Xi_{22} < 0, \\ &\Omega + \Xi_{12} + \Xi_{21} < 0, \qquad \Omega + \Xi_{12} + \Xi_{22} < 0. \end{aligned} \tag{6}$$

The reciprocally convex combination inequality provided in Park et al. [24] is used in this paper. This inequality has been reformulated by Seuret and Gouaisbaut [18] and is stated in Lemma 3.

Lemma 3. *For given positive integers n, m, a scalar δ in the interval $(0, 1)$, a given $n\times n$ matrix $R > 0$, and two matrices W_1 and W_2 in $\mathbb{R}^{n\times m}$. Define, for all vectors ξ in $\mathbb{R}^m$, the function $\Theta(\alpha, R)$ given by the following:*

$$\Theta(\delta, R) = \frac{1}{\delta}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\delta}\xi^T W_2^T R W_2 \xi. \tag{7}$$

Then, if there exists a matrix X in $\mathbb{R}^{n\times n}$ such that $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] > 0$, then the following inequality holds:

$$\min_{\delta\in(0,1)} \Theta(\delta, R) \geq \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}. \tag{8}$$

3. Main Result

The main objective of this section is to achieve a less conservative condition such that it can guarantee the stability of system (1) under the constraint C1. First, we estimate the derivative of Lyapunov functional less conservatively by constructing a new augmented LKF; then, with the Wirtinger inequality and the newly developed combined convex technique, the improved stability results are derived, which are less conservative than some existing ones.

For simplicity of matrix representation, we set block entry matrices e_i $(i = 1, \ldots, 7) \in \mathbb{R}^{7n\times n}$ (e.g., $e_2^T = [0\ I\ 0\ 0\ 0\ 0\ 0]$) and we define the following:

$$\eta^T(t) = \left[x^T(t) \;\; x^T(t-h) \;\; \int_{t-h}^{t} x^T(s)\,ds\right], \qquad \phi^T(t,s) = \left[x^T(t) \;\; x^T(s) \;\; \dot{x}^T(s)\right], \qquad \varphi^T(t) = \left[\dot{x}^T(t) \;\; 0 \;\; 0\right],$$
$$\xi^T(t) = \left[x^T(t) \;\; x^T(t-h(t)) \;\; x^T(t-h) \;\; \frac{1}{h(t)}\int_{t-h(t)}^{t} x^T(s)\,ds \;\; \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)} x^T(s)\,ds \;\; \dot{x}^T(t-h) \;\; \dot{x}^T(t-h(t))\right], \tag{9}$$

and construct the following LKF:

$$V(x_t) = \sum_{i=1}^{4} V_i(x_t), \tag{10}$$

where

$$\begin{aligned}
V_1(x_t) &= \eta^T(t) P \eta(t),\\
V_2(x_t) &= \int_{t-h(t)}^{t} \begin{bmatrix} \phi(t,s) \\ x(t-h) \end{bmatrix}^T Q \begin{bmatrix} \phi(t,s) \\ x(t-h) \end{bmatrix} ds\\
&\quad + \int_{t-h}^{t-h(t)} \begin{bmatrix} \phi(t,s) \\ x(t-h) \end{bmatrix}^T M \begin{bmatrix} \phi(t,s) \\ x(t-h) \end{bmatrix} ds,\\
V_3(x_t) &= \int_{t-h}^{t} \phi(t,s)^T Z \phi(t,s)\,ds,\\
V_4(x_t) &= h\int_{t-h}^{t}\int_{s}^{t} \dot{x}^T(u) R \dot{x}(u)\,du\,ds,
\end{aligned} \tag{11}$$

and here $P \in \mathbb{R}^{3n\times 3n}$, $Q \in \mathbb{R}^{4n\times 4n}$, $M \in \mathbb{R}^{4n\times 4n}$, $Z \in \mathbb{R}^{3n\times 3n}$, and $R \in \mathbb{R}^{n\times n}$.

Theorem 4. *For given scalar $h \geq 0$, μ_1 and μ_2 with C1, the system* (1) *is asymptotically stable, if there exist symmetric positive definite matrices $P \in \mathbb{R}^{3n\times 3n}$, $Q \in \mathbb{R}^{4n\times 4n}$, $M \in \mathbb{R}^{4n\times 4n}$, $Z \in \mathbb{R}^{3n\times 3n}$, and $R \in \mathbb{R}^{n\times n}$ and any matrices $S_{ij} \in \mathbb{R}^{n\times n}$ $(i, j = 1, 2)$, such that the following LMIs are feasible:*

$$\Sigma_{[\nabla_d^k]} + h\Sigma_1 + \Xi < 0, \quad \forall k = 1, 2, \tag{12}$$

$$\Sigma_{[\nabla_d^k]} + h\Sigma_2 + \Xi < 0, \quad \forall k = 1, 2, \tag{13}$$

$$\Phi > 0, \tag{14}$$

where

$$A_C^T = \left[A \;\; B \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T,$$

$$\Gamma = \left[e_1 - e_2 \;\; e_1 + e_2 - 2e_4 \;\; e_2 - e_3 \;\; e_2 + e_3 - 2e_5\right]^T,$$

$$\begin{aligned}
\Sigma_{[\nabla_d^k]} &= -\left(1-\nabla_d^k\right)\left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right] Q \left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right]^T\\
&\quad + \left(1-\nabla_d^k\right)\left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right] M \left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right]^T,\\
\Sigma_1 &= Sym\left(\left[0 \;\; 0 \;\; e_4\right] P \left[A_C^T \;\; e_6 \;\; e_1 - e_3\right]^T\right)\\
&\quad + Sym\left(\left[e_1 \;\; e_4 \;\; 0 \;\; e_3\right] Q \left[A_C^T \;\; 0 \;\; 0 \;\; e_6\right]^T\right)\\
&\quad + Sym\left(\left[0 \;\; e_4 \;\; 0\right] Z \left[A_C^T \;\; 0 \;\; 0\right]^T\right),\\
\Sigma_2 &= Sym\left(\left[0 \;\; 0 \;\; e_5\right] P \left[A_C^T \;\; e_6 \;\; e_1 - e_3\right]^T\right)\\
&\quad + Sym\left(\left[e_1 \;\; e_5 \;\; 0 \;\; e_3\right] M \left[A_C^T \;\; 0 \;\; 0 \;\; e_6\right]^T\right)\\
&\quad + Sym\left(\left[0 \;\; e_5 \;\; 0\right] Z \left[A_C^T \;\; 0 \;\; 0\right]^T\right),\\
\Xi &= Sym\left(\left[e_1 \;\; e_3 \;\; 0\right] P \left[A_C^T \;\; e_6 \;\; e_1 - e_3\right]^T\right)\\
&\quad + \left[e_1 \;\; e_1 \;\; A_C^T \;\; e_3\right] Q \left[e_1 \;\; e_1 \;\; A_C^T \;\; e_3\right]^T\\
&\quad + Sym\left(\left[0 \;\; 0 \;\; e_1 - e_2 \;\; 0\right] Q \left[A_C^T \;\; 0 \;\; 0 \;\; e_6\right]^T\right)\\
&\quad - \left[e_1 \;\; e_3 \;\; e_6 \;\; e_3\right] M \left[e_1 \;\; e_3 \;\; e_6 \;\; e_3\right]^T\\
&\quad + Sym\left(\left[0 \;\; 0 \;\; e_2 - e_3 \;\; 0\right] M \left[A_C^T \;\; 0 \;\; 0 \;\; e_6\right]^T\right)\\
&\quad + \left[e_1 \;\; e_1 \;\; A_C^T\right] Z \left[e_1 \;\; e_1 \;\; A_C^T\right]^T\\
&\quad - \left[e_1 \;\; e_3 \;\; e_6\right] Z \left[e_1 \;\; e_3 \;\; e_6\right]^T\\
&\quad + Sym\left(\left[he_1 \;\; 0 \;\; e_1 - e_3\right] Z \left[A_C^T \;\; 0 \;\; 0\right]^T\right)\\
&\quad + h^2 A_C^T R A_C - \Gamma^T \Phi \Gamma,
\end{aligned}$$

$$\Phi = \begin{bmatrix} \Theta & S \\ * & \Theta \end{bmatrix}, \quad \Theta = \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix}, \; S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \tag{15}$$

Proof. Taking the derivative of $V(x_t)$ with respect to t along the solutions of system (1) yields

$$
\begin{aligned}
\dot{V}_1(x_t) &= 2\eta^T(t)P\begin{bmatrix}\dot{x}(t)\\ \dot{x}(t-h)\\ x(t)-x(t-h)\end{bmatrix}\\
&= \xi^T(t)\left\{\mathrm{Sym}\left([e_1\ \ e_3\ \ h(t)e_4+(h-h(t))e_5]P\left[A_C^T\ \ e_6\ \ e_1-e_3\right]^T\right)\right\}\xi(t),\\
\dot{V}_2(x_t) &= \begin{bmatrix}\phi(t,t)\\ x(t-h)\end{bmatrix}^T Q\begin{bmatrix}\phi(t,t)\\ x(t-h)\end{bmatrix}-\left(1-\dot{h}(t)\right)\begin{bmatrix}\phi(t,t-h(t))\\ x(t-h)\end{bmatrix}^T Q\begin{bmatrix}\phi(t,t-h(t))\\ x(t-h)\end{bmatrix}\\
&\quad+2\int_{t-h(t)}^{t}\begin{bmatrix}\phi(t,s)\\ x(t-h)\end{bmatrix}^T Q\begin{bmatrix}\varphi(t)\\ \dot{x}(t-h)\end{bmatrix}ds+\left(1-\dot{h}(t)\right)\begin{bmatrix}\phi(t,t-h(t))\\ x(t-h)\end{bmatrix}^T M\begin{bmatrix}\phi(t,t-h(t))\\ x(t-h)\end{bmatrix}\\
&\quad-\begin{bmatrix}\phi(t,t-h)\\ x(t-h)\end{bmatrix}^T M\begin{bmatrix}\phi(t,t-h)\\ x(t-h)\end{bmatrix}+2\int_{t-h}^{t-h(t)}\begin{bmatrix}\phi(t,s)\\ x(t-h)\end{bmatrix}^T M\begin{bmatrix}\varphi(t)\\ \dot{x}(t-h)\end{bmatrix}ds\\
&= \xi^T(t)\left\{\left[e_1\ \ e_1\ \ A_C^T\ \ e_3\right]Q\left[e_1\ \ e_1\ \ A_C^T\ \ e_3\right]^T-\left(1-\dot{h}(t)\right)[e_1\ \ e_2\ \ e_7\ \ e_3]Q[e_1\ \ e_2\ \ e_7\ \ e_3]^T\right.\\
&\qquad+\mathrm{Sym}\left([h(t)e_1\ \ h(t)e_4\ \ e_1-e_2\ \ h(t)e_3]Q\left[A_C^T\ \ 0\ \ 0\ \ e_6\right]^T\right)\\
&\qquad+\left(1-\dot{h}(t)\right)[e_1\ \ e_2\ \ e_7\ \ e_3]\times M[e_1\ \ e_2\ \ e_7\ \ e_3]^T-[e_1\ \ e_3\ \ e_6\ \ e_3]M[e_1\ \ e_3\ \ e_6\ \ e_3]^T\\
&\qquad\left.+\mathrm{Sym}\left([(h-h(t))e_1\ \ (h-h(t))e_5\ \ e_2-e_3\ \ (h-h(t))e_3]M\left[A_C^T\ \ 0\ \ 0\ \ e_6\right]^T\right)\right\}\xi(t),\\
\dot{V}_3(x_t) &= \phi(t,t)^T Z\,\phi(t,t)-\phi(t,t-h)^T Z\phi(t,t-h)+2\int_{t-h}^{t}\phi(t,s)^T Z\varphi(t)\,ds\\
&= \xi^T(t)\left\{\left[e_1\ \ e_1\ \ A_C^T\right]Z\left[e_1\ \ e_1\ \ A_C^T\right]^T-[e_1\ \ e_3\ \ e_6]Z[e_1\ \ e_3\ \ e_6]^T\right.\\
&\qquad\left.+\mathrm{Sym}\left([he_1\ \ h(t)e_4+(h-h(t))e_5\ \ e_1-e_3]Z\left[A_C^T\ \ 0\ \ 0\right]^T\right)\right\}\xi(t).
\end{aligned}
\tag{16}
$$

Finally, $\dot{V}_4$ is easily obtained as

$$
\dot{V}_4(x_t) = h^2\dot{x}^T(t)R\dot{x}(t)-h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds. \tag{17}
$$

Using Lemmas 1 and 3 yields

$$
\begin{aligned}
&-h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds\\
&= -h\int_{t-h(t)}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds-h\int_{t-h}^{t-h(t)}\dot{x}^T(s)R\dot{x}(s)\,ds\\
&\le -\frac{h}{h(t)}[x(t)-x(t-h(t))]^T R[x(t)-x(t-h(t))]\\
&\quad-\frac{h}{h-h(t)}[x(t-h(t))-x(t-h)]^T\\
&\quad\times R[x(t-h(t))-x(t-h)]\\
&\quad-\frac{3h}{h(t)}\left[x(t)+x(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right]^T\\
&\quad\times R\left[x(t)+x(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right]\\
&\quad-\frac{3h}{h-h(t)}\\
&\quad\times\left[x(t-h(t))+x(t-h)-\frac{2}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)\,ds\right]^T\\
&\quad\times R\left[x(t-h(t))+x(t-h)-\frac{2}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)\,ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{h}{h(t)}\begin{bmatrix} x(t)-x(t-h(t)) \\ x(t)+x(t-h(t))-\dfrac{2}{h(t)}\displaystyle\int_{t-h(t)}^{t} x(s)\,ds \end{bmatrix}^T \\
&\quad \times \Theta \begin{bmatrix} x(t)-x(t-h(t)) \\ x(t)+x(t-h(t))-\dfrac{2}{h(t)}\displaystyle\int_{t-h(t)}^{t} x(s)\,ds \end{bmatrix} \\
&\quad - \frac{h}{h-h(t)} \\
&\quad \times \begin{bmatrix} x(t-h(t))-x(t-h) \\ x(t-h(t))+x(t-h)-\dfrac{2}{h-h(t)}\displaystyle\int_{t-h}^{t-h(t)} x(s)\,ds \end{bmatrix}^T \\
&\quad \times \Theta \begin{bmatrix} x(t-h(t))-x(t-h) \\ x(t-h(t))+x(t-h)-\dfrac{2}{h-h(t)}\displaystyle\int_{t-h}^{t-h(t)} x(s)\,ds \end{bmatrix} \\
&\le -\begin{bmatrix} x(t)-x(t-h(t)) \\ x(t)+x(t-h(t))-\dfrac{2}{h(t)}\displaystyle\int_{t-h(t)}^{t} x(s)\,ds \\ x(t-h(t))-x(t-h) \\ x(t-h(t))+x(t-h)-\dfrac{2}{h-h(t)}\displaystyle\int_{t-h}^{t-h(t)} x(s)\,ds \end{bmatrix}^T \\
&\quad \times \Phi \begin{bmatrix} x(t)-x3(t-h(t)) \\ x(t)+x(t-h(t))-\dfrac{2}{h(t)}\displaystyle\int_{t-h(t)}^{t} x(s)\,ds \\ x(t-h(t))-x(t-h) \\ x(t-h(t))+x(t-h)-\dfrac{2}{h-h(t)}\displaystyle\int_{t-h}^{t-h(t)} x(s)\,ds \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\dot{V}_4(x_t) \\
&\le \xi^T(t) \\
&\quad \times \left\{ h^2 A_C^T R A_C - \begin{bmatrix} e_1-e_2 \\ e_1+e_2-2e_4 \\ e_2-e_3 \\ e_2+e_3-2e_5 \end{bmatrix}^T \Phi \begin{bmatrix} e_1-e_2 \\ e_1+e_2-2e_4 \\ e_2-e_3 \\ e_2+e_3-2e_5 \end{bmatrix} \right\} \xi(t) \\
&= \xi^T(t)\left\{ h^2 A_C^T R A_C - \Gamma^T \Phi \Gamma \right\} \xi(t).
\end{aligned} \tag{18}
$$

Now, combining with (16)–(18), we get

$$
\dot{V}(x_t) \le \xi^T(t)\left(\Sigma_{[\dot{h}(t)]} + \beta h \Sigma_1 + (1-\beta) h \Sigma_2 + \Xi\right)\xi(t), \tag{19}
$$

where

$$
\begin{aligned}
\Sigma_{[\dot{h}(t)]} = &-\left(1-\dot{h}(t)\right)\left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right] Q \left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right]^T \\
&+\left(1-\dot{h}(t)\right)\left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right] M \left[e_1 \;\; e_2 \;\; e_7 \;\; e_3\right]^T; \\
&\beta = \frac{h(t)}{h}.
\end{aligned} \tag{20}
$$

It is clear that $\Sigma_{[\dot{h}(t)]} + \beta h \Sigma_1 + (1-\beta) h \Sigma_2 + \Xi < 0$ implies $\dot{V}(x_t) < 0$, which means that system (1) is asymptotically stable. Notice that $\Sigma_{[\dot{h}(t)]}$ is affine and a convex combination on $\dot{h}(t)$ satisfying $\mu_1 \le \dot{h}(t) \le \mu_2$, then $\Sigma_{[\dot{h}(t)]}$ can be expressed as convex combinations of the vertices $\Sigma_{[\dot{h}(t)]} = \alpha\Sigma_{[\nabla_d^1]} + (1-\alpha)\Sigma_{[\nabla_d^2]}$, $\alpha \in [0,1]$. Utilizing Lemma 2, then $\Sigma_{[\dot{h}(t)]} + \beta h \Sigma_1 + (1-\beta) h \Sigma_2 + \Xi < 0$ holds, if and only if (12) and (13) hold. This completes our proof. □

Remark 5. Recently, the reciprocally convex optimization technique and Wirtinger inequality to reduce the conservatism of stability criteria for linear systems with time-varying delay were proposed in [24, 25, 28] and [18, 21], respectively, and these methods were utilized in (18). In Lemma 1, it can be noticed that the term $(1/(b-a))(\omega(b)-\omega(a))^T R(\omega(b)-\omega(a))$ is equal to Jensen's inequality and that the newly appeared term $(3/(b-a))\Psi^T R \Psi$ can reduce the LKF enlargement of the estimation. The usage of reciprocally convex optimization method avoids the enlargement of $h(t)$ and $h-h(t)$ while only introducing matrix S. Then, the convex optimization method is used to handle $\dot{V}(x_t)$. During the proof procedure above, the dedicated constructed LKF (11) has full information on the systems.

Remark 6. Furthermore, we introduce terms $x(t)$, $x(t-h)$ in V_2. Therefore, more information on the cross terms in $x(t)$, $\dot{x}(t)$ and $x(t-h)$, $\dot{x}(t-h)$ is utilized. To reduce the conservatism, the term $\int_{t-h}^{t-h(t)} \left[\begin{smallmatrix} \phi(t,s) \\ x(t-h) \end{smallmatrix}\right]^T M \left[\begin{smallmatrix} \phi(t,s) \\ x(t-h) \end{smallmatrix}\right] ds$ is chosen as LKF when $\mu_1 \le \dot{h}(t) \le \mu_2$. These considerations highlight the main differences in the construction of the LKF candidate in this paper.

Remark 7. When $h(t)$ is not differentiable and $\dot{h}(t)$ is unknown, the state $\dot{x}(t-h(t))$ cannot be utilized as the augmented vector $\xi(t)$ by the methods presented in the proof of Theorem 4. Thus, we should modify the LKF which includes the term $\dot{x}(t-h(t))$, so we set $Q, M = 0$. Therefore, the corresponding stability criterion for C2 will be introduced as Corollary 8.

In Corollary 8, block entry matrices $\tilde{e}_i(t) \in \mathbb{R}^{6n\times n}$ will be used and the following notations are defined for the sake of simplicity of matrix notation:

Table 1: Delay bounds h with different μ_2 and $\mu_1 = -\mu_2$.

μ_2	0.0	0.1	0.2	0.5	0.8	$\mu_2 \geq 1$ or unknown
Fridman and Shaked [12]	4.472	3.604	3.303	2.008	1.364	0.999
He et al. [11]	4.472	3.605	3.039	2.043	1.492	1.345
Sun et al. (2010) [20]	4.472	3.611	3.047	2.072	1.590	1.529
Seuret and Gouaisbaut [18, Theorem 7]	6.059	4.703	3.834	2.420	2.137	2.128
Theorem 4	6.059	4.733	3.932	2.891	2.592	—
Corollary 8	—	—	—	—	—	2.113

$$\begin{aligned}
\widetilde{\xi}^T(t) &= \left[x^T(t) \;\; x^T(t-h(t)) \;\; x^T(t-h) \;\; \frac{1}{h(t)}\int_{t-h(t)}^{t} x^T(s)\,ds \;\; \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)} x^T(s)\,ds \;\; \dot{x}^T(t-h)\right],\\
\widetilde{\Sigma}_1 &= \operatorname{Sym}\left(\left[0 \;\; 0 \;\; \widetilde{e}_4\right] P\left[\widetilde{A}_C^T \;\; \widetilde{e}_6 \;\; \widetilde{e}_1-\widetilde{e}_3\right]^T\right) + \operatorname{Sym}\left(\left[0 \;\; \widetilde{e}_4 \;\; 0\right] Z\left[\widetilde{A}_C^T \;\; 0 \;\; 0\right]^T\right),\\
\widetilde{\Sigma}_2 &= \operatorname{Sym}\left(\left[0 \;\; 0 \;\; \widetilde{e}_5\right] P\left[\widetilde{A}_C^T \;\; \widetilde{e}_6 \;\; \widetilde{e}_2-\widetilde{e}_3\right]^T\right) + \operatorname{Sym}\left(\left[0 \;\; \widetilde{e}_5 \;\; 0\right] Z\left[\widetilde{A}_C^T \;\; 0 \;\; 0\right]^T\right),\\
&\qquad\qquad \widetilde{A}_C^T = \left[A \;\; B \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T,\\
\widetilde{\Xi} &= \operatorname{Sym}\left(\left[\widetilde{e}_1 \;\; \widetilde{e}_3 \;\; 0\right] P\left[\widetilde{A}_C^T \;\; \widetilde{e}_6 \;\; \widetilde{e}_1-\widetilde{e}_3\right]^T\right) + \left[\widetilde{e}_1 \;\; \widetilde{e}_1 \;\; \widetilde{A}_C^T\right] Z\left[\widetilde{e}_1 \;\; \widetilde{e}_1 \;\; \widetilde{A}_C^T\right]^T - \left[\widetilde{e}_1 \;\; \widetilde{e}_3 \;\; \widetilde{e}_6\right] Z\left[\widetilde{e}_1 \;\; \widetilde{e}_3 \;\; \widetilde{e}_6\right]^T\\
&\quad + \operatorname{Sym}\left(\left[h\widetilde{e}_1 \;\; 0 \;\; \widetilde{e}_1-\widetilde{e}_3\right] Z\left[\widetilde{A}_C^T \;\; 0 \;\; 0\right]^T\right) + h^2 A_C^T R A_C - \Gamma^T \Phi \Gamma.
\end{aligned} \tag{21}$$

Corollary 8. *For given scalar $h \geq 0$ with C2, the system* (1) *is asymptotically stable, if there exist symmetric positive definite matrices $P \in \mathbb{R}^{3n\times 3n}$, $Z \in \mathbb{R}^{3n\times 3n}$, and $R \in \mathbb{R}^{n\times n}$. and any matrices $S_{ij} \in \mathbb{R}^{n\times n}$ $(i, j = 1, 2)$, such that the following LMIs are feasible:*

$$\begin{aligned}
\widetilde{\Sigma}_1 + \widetilde{\Xi} &< 0,\\
\widetilde{\Sigma}_2 + \widetilde{\Xi} &< 0,\\
\Phi &> 0,
\end{aligned} \tag{22}$$

where $\widetilde{\Sigma}_1$, $\widetilde{\Sigma}_2$, $\widetilde{\Xi}$ are defined in (21) *and other matrices are defined in Theorem 4.*

Proof.

$$\widetilde{V}(x_t) = \sum_{i=1}^{3} \widetilde{V}_i(x_t), \tag{23}$$

where

$$\begin{aligned}
\widetilde{V}_1(x_t) &= \eta^T(t) P \eta(t),\\
\widetilde{V}_2(x_t) &= \int_{t-h}^{t} \phi(t,s)^T Z \phi(t,s)\,ds,\\
\widetilde{V}_3(x_t) &= h\int_{t-h}^{t}\int_{s}^{t} \dot{x}^T(u) R \dot{x}(u)\,du\,ds.
\end{aligned} \tag{24}$$

Finally, we can get

$$\dot{\widetilde{V}}(x_t) \leq \widetilde{\xi}^T(t)\left(\beta h\widetilde{\Sigma}_1 + (1-\beta) h\widetilde{\Sigma}_2 + \widetilde{\Xi}\right)\widetilde{\xi}(t) \tag{25}$$

with the augmented vector $\widetilde{\xi}^T(t)$ defined in (21) and it is easy to see that $\beta h\widetilde{\Sigma}_1 + (1-\beta)h\widetilde{\Sigma}_2 + \widetilde{\Xi}$ is a convex combination, so we can see that (22) can guarantee the asymptotic stability for system (1). □

4. Numerical Examples

In this section, four examples are given to show the effectiveness of the proposed method.

Example 9. Consider the linear system (1) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{26}$$

This system is a well-known delay-dependent stable system which has the analytical maximum allowable delay bound $h_{\max} = 6.1721$ when $\dot{h}(t) = 0$, $\forall t \geq 0$. With the conditions $0 \leq h(t) \leq h$ and $\mu_1 \leq \dot{h}(t) \leq \mu_2 < 1$, Table 1 shows that our results obtained by Theorem 4 improve the allowable maximum size of the delay for various μ_2. For the case μ_2 is unknown, they show less conservatism compared to the results of [11, 12, 20]; they also show that the combined convex technique and the Wirtinger inequality methods are effective but fall short compared to the results of [18] for this case.

Table 2: Delay bounds h with different μ_2 and $\mu_1 = 0$.

μ_2	0.00	0.05	0.10	0.50	$\mu_2 \geq 1$ or unknown
Park and Ko [9]	1.99	1.81	1.75	1.61	1.60
Kim [14]	2.52	2.17	2.02	1.62	1.60
Theorem 4	3.03	2.82	2.78	2.59	—
Corollary 8	—	—	—	—	1.64

Table 3: Delay bounds h with different μ_2 and $\mu_1 = -\mu_2$.

μ_2	0.2	0.4	0.6	0.8	$\mu_2 \geq 1$ or unknown
Li et al. [5]	1.6063	1.4119	1.2425	1.1077	—
Park et al. [29, Theorem 2]	1.6492	1.5297	1.4489	1.3899	—
Theorem 4	1.9464	1.7705	1.6646	1.5911	—
Corollary 8	—	—	—	—	1.4924

Table 4: Delay bounds h with different μ_2 and $\mu_1 = -\mu_2$.

μ_2	0.5	0.9	$\mu_2 \geq 1$ or unknown	Number of variables
Li et al. [5]	0.62	0.60	—	10
He et al. [11]	0.59	0.58	—	12
He et al. [23]	0.63	0.62	—	14
Theorem 4	0.68	0.67	—	9
Corollary 8	—	—	0.66	7

Example 10. Consider the linear system (1) with the parameters

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{27}$$

With the conditions $0 \leq h(t) \leq h$ and $\mu_1 \leq \dot{h}(t) \leq \mu_2 < 1$, our results obtained by Theorem 4 with the above systems are shown in Table 2. When $h(t)$ is not differentiable or $\dot{h}(t)$ is unknown, the corresponding results obtained by Corollary 8 are also included in Table 2. From Table 2, it can be seen that our results obtained both by Theorem 4 and by Corollary 8 improve the allowable maximum size of the delay for various μ_2.

Example 11. Consider the linear system (1) with the parameters

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}. \tag{28}$$

Our results obtained by Theorem 4 and by Corollary 8 are listed in Table 3. From Table 3, one can see that our results for Example 11 give larger upper bounds of time delay than the ones in [5, 29].

Example 12. Consider the linear system (1) with the parameters

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -0.009 & 0.09 & -0.04 & 0.04 \\ 0.09 & 0.09 & 0.04 & -0.06 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1.1789 & -1.3096 & -1.6629 & -7.3974 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

Table 4 lists the comparison results for μ_2 and unknown μ_2. It is clear that the results obtained in this paper are better than those in [5, 11, 23]. We can also see that the number of variables of our paper is less than that of others, so our methods can reduce the computational burden.

5. Conclusion

The problem of delay-dependent stability for linear system with time-varying delay is investigated in this paper. By using a novel combined convex technique and Wirtinger inequality to deal with the derivative of Lyapunov-Krasovskii functional, a less conservative delay-dependent stability criterion expressed in terms of LMIs has been presented. Four illustrative examples are given to demonstrate the reduced

conservativeness of the proposed method and improvements over some existing ones.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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